or

$$\begin{aligned}\psi_1(t)&=-[(\omega_0'^2+\omega_0^2B_2^2)/\omega_0']\\&\quad\times\exp[-\omega_0B_2t]\sin\omega_0't,\\\psi_2(t)&=\exp[-\omega_0B_2t]\\&\quad\times[\cos\omega_0't+(\omega_0)/(\omega_0')B_2\sin\omega_0't],\end{aligned}\tag{A.19}$$

for $\langle u_{0\mu}(t)\rangle^{(N)}$ or $\langle v_{0\mu}(t)\rangle^{(N)}$, respectively. $\omega_0'=(1-\delta)\omega_0$ is the shifted frequency, δ being given by expression (29).

Note added in proof. The expressions for $\langle u_{0\mu}(t)\rangle^{(N)}$ and $\langle v_{0\mu}(t)\rangle^{(N)}$ which can be constructed by the use of (A.18) or (A.19) in (A.11) would present the correct behavior at short times, and so would do the velocity autocorrelation function $\langle \mathbf{v}(t)\cdot\mathbf{v}(0)\rangle_T{}^I$. Unfortunately, these $\langle u_{0\mu}(t)\rangle^{(N)}$ and $\langle v_{0\mu}(t)\rangle^{(N)}$ do not satisfy the relation

$$(d/dt)\langle u_{0\mu}(t)\rangle^{(N)}=\langle v_{0\mu}(t)\rangle^{(N)}\tag{A.20}$$

at long times, so that these expressions are not useful for our purpose. In order to find an expression for $\langle v_{0\mu}(t)\rangle^{(N)}$ which presents the correct behavior at long times, we abandon the attempt to fit the short-time behavior, and consider (A.20) itself as the defining equation for $\langle v_{0\mu}(t)\rangle^{(N)}$ itself. In this case expressions (A.19) read

$$\begin{aligned}\psi_1(t)&=-[(\omega_0'^2+\omega_0^2B_2^2)/\omega_0']\exp[-\omega_0B_2t]\sin\omega_0't,\\\psi_2(t)&=\exp[-\omega_0B_2t]\\&\quad\times[\cos\omega_0't-(\omega_0/\omega_0')B_2\sin\omega_0't],\end{aligned}\tag{A.21}$$

and are seen to correspond to the W.U. expression (28) for $\langle v_{0\mu}(t)\rangle^{(N)}$.

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Theory of Dilute Mixtures of He^3 in Liquid He^4 at Low Temperatures*

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Dilute mixtures of He^3 in liquid He^4 are regarded as a set of He^3 atoms intereacting with elementary excitations of the He^4. The coupling between the He^3 and the He^4 is chosen phenomenologically and it is not weak but, for low temperatures, its main effects may be eliminated by means of two canonical transformations. The equations which determine the transformations describe the motion of a He^4 excitation in the presence of one or two He^3 atoms and, in the latter case, they resemble Faddeev's equations in the three-body problem. At low temperatures, there are few He^4 excitations and many of the properties of the mixture are determined by the effective He^3 Hamiltonian produced by the transformation. In this respect, the system behaves like a Fermi liquid. The He^3-He^3 potential is essentially momentum-independent and consists of the Van der Waals interaction together with an effectively repulsive potential induced by the He^4. It is suggested that a Fermi-liquid analysis is the best way of obtaining information about this potential, and the possible existence of a fermion superfluid phase transition is discussed from that point of view.

1. INTRODUCTION

It has been found[1] that mixtures of liquid He^3 and He^4 containing less than 6% of He^3 should be stable at arbitrarily low temperatures. Since the elementary excitations of the He^4 are phonons of rather high energy for a given momentum,[2] the low-lying excited states of the mixtures are characteristic of the He^3 and, at sufficiently low temperatures, they should have many of the properties of Fermi liquids. Therefore, just as for pure He^3, it will be interesting to compare the equilibrium and transport properties[3] with the predictions of Landau's theory of a Fermi liquid[4] and to search for a fermion superfluid phase transition,[5] similar to that which is observed in superconductors. Also, by changing the concentration, the Fermi momentum k_F may be varied over a wide range which, usually, is not possible in other systems.

The object of this paper is to discuss the dynamics of the mixture in order to see in detail how the Fermi-liquid picture emerges, to understand how the presence of the He^4 determines an effective He^3 Hamiltonian, and to see if a phase transition is likely to occur at an accessible temperature.

Once the effective He^3 Hamiltonian is known, it should be possible to carry out good microscopic calculations since, as we shall see, the number density of the

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[1] D. O. Edwards, D. F. Brewer, P. Seligman, M. Skertic, and M. Yaqub, Phys. Rev. Letters **15**, 773 (1965); A. C. Anderson, W. R. Roach, R. E. Sarwinski, and J. C. Wheatley, *ibid.* **16**, 263 (1966).

[2] K. R. Atkins, *Liquid Helium* (Cambridge University Press, Cambridge, England, 1959).

[3] W. R. Abel, A. C. Anderson, W. C. Black, and J. C. Wheatley, Physics **1**, 337 (1965) (and references therein); V. P. Peshkov, Zh. Eksperim. i Teor. Fiz. **46**, 1510 (1964) [English transl.: Soviet Phys.—JETP **19**, 1023 (1964)]; *Proceedings of the Ninth International Conference on Low Temperature Physics*, edited by J. G. Daunt, D. O. Edwards, F. J. Milford, and M. Yaqub (Plenum Press, New York, 1965) p. 79; Zh. Eksperim. i Teor. Fiz. **48**, 997 (1965) [English transl. Soviet Phys.—JETP **21**, 663 (1965)].

[4] A. A. Abrikosov and I. M. Khalatnikov, Rept. Progr. Phys. **22**, 329 (1959).

[5] L. P. Pitaevskii, Zh. Eksperim. i Teor. Fiz. **37**, 1794 (1959) [English transl.: Soviet Phys.—JETP **10**, 1267 (1960)]; K. A. Brueckner, P. W. Anderson, T. Soda, and P. Morel, Phys. Rev. **118**, 1442 (1960); V. J. Emery and A. M. Sessler, *ibid.* **119**, 43 (1960).

He^3 is low enough for an independent-pair approximation to be used.

