Relativistic $U(6,6)$ Theory

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Infinite-dimensional unitary representations of the noncompact group $U(6,6)$ are employed to classify elementary particles and, following ideas related to those of Fronsdal, are used to construct relativistic S-matrix elements. Like the previously treated relativistic theories where finite-dimensional representations of $U(6,6)$ were used, a particular S-matrix element shows no symmetry higher than that of the appropriate hybrid subgroup. The over-all $U(6,6)$ symmetry may give new relations between form factors for different processes but will not, in general, give anything beyond the results of the previous formulations for the scattering processes. The unitarity of the S matrix is compatible with the subgroup hierarchy, provided that an infinity of multiplets for elementary particles exists and provided that all such multiplets possess the same mass. The crucial point of our formulation is that if mass differences are introduced, these affect not the relativistic invariance but the unitarity of the S matrix.

1. INTRODUCTION

IN a series of earlier papers,[1] it was suggested that elementary particles may be classified as multiplets of a rest symmetry group $U(6)\times U(6)$. It was shown that moving multiplets of momentum p could be covariantly represented by using the finite-dimensional representations of a larger group $U(6,6)$ for construction of a (covariant) algebra of the structure $U(6)\times U(6)|_p$. This construction procedure involved a covariant projection[2] method from the larger group $U(6,6)$ based on Bargmann-Wigner equations. The problem of coupling of $U(6)\times U(6)|_p$ multiplets could then be reduced to the comparatively simpler problem of coupling of $U(6,6)$ representations. Unfortunately, since the projection procedure (the Bargmann-Wigner relation) was not $U(6,6)$ covariant, the over-all symmetry of the resulting S matrix was considerably smaller than the symmetry started with. It was in fact shown by Harari and Lipkin[3] and Dashen and Gell-Mann[4] that the maximal symmetry one might expect for S-matrix elements in such a theory could be classified in the following hierarchy:

(1) $U(6)\times U(6)|_p$ for one-momentum processes;

(2) $U_w(6)|_{p_1p_2}$ for two independent momenta;

(3) $U(3)\times U(3)|_{p_1p_2p_3}$ for three independent momenta;

and $U(3)$ for 4 or more momenta. It was also shown by a number of authors (for references see the review article in the Trieste Seminar Volume[2]) that the unitarity equation for the S matrix

$$\mathrm{Im}T_{if}=\sum\nolimits_n T_{in}T_{nf}^\dagger \tag{1.1}$$

would in general be compatible with the maximal symmetry specified above, provided the intermediate states in the sum on the right of (1) themselves were restricted so as to belong to the relevant subgroup in the hierarchy.

Subsequent to these developments, a suggestion was made that to obviate some of the problems mentioned above, one might employ infinite-dimensional representations of the group[5,6] $U(6,6)$ for classifying particles. One would then start with the assumption that there are in nature an infinite number of $U(6)\times U(6)$ multiplets, all of the same mass in the exact symmetry limit. In a given representation of $U(6,6)$, these multiplets would be grouped together constituting as it were different "rungs" of a given $U(6,6)$ "tower." Each such tower would carry in addition to the labels m characterizing individual "rungs" also a momentum parameter p. A tower of momentum p would be carried to one of momentum p' by Lorentz transformations with each "rung" being carried to essentially the same rung in the new tower.[7] The noncompact transformations contained in $U(6,6)$ would however induce transitions between distinct "rungs."

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[2] For a discussion of the group $U(6)\times U(6)|_p$ see K. J. Barnes, Phys. Rev. Letters **14**, 798 (1965), and the contribution by R. Delbourgo, M. A. Rashid, Abdus Salam, and J. Strathdee, in *Proceedings of the Seminar on High Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965), p. 486.

[3] H. Harari and H. J. Lipkin, Phys. Rev. **140**, B1617 (1965).

[4] R. Dashen, M. Gell-Mann, Phys. Letters **17**, 142 (1965).

[5] A. O. Barut, in *Proceedings of the Seminar on High Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965), p. 679; P. Budini and C. Fronsdal, Phys. Rev. Letters **14**, 968 (1965); C. Fronsdal, in *Proceedings of the Seminar on High Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Energy Agency, Vienna, 1965), p. 665; Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters **17**, 148 (1965).

[6] R. Delbourgo, Abdus Salam, and J. Strathdee, Proc. Roy. Soc. (London) **289A**, 177 (1966); Abdus Salam and J. Strathdee, *ibid.* **292A**, 314 (1966); Abdus Salam, in *Proceedings of the Oxford Conference on Elementary Particles* (Rutherford High Energy Laboratory, Harwell, England, 1966).

[7] In the new rung, some rotational shuffling may of course have occurred but there is no admixture of distinct rungs m and m'.

In the present paper we show that one can indeed consistently start with such towers, and that using these one can construct S-matrix theories where both symmetry and unitarity are fully compatible. We find to our surprise, however, that the resulting S-matrix elements share with the earlier theory the feature that the over-all invariance of the S matrix for the coupling of given "rungs" is again governed by the same hierarchy of subgroups $U(6)\times U(6)|_{p_1}$, $U_w(6)|_{p_1p_2}$, $U(3)\times U(3)|_{p_1p_2p_3}$, and $U(3)$ for one-, two-, three-, or four-momentum processes. The starting "tower-symmetry" $U(6,6)$ itself appears to play no experimentally accessible role in four-particle scattering amplitudes. Its major predictive value seems to lie (1) in the prediction of new relations connecting vertex-function coupling constants of one set of rungs to the coupling of other sets, (2) in the possibility of providing more powerful and universal mass formulas, and (3) in assuring unitarity in the limit of there existing an infinity of "rungs" in the elementary particle spectrum—all with the same mass.

In Sec. 2, we give the formalism and establish its consistency. The chief result of the paper is the formula (2.13) and the rather transparent expression it provides for the kinematic momentum-dependent factors in the theory. In Secs. 3 and 4 we exploit some well-known techniques of Feynman's[8] operator calculus, developing further the work of Ref. 6, to evaluate these kinematic factors for the case of *3-meson* coupling. We have not yet been able to find a suitable $U(6,6)$ representation for baryons and thus the question whether the present theory reproduces the results of the earlier papers[1,2]—and in particular if it relates the electric and magnetic form factors for the proton in the desired manner—remains unanswered. We intend to turn to this problem in subsequent work.

We believe the theory presented here has close similarities to the recent work of Fronsdal[5] and Rühl,[9] though it is not exactly easy to trace the relationship of the ideas, or the techniques. To these authors and to Professor M. Gell-Mann, who has been working on related lines, we offer our appreciation for stimulating discussions.

2. GENERAL FORMULATION

Suppose that we are given in addition to the generators $J_{\mu\nu}$, P_μ of space-time transformations a set of generators F^α of some internal symmetry G. Suppose, moreover, that the F^α transform covariantly (and nontrivially) under the space-time group[10]

$$[J_{\mu\nu},F^\alpha]=\sum_\beta c_\beta{}^\alpha F^\beta\,, \tag{2.1}$$

$$[P_\mu,F^\alpha]=0\,. \tag{2.2}$$

If G is a spin-containing symmetry then at least some of the $C_\beta{}^\alpha$ will be nonvanishing and the relation (2.1) implies that the generators F^α must include the generators of $SL(2,c)$. This means that G must be a noncompact symmetry. Given these properties it is possible to set up a covariant theory with higher symmetry.

In general terms the procedure is very straightforward. After the fashion of Wigner one specifies a family of Lorentz boosts L_p which serve to carry a fixed 4-vector $\hat{p}$ into p,

$$(L_p)_\mu{}^\nu\hat{p}_\nu=p_\mu\,. \tag{2.3}$$

Corresponding to L_p there is a unitary operator $U_J(L_p)$ —made with the $J_{\mu\nu}$—which acts upon the physical states. A complete set of physical states may be generated formally by applying the $U_J(L_p)$ to a set of *rest states* which we denote by $|\hat{p},m\rangle$. The label m ranges over the basis of a unitary representation of the internal symmetry group G. These representations are of course infinite-dimensional.