The mixture will be regarded as a set of He^3 atoms and He^4 elementary excitations and their interaction will be chosen phenomenologically. In contrast to the electron-phonon system, the coupling is not weak and typical He^3 velocities v are much smaller than the He^4 sound velocity c so that retardation effects[6] are unimportant and v/c may be used as an expansion parameter.

The method of calculation will be described in Sec. 3. It consists in carrying out successive canonical transformations to eliminate the main effects of the coupling between the He^3 and He^4 to zero and first order in v/c. The equations which determine the transformations describe the multiple scattering of one He^3 atom and a He^4 excitation or two He^3 atoms and a He^4 excitation. The latter may be cast into a form in which they resemble Fadeev's equations in the three-body problem.[7] The transformed Hamiltonian is an effective fermion Hamiltonian together with a part which involves boson variables and may be treated as a small perturbation at very low temperatures. This is the basis of the Fermi-liquid model of the mixture.

The following picture emerges. Each He^3 atom polarizes the He^4 and, as it moves, it acquires an effective mass m^* from the motion of the polarization cloud. Experimentally,[1,8] m^* is known to be about $2.5m$, where m is the mass of a He^3 atom. When two He^3 atoms come close together, each one interacts with and distorts the polarization cloud of the other with the result that there is an induced force between the He^3 atoms which turns out to be essentially momentum-independent. In Sec. 3 it will be argued that the induced force has to be repulsive or two He^3 atoms will form bound states which have not been observed experimentally. If there is a fermion superfluid phase transition, it will be brought about by the van der Waals attraction, aided by the increased mass and opposed by the van der Waals repulsion and by the induced force.

This picture is qualitatively quite different from that of the mixture of hard spheres discussed by Van Leeuwen and Cohen.[9] From Bogoliubov's method[10] for a low-density Bose gas they calculated the effective fermion-fermion force induced by the exchange of one phonon and showed that in some circumstances it would produce a fermion superfluid phase transition. They suggested that the same might be true for mixtures of He^3 and He^4. However, the present calculation suggests that, in helium, the induced potential is effectively repulsive and that, in addition, the van der Waals attraction, the He^3 effective mass correction, the scattering of He^4 excitations by He^3 atoms, and the overlap of the van der Waals repulsion and the induced potential may not be neglected.

In Sec. 3, we shall obtain a Fermi-liquid Hamiltonian and a qualitative picture of the mixture but, since the effective potential between He^3 atoms consists of several contributions which partially cancel each other, it is necessary to know the coupling between the He^3 atoms and the He^4 excitations with quite high accuracy in order to do good quantitative calculations. Therefore, it will be more profitable to learn directly about the effective forces by using the results of Landau's Fermi-liquid theory[4] to interpret the low-temperature properties of the mixture, and this will be discussed in Sec. 4.

2. HAMILTONIAN OF THE MIXTURE

If r_c is the radius of the repulsive region of the van der Waals potential and k_F is the He^3 Fermi momentum, then $k_F r_c$ is about 0.8 for a 6% mixture. Therefore, just as in nuclear matter, for which $k_F r_c$ is 0.6 to 0.7 (depending on the assumed core radius), it should be sufficient to use a low-order cluster expansion[11] for the He^3.

It is much more difficult to carry out a good microscopic calculation for the He^4 since its number density is almost the same as that of the pure liquid. Therefore, we shall use a phenomenological description in which the excited states of the He^4 are regarded as sets of elementary excitations[2] with momenta $\mathbf{q}$ and energy $\omega_{\mathbf{q}}$. (Units will be chosen so that $\hbar=1$.) For low momenta, the excitations are phonons of velocity c; at higher momenta they are rotons.

The Hamiltonian will be written in the form

$$H=H_3+H_4+H_I\,, \tag{2.1}$$

where

$$H_3=\sum_{i=1}^{N}\frac{\mathbf{p}_i^2}{2m}+\tfrac{1}{2}\sum_{\mathbf{k}} v_{\mathbf{k}}\rho_{\mathbf{k}}\rho_{-\mathbf{k}} \tag{2.2}$$

describes the He^3,

$$H_4=\sum_{\mathbf{q}} \omega_{\mathbf{q}} b_{\mathbf{q}}^{\dagger} b_{\mathbf{q}} \tag{2.3}$$

refers to the He^4, and

$$H_I=\sum_{\mathbf{q}} f_{\mathbf{q}}\rho_{\mathbf{q}}(b_{\mathbf{q}}^{\dagger}+b_{-\mathbf{q}})+\sum_{\mathbf{q}}\sum_{\mathbf{k}} h_{\mathbf{q}}\rho_{\mathbf{q}} b_{\mathbf{k}+\mathbf{q}}^{\dagger} b_{\mathbf{k}} \tag{2.4}$$

gives their interaction.

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[6] P. Morel and P. W. Anderson, Phys. Rev. **125**, 1263 (1962).

[7] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)]; Dokl. Akad. Nauk SSR **138**, 565 (1961); **145**, 301 (1962) [English transls.: Soviet Phys.—Doklady **6**, 384 (1961); **7**, 600 (1963)].

[8] D. J. Sandiford and H. A. Fairbank, in *Proceedings of the Seventh International Conference on Low Temperature Physics, Toronto, 1960*, edited by G. M. Graham and A. C. Hollis-Hallett (University of Toronto Press, Toronto, Canada, 1961), p. 641; C. G. Niels-Hakkenberg, L. Meermans and H. C. Kramers, in *Proceedings of the Eighth International Conference on Low Temperature Physics, London 1962*, edited by R. O. Davies (Butterworths Scientific Publications, Ltd., London 1962), p. 45.

[9] J. M. J. Van Leeuwen and E. G. D. Cohen, Physica **27**, 1157 (1961); *Proceedings of the Eighth International Conference on Low Temperature Physics, London, 1962*, edited by R. O. Davies (Butterworths Scientific Publications, Ltd., London, 1962), p. 43.

[10] N. N. Bogoliubov, J. Phys. USSR **11**, 23 (1947).

[11] H. A. Bethe, Phys. Rev. **138**, B804 (1965).