Corresponding to each rest state $|\hat{p},m\rangle$ there is a continuum of boosted states $|p,m\rangle$ defined by the relation

$$U_J(L_p)|\hat{p}m\rangle=|pm\rangle\,. \tag{2.4}$$

Note that the index m is unaffected.

Clearly, from (3)

$$\langle\hat{p}m'|F^\alpha|\hat{p}m\rangle=\langle pm'|U_J(L_p)F^\alpha U_J{}^{-1}(L_p)|pm\rangle\,,$$

and by virtue of assumption (2.1), (expressed for finite Lorentz transformations)

$$U_J(\Lambda)F^\alpha U_J{}^{-1}(\Lambda)=\sum_\beta F^\beta T_\beta{}^\alpha(\Lambda)\,, \tag{2.5}$$

one finds that the matrix elements of the F^α between boosted and unboosted states possess a simple linear relationship:

$$\langle\hat{p}m'|F^\alpha|\hat{p}m\rangle=\sum_\beta\langle pm'|F^\beta|pm\rangle T_\beta{}^\alpha(L_p)\,. \tag{2.6}$$

In the sequel, whenever the p dependence is not explicitly shown in $\langle m'|F^\alpha|m\rangle$ we shall always mean the matrix element $\langle\hat{p}m'|F^\alpha|\hat{p}m\rangle$.

Now, as is well known, if the F^α's are generators of [an $SL(2,c)$-containing] noncompact symmetry G [e.g., $U(6,6)$], the unitary representations of G must be infinite-dimensional. An irreducible representation of G consists of a *tower*, each rung of the tower corresponding to an irreducible representation of the maximal compact subgroup [e.g., $U(6)\times U(6)$]. The lables m for each rung characterize these $U(6)\times U(6)$ representations. The $U(6)\times U(6)$ generators produce a linear mixing of states at each rung while the remaining generators make transitions between rungs m and m'. We shall on occasion refer to $U(6,6)$ as the *tower symmetry* and $U(6)\times U(6)$ as the *rung symmetry*.

Each tower carries a momentum label p, and the boosts $U_J(L_p)$ carry a tower at rest to a tower with momentum p, each rung of the rest-tower being carried to the corresponding rung of the moving tower. As one may expect, the operation of the generators F^α of $U(6,6)$ on the moving tower is p-dependent and is specified in (2.6).

[8] R. P. Feynman, Phys. Rev. 84, (1951).

[9] W. Rühl, Nuovo Cimento 42, 619 (1966). We are indebted to Professor Rühl for sending us his papers prior to publication.

[10] This set of commutation relations is also the starting point of Fronsdal's work (Ref. 5).

Note however, that if we write $F^\alpha(p)=U(L_p)F^\alpha U^{-1}(L_p)$, we have the transparent relation

$$\langle pm'|F^\alpha(p)|pm\rangle=\langle\hat{p}m'|F^\alpha|\hat{p}m\rangle. \quad (2.7)$$

The behavior of the towers under the space-time group is perfectly straightforward. Explicitly, and following Wigner's classical method closely, we obtain

$$U_J(\Lambda)|pm\rangle=U_J(\Lambda L_p)|\hat{p}m\rangle$$
$$=U_J(L_{\Lambda p}R)|\hat{p}m\rangle,$$

where R denotes an ordinary little group rotation—one which leaves $\hat{p}$ invariant—

$$\begin{aligned}U_J(\Lambda)|pm\rangle&=U_J(L_{\Lambda p})U_J(R)|\hat{p}m\rangle\\&=\textstyle\sum_{m'} U_J(L_{\Lambda p})|\hat{p}m'\rangle\langle m'|R|m\rangle\\&=\textstyle\sum_{m'}|\Lambda pm'\rangle\langle m'|R|m\rangle. \quad (2.8)\end{aligned}$$

Since R is a compact rotation, one can always bring the infinite matrix $\langle m'|R|m\rangle$ into block-diagonal form, each block being finite-dimensional. With the space-time transformations therefore we move from a given rung of a tower of momentum p to the corresponding rung of the tower with momentum $p'=\Lambda p$, the indices m being shuffled by an ordinary rotation R.