Here $\mathbf{r}_i$ and $\mathbf{k}_i$ are respectively the coordinates and momenta of the He^3,

$$\rho_{\mathbf{k}}=\sum_{i=1}^{N} e^{-i\mathbf{k}\cdot\mathbf{r}_i} \tag{2.5}$$

is the Fourier transform of the number density operator of the He^3, m is the mass of a He^3 atom, and $v_{\mathbf{k}}$ is the Fourier transform of the van der Waals potential divided by the volume of definition of the periodic boundary conditions which are to be used. Since the van der Waals potential is singular at short distances, it is assumed that $v_{\mathbf{k}}$ is defined with a cutoff which may be removed to obtain a finite result at the end of the calculation. The operators $b_{\mathbf{q}}^{\dagger}$ create a He^4 excitation with momentum $\mathbf{q}$ and obey Bose commutation relations. It is not essential to use the mixture of second quantization notation for the He^4 and coordinates and momenta for the He^3 but it turns out to be the most convenient for the method of calculation which will be employed.

The interaction H_I describes the creation, annihilation, and scattering of He^4 excitations by He^3 atoms. The creation and annihilation of several excitations will appear as a high-order process although, in principle, it should occur in H_I also. This will be reflected in the values of $f_{\mathbf{q}}$ and $h_{\mathbf{q}}$ which are to be chosen to fit experiment. There is reason to believe that this may not be a bad approximation since it has been shown[12] that a large part of the effective mass of a single He^3 atom in He^4 is given by processes which involve only one phonon.

The scattering of one excitation by another has been omitted as it has a smaller cross section than the scattering of an excitation by a He^3 atom.[2,13] It will be shown that He^3 momenta are small and so H_I has been chosen to depend upon He^3 coordinates only. It is assumed that $f_{\mathbf{q}}$ and $h_{\mathbf{q}}$ are real and depend upon $|\mathbf{q}|$ only.

The Hamiltonian should contain the self-interaction of the He^3 atoms [which is included incorrectly in Eq. (2.2)], the ground-state energy of the He^4 and an expression which, roughly speaking, gives the interaction of the He^3 with the unexcited He^4. These terms are not important for the present calculation but, for example, they would be essential in a discussion of phase separation. We shall assume that the equilibrium concentration and density are given and then attempt to understand which processes are most important for the behavior of the mixture and to establish the Fermi-liquid model.

Some knowledge of $h_{\mathbf{q}}$ may be obtained from transport properties in mixtures but it is not sufficiently detailed for our purpose. For this reason, it will be more profitable to use the Fermi-liquid model to obtain information about the effective He^3 Hamiltonian and to use this in turn to discuss the properties of the mixture and to learn about $f_{\mathbf{q}}$ and $h_{\mathbf{q}}$. In the next section, it will be seen that the experimental value of m^* alone is enough to give a very good indication of the way in which the problem has to be approached.

3. CANONICAL-TRANSFORMATION METHOD

For a given momentum, the phonon modes in the He^4 have a much higher energy than the quasiparticle states in the He^3 and so it should be a good approximation to assume that, at very low temperatures, the He^4 is in its ground state except for virtual excitations which are created by the He^3 and are responsible for the effective mass and effective potential of He^3 atoms.

The interaction H_I is not weak. It will be seen that, since m^* is as large as $2.5m$, it cannot be calculated from a low-order perturbation expansion in $f_{\mathbf{q}}$,even when multiple scattering produced by $h_{\mathbf{q}}$ is taken into account.

For particles of zero momentum, however, there is a considerable cancellation in each order of perturbation theory which is most clearly revealed by using a canonical transformation to remove from H those terms which allow He^3 atoms of zero momentum to create a single He^4 excitation. The transformation can be carried out without approximation and it produces an effective He^3–He^3 potential v' which is of second order in $f_{\mathbf{q}}$. In the static limit, $m\rightarrow\infty$, all terms which create He^4 excitations would be removed. When m is finite, He^3 atoms of zero momentum can exchange two excitations but, for helium mixtures, the correction to v' is less than 5%.

The coupling for the creation of a single excitation is now proportional to the momentum of a He^3 atom. Now typical He^3 momenta p are small compared to m^*c. As two He^3 atoms scatter, the most important intermediate-state momenta are given by the van der Waals repulsion which has radius r_c and[14] they are of order $\pi/2r_c$ which is about 0.6 Å^{-1}. For a 6% mixture, k_F is 0.3 Å^{-1}, and since m^*c is about 2.75 Å^{-1}, p/m^*c is about 0.22 for typical intermediate states and less than 0.11 for occupied states.

For this reason, the momentum-dependent potential which is produced by the exchange of a single excitation is small compared to v' and so, after the transformation has been carried out, the only important effect of the He^4 is to give an effective mass to the He^3 atoms. A He^4 excitation is strongly scattered by a He^3 atom and this gives rise to the dependence of m^* upon high orders of $f_{\mathbf{q}}$. To take this into account, we shall carry out a second canonical transformation to remove from the transformed Hamiltonian the terms which are bilinear in the $b_{\mathbf{q}}^{\dagger}$ and $\mathbf{p}_j$. The resulting expression for m^* is an

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[12] A. Miller, D. Pines, and P. Nozières, Phys. Rev. **127**, 1452 (1962).

[13] I. M. Khalatnikov and V. N. Zharkov, Zh. Eksperim. i Teor. Fiz. **32**, 1108 (1957) [English transl.: Soviet Phys.—JETP **5**, 905 (1957)].

[14] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963).

extension of that obtained by Lee, Low, and Pines[15] for the polaron with intermediate coupling.

The principal correction to m^* comes from the emission and absorption of two excitations—for momentum $\mathbf{p}$, expansion of the energy denominator gives a contribution to the He^3 self-energy proportional to p^2/m^{*2}. As the coupling increases, m^* becomes larger and helps to diminish the correction. For helium mixtures, it turns out to be less than 23% of m^* and, if required, it could be retained so that the remaining error is rather small.

The rest of this section will be concerned with setting out the calculation in more detail.

A. Static Transformation

Since H_I has one term which is linear and another which is quadratic in boson variables, the generator S_1 of the first transformation has one part which involves single He^3 atoms and another which refers to pairs. Let

$$S_1=\sum_{\mathbf{q}} (b_{\mathbf{q}}^\dagger-b_{-\mathbf{q}})(\phi_{\mathbf{q}}\rho_{\mathbf{q}}+\sum_{\mathbf{k}} \psi_{\mathbf{k},\mathbf{q}-\mathbf{k}}\rho_k\rho_{\mathbf{q}-\mathbf{k}}) \tag{3.1}$$

be the generator of the transformation. Without loss of generality, it can be required that

$$\psi_{\mathbf{k}_1,\mathbf{k}_2}=\psi_{\mathbf{k}_2,\mathbf{k}_1} \tag{3.2}$$

and, in order that the transformation will be canonical, $S_1{}^\dagger=-S_1$ and so

$$\begin{gathered}\phi_{\mathbf{q}}=\phi_{-\mathbf{q}}{}^*,\\ \psi_{\mathbf{k}_1,\mathbf{k}_2}=\psi^*{}_{-\mathbf{k}_1,-\mathbf{k}_2}.\end{gathered} \tag{3.3}$$

Then

$$\begin{aligned}b_{\mathbf{q}}'&=e^{-S_1}b_{\mathbf{q}}e^{S_1}\\ &=b_{\mathbf{q}}+\phi_{\mathbf{q}}\rho_{\mathbf{q}}+\sum_{\mathbf{k}} \psi_{\mathbf{k},\mathbf{q}-\mathbf{k}}\rho_{\mathbf{k}}\rho_{\mathbf{q}-\mathbf{k}},\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\mathbf{p}_j'&=e^{-S_1}\mathbf{p}_je^{S_1}\\ &=\mathbf{p}_j-\boldsymbol{\pi}_j-\boldsymbol{\pi}_j{}^\dagger,\end{aligned} \tag{3.5}$$

where

$$\boldsymbol{\pi}_j{}^\dagger=\sum_{\mathbf{q}} b_{\mathbf{q}}{}^\dagger(\mathbf{q}\phi_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{r}_j}+2\sum_{\mathbf{k}} \mathbf{k}\psi_{\mathbf{k},\mathbf{q}-\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}_j}\rho_{\mathbf{q}-\mathbf{k}}). \tag{3.6}$$

We shall refer to the transformed particles loosely as "He^4 excitations" and "He^3 atoms" as before. Since S_1 is symmetric in He^3 coordinates, the new He^3 atoms are fermions.

The transformation does not change $\rho_{\mathbf{q}}$ and so the new Hamiltonian H' may be determined exactly with the aid of Eqs. (3.4), (3.5), and (3.6). It is straightforward to find the terms in the transformed Hamiltonian H' which allow He^3 atoms of zero momentum to create a single He^4 excitation. They will vanish if both the coefficient of $b_{\mathbf{q}}{}^\dagger\rho_{\mathbf{q}-\mathbf{k}}\rho_{\mathbf{k}}$ is zero :

$$\left(\omega_{\mathbf{q}}+\frac{(\mathbf{q}-\mathbf{k})^2}{2m}+\frac{k^2}{2m}\right)\psi_{\mathbf{k},\mathbf{q}-\mathbf{k}}+\tfrac{1}{2}(h_{\mathbf{k}}F_{\mathbf{q}-\mathbf{k}}+h_{\mathbf{q}-\mathbf{k}}F_{\mathbf{k}})+\sum_{\mathbf{K}} h_{\mathbf{K}}(\psi_{\mathbf{k},\mathbf{q}-\mathbf{k}-\mathbf{K}}+\psi_{\mathbf{k}-\mathbf{K},\mathbf{q}-\mathbf{k}})=0 \tag{3.7}$$

and the coefficient of $b_{\mathbf{q}}{}^\dagger\rho_{\mathbf{q}}$ is zero:

$$f_{\mathbf{q}}+(\omega_{\mathbf{q}}+q^2/2m)F_{\mathbf{q}}+\sum_{\mathbf{k}} h_{\mathbf{q}-\mathbf{k}}F_{\mathbf{k}}=0. \tag{3.8}$$

Here

$$F_{\mathbf{q}}=\phi_{\mathbf{q}}+\sum_{\mathbf{k}} \psi_{\mathbf{k},\mathbf{q}-\mathbf{k}} \tag{3.9}$$

is the complete coefficient of $\rho_{\mathbf{q}}$ in Eq. (3.1). Equation (3.8) may be rewritten

$$F_{\mathbf{q}}=-\tilde{f}_{\mathbf{q}}/[\omega_{\mathbf{q}}+(q^2/2m)], \tag{3.10}$$

where

$$\tilde{f}_{\mathbf{q}}=f_{\mathbf{q}}-\sum_{\mathbf{k}} h_{\mathbf{q}-\mathbf{k}}[\omega_{\mathbf{k}}+(k^2/2m)]^{-1}\tilde{f}_{\mathbf{k}}. \tag{3.11}$$

This equation describes the motion of one He^3 atom and a He^4 excitation with total momentum zero. It gives the multiple-scattering correction to $f_{\mathbf{q}}$ and may be solved numerically, given $f_{\mathbf{q}}$ and $h_{\mathbf{q}-\mathbf{k}}$.

Equation (3.7) is concerned with the creation of a He^4 excitation and its subsequent motion in the field of the two He^3 atoms. There is no interaction between the He^3 atoms and the total momentum is zero.

It is possible to go some way towards a formal solution of Eq. (3.7) by rewriting it as an operator equation. Let

$$\Delta=\omega_{\mathbf{p}_1+\mathbf{p}_2}+\mathbf{p}_1{}^2/2m+\mathbf{p}_2{}^2/2m, \tag{3.12}$$

and

$$T_{12}=-h_1F_2-h_1(1/\Delta)(T_{12}+T_{21}), \tag{3.13}$$

where subscripts 1 and 2 distinguish particles 1 and 2 and h and F are operators which have matrix elements in momentum representation $\langle\mathbf{q}+\mathbf{k}|h|\mathbf{q}\rangle$ equal to $h_{\mathbf{k}}$ and $\langle\mathbf{q}+\mathbf{k}|F|\mathbf{q}\rangle$ equal to $F_{\mathbf{k}}$. Here, only He^3 momenta are shown, and it is implied that there is a He^4 excitation being created and scattered to keep the total momentum zero. Then if $|\mathbf{k},\mathbf{q}-\mathbf{k}\rangle$ is a product of one-particle plane-wave states with momenta $\mathbf{k}$ and $\mathbf{q}-\mathbf{k}$, Eqs. (3.7) and (3.13) give

$$\psi_{\mathbf{k},\mathbf{q}-\mathbf{k}}=\tfrac{1}{2}\langle\mathbf{k},\mathbf{q}-\mathbf{k}|(1/\Delta)(T_{12}+T_{21})|0,0\rangle. \tag{3.14}$$