As an aid in the discussion of coupling problems it is useful to have an alternative set of basis vectors for the physical states. These we now define. Since G contains the homogeneous Lorentz group we have at our disposal the unitary matrices $\langle m'|\Lambda|m\rangle$ and, in particular, $\langle m'|L_p|m\rangle$. Let us use these to define a new orthonormal set[11]:

$$|pm\rangle_s=\textstyle\sum_{m'}|pm'\rangle\langle m'|L_p^{-1}|m\rangle. \quad (2.9)$$

These states have very simple transformation properties, namely,

$$\begin{aligned}U_J(\Lambda)|p,m\rangle_s&=\textstyle\sum_{m'}|\Lambda pm'\rangle_s\langle m'|\Lambda|m\rangle,\\F^\alpha|pm\rangle_s&=\textstyle\sum_{m'}|p,m'\rangle_s\langle m'|F^\alpha|m\rangle. \quad (2.10)\end{aligned}$$

The new states $|p,m\rangle_s$ mix up the rungs of the Wigner set, $|p,m\rangle$ through the operation of the matrix $\langle m'|L_p|m\rangle$ and therefore are rather difficult to interpret physically.[12]

In the basis (2.9) the trilinear invariants (if any) would take the form

$$\begin{aligned}I=\int dp_1dp_2dp_3\,&\theta(\pm p_1)\delta(p_1^2-\kappa_1^2)\\\times&\theta(\pm p_2)\delta(p_2^2-\kappa_2^2)\theta(\pm p_3)\delta(p_3^2-\kappa_3^2)\\\times&\sum_{m_1m_2m_3}(m_1m_2m_3)|p_1m_1\rangle_s|p_2m_2\rangle_s|p_3m_3\rangle_s, \quad (2.11)\end{aligned}$$

[11] To see the connection with the $U(6)\times U(6)|_p$ theory of Ref. 1, take for $|pm\rangle_s$ states, a finite-dimensional nonunitary representation of $U(6,6)$ and for $|pm'\rangle$ the (unitary) boosted states of $U(6)\times U(6)$. The matrix $\langle m'|L_p^{-1}|m\rangle$ is then nonunitary. Here $m'=1, 2, \cdots, 6$; $m=1, 2, \cdots, 12$ and $L_p^{-1}=p+\kappa$ (for example) for a quark. Clearly $|pm\rangle_s$ satisfies the Bargmann-Wigner equation.

[12] We believe that the states introduced by Fronsdal and Rühl are the states $|pm\rangle_s$.

where the numbers $(m_1m_2m_3)$ are coupling coefficients appropriate to the tower symmetry G. This may be expressed in the less cumbersome form:

$$\begin{aligned}&{}_s\langle p_1m_1|p_2m_2p_3m_3\rangle_s\\&\quad=\delta(p_1-p_2-p_3)(m_1|m_2m_3)F(p_1^2,p_2^2,p_3^2), \quad (2.12)\end{aligned}$$

where $F(p_1^2,p_2^2,p_3^2)$ is an unknown amplitude function.

The formulas (2.11) and (2.12), which follow directly from (2.10), may be translated back into the original basis, yielding

$$\begin{aligned}\langle p_1m_1|p_2m_2p_3m_3\rangle=&\,\delta(p_1-p_2-p_3)F(p_1^2,p_2^2,p_3^2)\\&\times\textstyle\sum\langle m_1|L_{p_1}^{-1}|m_1'\rangle(m_1'|m_2'm_3')\\&\times\langle m_2'|L_{p_2}|m_2\rangle\langle m_3'|L_{p_3}|m_3\rangle. \quad (2.13)\end{aligned}$$

Thus, we have the relative p dependence of an infinite set of form factors made explicit in terms of the kinematic factors $\langle m'|L_p|m\rangle$ and, moreover, as we shall show, this form is suited to practical calculations.