Now Eq. (3.13) may be rewritten

$$\begin{aligned}T_{12}&=-[1+h_1(1/\Delta)]^{-1}h_1[F_2+(1/\Delta)T_{21}]\\ &=-\tilde{h}_1[F_2+(1/\Delta)T_{21}],\end{aligned} \tag{3.15}$$

where $\tilde{h}_1$ is given by

$$\tilde{h}_1=h_1-h_1(1/\Delta)\tilde{h}_1 \tag{3.16}$$

which takes account of the multiple scattering of the He^4 excitation by one He^3 atom with the other a bystander in an excited state. In momentum representation, Eq. (3.16) is very similar to Eq. (3.11) and may be solved in the same way.

Since T_{12} is concerned with a process in which the second He^3 atom does not take part in the last scattering, Eq. (3.15) is reminiscent of Fadeev's equations[7] in the three-body problem.

If arguments 1 and 2 are interchanged in Eq. (3.15) to obtain an equation for T_{21} in terms of T_{12}, and the

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[15] T. D. Lee, F. E. Low, and D. Pines, Phys. Rev. **90**, 297 (1953).

result is substituted back into Eq. (3.15) we find

$$T_{12}=-\tilde{h}_1F_2+\tilde{h}_1(1/\Delta)\tilde{h}_2F_1+\tilde{h}_1(1/\Delta)\tilde{h}_2(1/\Delta)T_{12}. \quad (3.17)$$

Since $\tilde{h}$ and F include multiple scattering, they are not singular but, when we come to calculate m^*, it will be seen that F is not weak and so it is possible that Eq. (3.17) cannot be solved by low order perturbation theory.

The transformed Hamiltonian H' can be written down exactly. It contains terms which depend upon the coordinates of three or four He^3 atoms but, in view of the low number density of the He^3, they should be unimportant and so they will be omitted. Then

$$\begin{aligned} H' &= e^{-S_1}He^{S_1} \\ &= H_0'+H_1'+H_2', \end{aligned} \quad (3.18)$$

where

$$H_0'=\sum_{i=1}^{N}\frac{\mathbf{p}_i^2}{2m}+\tfrac{1}{2}\sum_{\mathbf{q}} v_{\mathbf{q}}'\rho_{\mathbf{q}}\rho_{-\mathbf{q}}+\text{const}, \quad (3.19)$$

$$H_1'=-\sum_{j=1}^{N}\frac{\boldsymbol{\pi}_j{}^{\dagger}\mathbf{p}_j}{m}+\text{H.c.}, \quad (3.20)$$

$$\begin{aligned} H_2' &= \sum_{\mathbf{q}} \omega_{\mathbf{q}}b_{\mathbf{q}}{}^{\dagger}b_{\mathbf{q}}+\sum_{\mathbf{q}}\sum_{\mathbf{k}} h_{\mathbf{q}}\rho_{\mathbf{q}}b_{\mathbf{k}+\mathbf{q}}{}^{\dagger}b_{\mathbf{k}} \\ &\quad +\sum_{j=1}^{N}\frac{(\boldsymbol{\pi}_j{}^{+2}+2\boldsymbol{\pi}_j{}^{\dagger}\cdot\boldsymbol{\pi}_j+\boldsymbol{\pi}_j{}^2)}{2m}. \end{aligned} \quad (3.21)$$

Here, $\boldsymbol{\pi}_j$ is given in Eq. (3.6) and, using Eqs. (3.7) and (3.9)

$$v_{\mathbf{q}}'=v_{\mathbf{q}}+u_{\mathbf{q}}+w_{\mathbf{q}}, \quad (3.22)$$

where

$$u_{\mathbf{q}}=-\frac{2\tilde{f}_{\mathbf{q}}{}^2}{\omega_{\mathbf{q}}+q^2/2m}-2\sum_{\mathbf{k}} F_{\mathbf{k}}\langle-\mathbf{q},\mathbf{k}+\mathbf{q}|T|00\rangle, \quad (3.23)$$

and

$$\begin{aligned} w_{\mathbf{q}} &= -F_{\mathbf{q}}{}^2\mathbf{q}^2/m-4\sum_{\mathbf{k}}\frac{\mathbf{q}\cdot\mathbf{k}}{m}F_{\mathbf{k}}\psi_{\mathbf{q},\mathbf{k}-\mathbf{q}} \\ &\quad +4\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\frac{\mathbf{q}\cdot\mathbf{k}}{m}\psi_{\mathbf{k},\mathbf{k}'}\psi_{\mathbf{k}+\mathbf{q},\mathbf{k}'-\mathbf{q}}. \end{aligned} \quad (3.24)$$

The exclusion principle in intermediate states does not appear in $v_{\mathbf{q}}'$, since terms involving three or four He^3 atoms have been omitted.

The induced potentials $u_{\mathbf{q}}$ and $w_{\mathbf{q}}$ arise when two He^3 atoms exchange a He^4 excitation and scatter it back and forth between them. They are of second order in $f_{\mathbf{q}}$. If $h_{\mathbf{q}}$ were zero, the second terms on the right-hand side of Eqs. (2.23) and (3.24) would be missing and $u_{\mathbf{q}}$ and $w_{\mathbf{q}}$ would come from simple exchange without scattering. Later, it will be seen that this would not be a good description when the He^3 atoms come close together.

The third term in H_2' will give rise to an effective potential through the exchange of two excitations. In second-order perturbation theory it turns out to be less than 5% of $u_{\mathbf{q}}$ and $w_{\mathbf{q}}$ for the strength of coupling which fits the experimental value of m^*, and it will be disregarded. It is small because there is a cancellation among terms of each order in perturbation theory, which has been shown up automatically by the transformation.

In a scattering event in which p_i and p_f are typical initial and final He^3 momenta, the exchange of a single He^3 excitation, as given by H_1', will induce a momentum-dependent potential which is of order $p_ip_f/(mc)^2$ compared to $u_{\mathbf{q}}$ and $w_{\mathbf{q}}$. The multiple scattering produced by H_2' gives a factor m/m^*. The importance of this potential may be estimated in detail by means of a separation method[16] and it is found to be less than 2% of $u_{\mathbf{q}}$ and $w_{\mathbf{q}}$. It is small because p/m^*c is small.

Accordingly, H_0' already contains the major part of the induced potential and we have only to calculate the self-energy of a He^3 atom. This is the main advantage of carrying out the first transformation.

B. Effective Mass of the He^3 Atoms

The momentum-dependent part of the self-energy of a He^3 atom comes from the emission and absorption of He^4 excitations given by H_1'. The effective-mass approximation consists in expanding to second order in p/m^*c which is equivalent to second order in H_1'.

However, in calculating m^*, the multiple scattering produced by H_2' is very important. In particular, the term proportional of $\boldsymbol{\pi}_j{}^{\dagger}\cdot\boldsymbol{\pi}_j$ gives rise to the dependence of m^* on all orders of $f_{\mathbf{q}}$. To take account of these effects, we shall carry out a second canonical transformation. The object will be to remove from H' the terms which are bilinear in the $b_{\mathbf{q}}{}^{\dagger}$ and $\mathbf{p}_j$.

Since the change in the induced potential will be small and since the number density of the He^3 is low, we shall retain $v_{\mathbf{q}}'$ but, otherwise, omit all parts of the transformed Hamiltonian which involve two or more He^3 atoms.

Let

$$H''=e^{-S_2}H'e^{S_2}, \quad (3.25)$$

where

$$S_2=\sum_{i=1}^{N}\sum_{\mathbf{q}} G_{\mathbf{q}}e^{-i\mathbf{q}\cdot r_{\mathbf{i}}}(b_{\mathbf{q}}{}^{\dagger}+b_{-\mathbf{q}})\frac{\mathbf{q}\cdot(\mathbf{p}_i+\mathbf{P})}{m}, \quad (3.26)$$

and

$$\mathbf{P}=\sum_{\mathbf{q}} \mathbf{q}b_{\mathbf{q}}{}^{\dagger}b_{\mathbf{q}} \quad (3.27)$$

is the total-momentum operator for the He^4 excitations. The exponentials in Eq. (3.25) may be expanded in multiple commutators to give

$$H''=H'+[H',S_2]+\tfrac{1}{2}[[H',S_2]S_2]+\cdots. \quad (3.28)$$

If terms involving two or more He^3 atoms are neglected, the expansion terminates at second order and

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[16] R. J. Eden and V. J. Emery, Proc. Roy. Soc. (London) **A248**, 266 (1958).

it is not necessary to make any further approximation in evaluating H''. This is a consequence of the fact that, if atom i were alone in the He⁴, $\mathbf{k}_i+\mathbf{P}$ would commute with the individual parts of H' and it was for this reason that $\mathbf{P}$ was included in S_2. In contrast to the first transformation, it is not possible to evaluate H'' in closed form if the approximation is not made.

The terms which are bilinear in $b_{\mathbf{q}}^{\dagger}$ and $\mathbf{p}_j$ are contained entirely in H_1' and $[H_0'+H_2',S_2]$ and, using Eqs. (3.19), (3.20), (3.21), and (3.26), they will vanish if $G_{\mathbf{q}}$ is determined by

$$G_{\mathbf{q}}(\omega_{\mathbf{q}}+q^2/2m)=F_{\mathbf{q}}\left(1-\tfrac{2}{3}\sum_{\mathbf{q}'}F_{q'}G_{q'}\frac{q'^2}{m}\right) -\sum_{\mathbf{q}'}\frac{\mathbf{q}\cdot\mathbf{q}'}{q^2}h_{\mathbf{q}-\mathbf{q}'}G_{\mathbf{q}'}. \tag{3.29}$$

If we define

$$\tilde{F}_{\mathbf{q}}=F_{\mathbf{q}}-\sum_{q'}\frac{\mathbf{q}\cdot\mathbf{q}'}{q^2}h_{\mathbf{q}-\mathbf{q}'}\frac{\tilde{F}_{\mathbf{q}'}}{\omega_{\mathbf{q}'}+q'^2/2m}, \tag{3.30}$$

and

$$\eta=\tfrac{2}{3}\sum_{\mathbf{q}'}F_{\mathbf{q}'}G_{\mathbf{q}'}\frac{q'^2}{m}, \tag{3.31}$$

then Eq. (3.29) is satisfied if

$$G_{\mathbf{q}}=\frac{\tilde{F}_{\mathbf{q}}(1-\eta)}{\omega_{\mathbf{q}}+q^2/2m}. \tag{3.32}$$

Combining Eqs. (3.31) and (3.32),

$$\frac{\eta}{1-\eta}=\tfrac{2}{3}\sum_{\mathbf{q}'}\frac{F_{\mathbf{q}}\tilde{F}_{\mathbf{q}'}(q'^2/m)}{\omega_{q'}+q'^2/2m}. \tag{3.33}$$

Equation (3.30) has to be solved for $\tilde{F}_{\mathbf{q}}$ and gives a multiple scattering correction to $F_{\mathbf{q}}$. Equation (3.33) determines η and then Eq. (3.32) gives $G_{\mathbf{q}}$ directly.

Now define

$$\bar{H}_1'=H_1'+[H_0'+H_2', S_2]. \tag{3.34}$$

Using Eqs. (3.30), (3.32), and (3.33),

$$\bar{H}_{1'}=\sum_{j=1}^{N}\sum_{\mathbf{q}}b_{\mathbf{q}}^{\dagger}e^{-i\mathbf{q}\cdot\mathbf{r}_j}\left(\frac{\mathbf{q}\cdot\mathbf{p}_j}{m}\;\frac{\mathbf{q}\cdot(\mathbf{p}_j+\mathbf{P})}{m}G_{\mathbf{q}}+\frac{\mathbf{q}\cdot\mathbf{P}}{m}F_{\mathbf{q}}\right)+\text{H.c.} \tag{3.35}$$

Then, from Eqs. (3.28) and (3.34),

$$H''=H_0'+H_2'+\bar{H}_1'+\tfrac{1}{2}[H_1'+\bar{H}_1', S_2] +\tfrac{1}{2}[[\bar{H}_1',S_2],S_2]. \tag{3.36}$$

In writing down Eqs. (3.35) and (3.36), terms involving two or more He³ atoms have been neglected, otherwise there is no approximation.