The expression (2.13) which relates all the vertex parts to a single unknown function (or at least to a limited number of them) is the strongest result that can be expected from the relativistic symmetry theory. When it comes to four-point functions it will generally be found that the number of unknown functions F is infinite. This is because the product of two irreducible unitary representations of a noncompact group G in general leads to an infinite sum or integral of irreducible representations.[5] For the four-point function one expects therefore, that the number of tower symmetry factors $(m_1m_2|m_3m_4)$ is infinite, so that a manifestation of the tower symmetry would be difficult to pin down. The situation is different for the rung level symmetries however. It is important to emphasize that one of the principal results of previous work towards the construction of relativistic symmetry theories, namely, the emergence of a hierarchy of hybrid, p-dependent, compact subgroups is again discovered here. These special symmetries which heretofore were suspect because of unitarity considerations, are now founded upon a manifestly unitary theory.

The hybrid symmetry groups $\bar{G}(p_1,p_2,\cdots)$ are defined as the compact subgroups of G which commute with the boosts $L_{p_1}, L_{p_2}, \cdots$. The action of these transformations on the boosted states will evidently be the same as upon rest states,

$$\bar{G}(p)|pm\rangle=\textstyle\sum_{m'}|pm'\rangle\langle m'|\bar{G}(p)|m\rangle, \quad (2.14)$$

where $\langle m'|\bar{G}(p)|m\rangle=\langle\hat{p}m'|\bar{G}(p)|\hat{p}m\rangle$. The summation over m' in (2.14) is a finite one since $\bar{G}(p)$ must be contained in the maximal compact subgroup, by reference to which the rungs were labled.

To illustrate, if one picks out from G those compact transformations, $\bar{G}(p_1,p_2,p_3)$, which commute with L_{p_1}, L_{p_2}, and L_{p_3}, then, for the vertex (2.13),

$$\begin{aligned}\sum_{m_1'm_2'm_3'}&\langle m_1|\bar{G}^{-1}|m_1'\rangle\langle p_1m_1'|p_1'|p_2m_2'p_3m_3'\rangle\\&\times\langle m_2'|\bar{G}|m_2\rangle\langle m_3'|\bar{G}|m_3\rangle=\langle p_1m_1|p_2m_2p_3m_3\rangle\end{aligned}$$

which shows that $\bar{G}$ is a symmetry of the matrix element $\langle p_1m_1|p_2m_2p_3m_3\rangle$. Similar considerations apply to matrix elements with any number of momenta. If $G=U(6,6)$ then the hierarchy reads

$U(6)\times U(6)$ for two-point functions (rung symmetry),
$U(6)$ for three-point functions (collinear symmetry),
$U(3)\times U(3)$ for four-point functions (coplanar symmetry),
and
$U(3)$ for five-point or higher functions.

The price for having made these symmetries compatible with unitarity is of course the infinite numbers of particles which must fill all the higher rungs of a $U(6,6)$ tower. The infinity is easily seen to be necessary if one allows for the inclusion in the intermediate states of the unitarity relations of particles with arbitrary momenta because, if $q\neq p, \cdots$ then

$$\bar{G}(p,\cdots)|qm\rangle=\sum_{m'}|qm'\rangle\langle m'|L_q^{-1}\bar{G}(p,\cdots)L_q|m\rangle,$$

where the summation generally extends over all the levels.

Further, all of these particles *must* possess the same mass if the tower symmetry—and with it the unitarity of the S matrix—is to survive. A mass-breaking immediately produces an incompatibility of the hierarchy of symmetry groups above with the unitarity condition

$$\mathrm{Im}T=T\rho T^{\dagger}$$

except in the situation that only such intermediate states are allowed in the sum on the right which themselves possess the symmetry of the hierarchy. Since we expect from physical evidence[13] that the mass differences between the rungs of the towers are quite large, there is little hope of being able to define a "mean-mass" for a given tower which could realistically approximate to the different "rung masses." This may therefore present a serious difficulty in giving credence to the results of this theory.