The part of H'' which refers to the He³ only is given by

$$H_0''=\sum_{i=1}^{N}\frac{p_i^2}{2m^*}+\tfrac{1}{2}\sum_{\mathbf{q}}v_{\mathbf{q}'}\rho_{\mathbf{q}}\rho_{-\mathbf{q}}, \tag{3.37}$$

where

$$m/m^*=1-\eta. \tag{3.38}$$

Here H_0'' consists of H_0' together with a contribution

$$-\eta\sum_{i=1}^{N}\frac{p_i^2}{2m}$$

from $\tfrac{1}{2}[H_1',S_2]$.

The remainder of H'' involves the He⁴ variables. It is assumed that the excitation scattering terms do not make the He⁴ ground state different from the vacuum. Then, at very low temperatures, real excitations may be ignored and we have only to estimate the corrections produced by the creation of virtual excitations. Exchange of a single excitation is of fourth order in the $\mathbf{p}_i$ and so may be neglected; and the largest correction to the He³ self-energy comes from the emission and reabsorption of two He⁴ excitations. This process is of fourth order in $f_{\mathbf{q}}$ and depends upon the momentum $\mathbf{p}$ of the He³ atom through the energy denominator, which may be expanded to give a term which is proportional to p^2/m^{*2}. Since m^* becomes larger as the coupling is increased, it helps to diminish the correction. For the strength of coupling which gives the experimental value of m^*, it is found that m/m^* should be reduced by less than 23%. If required, this leading correction may be included explicitly in H_0'' and then the over-all error is rather small. With this proviso, the difference between H'' and H_0'' may be regarded as a small perturbation and many of the properties of the mixture are determined by the fermion Hamiltonian H_0''. Since the fermion number density is low, it is sufficient to use an independent pair approximation to calculate quasiparticle scattering amplitudes. This is the basis of the Fermi-liquid model of the mixture.

Equations (3.33) and (3.38) are extensions of the results obtained by Lee, Low, and Pines[15] for the polaron with intermediate coupling. The errors are very similar in both problems.

The effective mass m^* is related to the He³-He⁴ coupling via η. The interpretation is complicated by the fact that He³-He⁴ scattering contributes to the observed value of m^* although, as in nuclear matter,[12] the effect should not be large. Thus η is about $\tfrac{1}{2}$ and the right hand side of Eq. (33.3) is about 1, so it is clear that low-order perturbation theory cannot be used to find η.

C. Qualitative Discussion of the Effective Potential

The van der Waals potential is just about strong enough[17] to bind two isolated He⁴ atoms. They have

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[17] J. E. Kilpatrick, W. E. Keller, and E. F. Hammel, Phys. Rev. 97, 9 (1955).

mass $4m/3$. Thus, in a mixture, if the induced potential were negligible, the effective mass of $2.5m$ should be sufficient to make the He^3 atoms form bound pairs and an estimate of the binding energy suggests that it would be of the order of 1°K. No evidence of this has been seen experimentally down to 0.008°K, so that, either there are no bound pairs or their binding energy is much less than would be given by the van der Waals force alone. The exclusion principle does have the effect of cutting down the force at distances somewhat greater than about k_F^{-1} but, for 6% mixture, this should not be too significant since k_F^{-1} is about 3.3 Å. Therefore the potential $u_{\mathbf{q}}+w_{\mathbf{q}}$, induced by the He^4, must be effectively repulsive.

In order to see how this comes about, it is desirable to keep in mind what is happening in configuration space. A He^3 atom polarizes the He^4 by creating a hole in its density distribution. The nearest He^4 atoms are about 4 Å away and beyond that there are small ripples in the He^4 density which tend to die out quite quickly. When two He^3 atoms come close together, each one distorts and interacts with the hole produced by the other and it is more accurate to talk of their joint polarization of the He^4. This changes the energy of interaction of the He^3 atoms with the He^4, and is the source of the induced potential.

As the distance r between the He^3 atoms increases, the overlap of their polarization distributions becomes smaller or, equivalently, there is a decreasing probability that a He^4 excitation emitted by one He^3 atom will reach the other. Accordingly, for large r, simple-phonon exchange, as given by the first terms on the right-hand sides of Eqs. (3.23) and (3.24) will dominate the induced potential and it should be attractive but weak.

When r is 3 or 4 Å, which is the region of the direct van der Waals attraction, the polarization distributions strongly overlap and He^4 excitations are easily passed between the He^3 atoms. Then, unless $\tilde{h}_{\mathbf{q}}$ is much smaller than $\tilde{f}_{\mathbf{q}}$, the scattering terms in Eqs. (3.23) and (3.24) should be comparable to the simple exchange terms. For these values of r, each He^3 atom on average lacks one of its He^4 neighbors to which it was bound, and so the induced potential should be repulsive. The first terms of $u_{\mathbf{q}}$ and $w_{\mathbf{q}}$ are negative but their Fourier transforms could oscillate and be repulsive for r equal to 3 or 4 Å. The scattering terms could give either sign. In order to avoid bound states the total induced potential should be repulsive in this region.

For smaller values of r, the induced potential is dominated by the van der Waals repulsion so that, even if it were attractive, it would not be effective.

What is required is the difference between the scattering of two He^3 atoms and the scattering of a He^3 atom and a He^4 atom in the medium. The difference is caused partly by the change in mass but mainly by the fact that virtual states of the He^4 are phonon and roton modes so that it is essential to do the sort of calculation set out in Secs. (3A) and (3B).

To determine the effective potential, it is necessary to know $\tilde{f}_{\mathbf{q}}$ and $\tilde{h}_{\mathbf{q}}$ well enough to obtain the Fourier transform of $u_{\mathbf{q}}$ and $w_{\mathbf{q}}$ accurately for r equal to 3 or 4 Å where it partly cancels the van der Waals attraction. The information is not available at present and it will be difficult to obtain it with the required accuracy. Since the Fermi-liquid picture has been established in Secs. (3A) and (3B), it seems more profitable to use that, to learn directly about $v_{\mathbf{q}}'$.

4. EXPERIMENTAL DETERMINATION OF THE EFFECTIVE POTENTIAL

Landau's theory[4] shows how information about the effective interaction between quasiparticles in a Fermi liquid may be obtained from the equilibrium and transport properties of the system. The velocity of a density wave in a mixture depends partly upon the compressibility of the He^4 and the specific heat is determined by m^* which comes mainly from the He^3–He^4 interaction and so neither of these quantities is directly useful.

We shall concentrate upon the spin susceptibility χ and the spin diffusion coefficient D which are given entirely by the He^3 atoms. Corresponding formulas for the viscosity and the thermal conductivity are given elsewhere.[4,18]

The effective He^3–He^3 potential is most directly related[18] to the physical scattering amplitude $A(\mathbf{k},\phi)$ for two particles on the Fermi surface with relative momentum $\mathbf{k}$ and scattering angle ϕ. The spin susceptibility[19] depends upon Landau's forward scattering amplitude $f(\mathbf{k}\sigma;\mathbf{k}',\sigma')$ for particles of momenta and spin $\mathbf{k},\sigma$ and $\mathbf{k}'$, σ' and, at very low temperatures, this in turn is obtained[20] from $A(\mathbf{k},0)$. The result may be expressed in terms of the odd-state amplitude $A_O(\mathbf{k},\phi)$ and the even-state amplitude $A_E(\mathbf{k},\phi)$:

$$1-\frac{\chi}{\chi_0}=\frac{m^*}{k_F\pi^2}\int_0^{k_F}k\,dk[A_O(\mathbf{k},0)-A_E(\mathbf{k},0)],\qquad(4.1)$$

where χ is the measured susceptibility and χ_0 the susceptibility of an ideal Fermi gas with particles of mass m*.

The spin-diffusion coefficient D is given by[21]

$$D=\tfrac{1}{3}(\chi_0/\chi)v_F{}^2\tau_D\,,\qquad(4.2)$$

where v_F is the Fermi velocity and

$$\tau_D{}^{-1}=\frac{m^{*3}T^2}{8\pi^2k_F{}^4}\int_0^{2\pi}d\phi(1-\cos\phi)$$
$$\times\int_0^{k_F}dk\frac{k^3}{(1-k^2/k_F{}^2)^{1/2}}|A_E(\mathbf{k},\phi)+A_O(\mathbf{k},\phi)|^2.\qquad(4.3)$$

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[18] V. J. Emery, Ann. Phys. (N.Y.) **28**, 1 (1964).

[19] Reference 4, p. 335.

[20] Reference 4, p. 366.

[21] D. Hone, Phys. Rev. **121**, 669 (1961); V. J. Emery, *ibid.* **133**, A661 (1964).

Relative momenta are less than k_F which is $0.3\ \text{Å}^{-1}$ for a 6% mixture and so the D-state impact parameter lies outside the range of the interparticle force. Thus the scattering amplitude is dominated by S snd P states and we may take $A_E(\mathbf{k},\phi)$ to be $A_E(\mathbf{k},0)$ and $A_0(\mathbf{k},\phi)$ to be $A_0(\mathbf{k},0)\cos\phi$. Measurement of χ and D as functions of concentration and hence k_F at temperatures of the order of 0.01°K would give $A_E(\mathbf{k},0)$ and $A_0(\mathbf{k},0)$, provided rearrangement effects[22] could be estimated. The method of obtaining a fermion superfluid transition temperature T_c from the scattering amplitudes is given in Ref. 18.

Present indications[23,24] are that χ is very close to χ_0 for concentrations of a few percent so that the right-hand side of Eq. (4.1) is small. This result does not necessarily imply that the effective forces are weak since $A_0(\mathbf{k},0)$ and $A_E(\mathbf{k},0)$ can cancel each other on average. In addition, the scattering amplitudes reflect the behavior of the potential and change sign as k increases so that the integrals of $A_0(\mathbf{k},0)$ and $A_E(\mathbf{k},0)$ can separately be small even though the amplitudes themselves are large for some values of k.

However, measurements of the spin diffusion in dilute mixtures[24] show that the lifetimes of the He^3 quasiparticles are 50–100 times larger than in pure He^3. It is not easy to attribute this to cancellations in the right-hand side of Eq. (4.3) and so the scattering amplitudes appear to be much smaller in magnitude than in pure He^3. Detailed estimates, based upon the value of D only, depend upon assumptions about the relative values of the S- and P-state scattering amplitudes. Nevertheless it seems unlikely that dilute mixtures will undergo a fermion superfluid phase transition at presently attainable temperatures.

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Behavior of Ions and Quantized Vortex Rings of Low Energy in Helium II around 0.3°K

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Quantized vortex rings of low energy down to 0.1 eV have been produced in specially purified liquid helium at 0.3°K and the deflection of rings of 0.15- to 0.5-eV energy in a transverse magnetic field measured. In the same apparatus the Hall mobility of positive ions has been measured at 0.3°K, yielding after extrapolation to zero electric field a value $\mu=3.0\times10^4\ \text{cm}^2\text{V}^{-1}\ \text{sec}^{-1}$. The drift velocity of ions can reach $40\ \text{m sec}^{-1}$ before vortex rings are produced, also at 0.3°K.

INTRODUCTION

RECENTLY, Rayfield and Reif[1] have shown that ions in He II, accelerated beyond thermal velocity, produce vortex rings of one quantum of circulation. They studied the behavior of these rings in detail in the energy range of 5–50 eV, the rings having a radius between about 0.2–2 μ. Even if the circulation of these rings is quantized, their hydrodynamic behavior is essentially classical.[2] Therefore, it seemed of interest to make rings of as low an energy as possible with correspondingly small radius, in the hope of getting some insight into the behavior of the rings as they approach the quantum hydrodynamic region. This should also yield information on the mechanism producing such rings.

The energy E of a vortex ring of radius R (assuming the core with radius a is solid) is given by[2–6]

$$E=\tfrac{1}{2}\rho\kappa^2R[\eta-7/4]\,, \tag{1}$$

where ρ is the density of the fluid; κ, the circulation, is h/m (m is the mass of the He atom) in this case; and $\eta=\ln 8R/a$.

Since each part of the vortex ring is in the velocity field of the other parts, a net velocity v results, perpen-

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