Summarizing: Let us assume the existence of an infinite number of multiplets, constituting the rungs of a tower of noncompact symmetry, and assume that all such multiplets possess the same mean mass. Provided that there is a unique coupling of three such towers, the over-all symmetry ensures that there is one scalar form factor $f(p_1^2,p_2^2,p_3^2)$. It also specifies unambiguously the relationships between the coefficients $(m_1|m_2m_3)$. For the coupling of the rungs themselves an $SU_w(6)$ symmetry exists, with unitarity of the theory automatically guaranteed. For the four-point function, one expects in general, an infinite number of amplitudes $f(s,t)$ so that the tower-symmetry is unlikely to give meaningful relationships between scatterings of different types of (rung) multiplets. However, the coplanar symmetry for specified rungs scattering from each other will survive.

It is instructive to compare the present scheme with an earlier one which employed finite-dimensional nonunitary representations of $U(6,6)$. Once it is accepted that realistic theories should avoid infinite degeneracies then there exists a good deal of common ground between the two approaches. If the infinite-dimensional unitary representations are used then it is necessary to lift their mass degeneracy so that only a finite number of rungs can contribute in any unitarity calculation. That is to say, the imposition of a realistic unitarity condition serves to violate the symmetry. On the other hand, if finite-dimensional nonunitary representations are employed, then, in order to avoid difficulty with the metric it is necessary to impose conditions (the Bargmann-Wigner equations) which project out just one $U(6)\times U(6)$ rung. These conditions violate the $U(6,6)$ symmetry. Both approaches lead to the same hierarchy of hybrid subgroups for the two-, three-, and four-point functions (prior to the imposition of the realistic unitarity conditions). It may be that for the three-point functions the present method will be able to make stronger predictions—i.e., reduce everything to one unknown amplitude—but for the four-point and higher functions there seems to be nothing to choose between them.

[13] P. G. O. Freund, Phys. Rev. Letters **14**, 803 (1965).

3. COUPLING OF THREE TOWERS; THE COEFFICIENTS $(m_1|m_2m_3)$

For illustration we construct the invariant coupling between three meson-like Feynmann towers [i.e., degenerate discrete representations of $U(6,6)$]. This is not a particularly realistic case since, as will be seen, one of these representations must be different from the other two. That is, the mesons would have to be distributed over at least two distinct representations. The virtue of this example lies, however, in its simplicity, and it illustrates the calculational techniques that we intend to employ in future computations.

We begin with the construction of some discrete representations. Let us define the 12-component $U(6,6)$ spinors

$$\psi_A=\begin{pmatrix}\bar{a}_i\\ b_{\bar{i}}\end{pmatrix};\quad i=1,\cdots,6;\quad \bar{i}=7,\cdots,12,\qquad(3.1)$$

where the entries $\bar{a}_i$ and $b_{\bar{i}}$ are algebraic entities satisfying Bose-like commutation relations:

$$[a^i,\bar{a}_j]=\delta_j{}^i,\quad [b_{\bar{i}},\bar{b}^{\bar{j}}]=\delta_{\bar{i}}{}^{\bar{j}},\qquad(3.2)$$

where

$$a^i=(\bar{a}_i)^{\dagger},\quad \bar{b}^{\bar{i}}=(b_{\bar{i}})^{\dagger},\qquad(3.3)$$

all other commutators vanishing. Corresponding to ψ there is the adjoint $\bar{\psi}$ defined by

$$\bar{\psi}^B=\psi^{\dagger}\gamma_0=(-a^i\ \bar{b}^{\bar{i}}).\qquad(3.3)$$

The γ_0 has been inserted here in order to make the commutator of ψ and $\bar{\psi}$ a $U(6,6)$ invariant

$$[\psi_A,\bar{\psi}^B]=\delta_A{}^B\,,\quad A,B,\cdots=1,\cdots,12. \tag{3.4}$$

The quantities $M_A{}^B$ defined by

$$M_A{}^B=\psi_A\bar{\psi}^B-\tfrac{1}{12}\delta_A{}^B\psi_C\bar{\psi}^C \tag{3.5}$$

satisfy the commutation rules

$$[M_A{}^B,M_C{}^D]=\delta_A{}^D M_C{}^B-\delta_C{}^B M_A{}^D\,, \tag{3.6}$$

and the Hermiticity condition

$$(M_A{}^B)^\dagger=(\gamma_0)_B{}^{B'}M_{B'}{}^{A'}(\gamma_0)_{A'}{}^A\,, \tag{3.7}$$

and thus may be employed as generators of unitary representations of $U(6,6)$.

We take for the annihilation operators a^i, $b_{\bar{i}}$ only those representations which admit a vacuum state

$$a^i|0\rangle=0\,,\quad b_{\bar{i}}|0\rangle=0. \tag{3.8}$$

Then the adjoint operators $\bar{a}_i$, $\bar{b}^{\bar{i}}$ will create states of positive norm. A complete set of normalized states is given by

$$\begin{aligned}|m\rangle&=c_m\bar{a}_{i_1}\cdots\bar{a}_{i_m}\bar{b}^{\bar{j}_1}\cdots\bar{b}^{\bar{j}_m}|0\rangle\\&=\prod_{i=1}^{6}\frac{1}{(m_i!)^{1/2}}(\bar{a}_i)^{m_i}\prod_{j=7}^{12}\frac{1}{(m^j!)^{1/2}}(\bar{b}^{\bar{j}})^{m_{\bar{j}}}|0\rangle\,,\end{aligned} \tag{3.9}$$

where m_i denotes the number of times i occurs in the sequence $i_1,\cdots i_m$, etc. and c_m is a normalization constant. These can all be generated by applying $M_i{}^{\bar{j}}$ to the vacuum. They constitute the basis for an irreducible unitary representation of $U(6,6)$. It is a simple matter to evaluate in this basis the matrix elements $\langle n|M_A{}^B|n'\rangle$. The Casimir operators may all be expressed in terms of the single invariant $\psi_A\bar{\psi}^A$ and this is fixed by

$$\psi_A\bar{\psi}^A|0\rangle=6|0\rangle. \tag{3.10}$$

If we introduce an independent set of operators ψ_A' such that

$$[\psi'_A,\psi_B]=[\psi'_A,\bar{\psi}^B]=0\,, \tag{3.11}$$

and put

$$M_A{}^B=\psi_A\bar{\psi}^B+\psi_A'\bar{\psi}'^B-\tfrac{1}{12}\delta_A{}^B(\psi_C\bar{\psi}^C+\psi'_C\bar{\psi}'^C)\,, \tag{3.12}$$

then, if they annihilate a common vacuum,

$$a^i|0\rangle=b_{\bar{i}}|0\rangle=a'^i|0\rangle=b'_{\bar{i}}|0\rangle=0\,, \tag{3.13}$$

we can construct a new representation by repeated applications of $M_A{}^B$ to the vacuum. This representation is irreducible since the lowest state $|0\rangle$ is evidently an eigenstate of the Casimir operators which are all expressible in terms of the invariants

$$\psi_A\bar{\psi}^A,\quad \psi_A\bar{\psi}'^A,\quad \psi'_A\bar{\psi}^A,\quad\text{and}\quad \psi'_A\bar{\psi}'^A.$$

One may verify that the Casimir operators take values different from those obtaining in the simple Feynman tower discussed above. The new representation will be spanned by the set of (unnormalized) states

$$\begin{aligned}\Psi_{i_1\cdots i_m}{}^{\bar{j}_1\cdots\bar{j}_m}&=M_{i_1}{}^{\bar{j}_1}\cdots M_{i_m}{}^{\bar{j}_m}|0\rangle\\&=(\bar{a}_{i_1}\bar{b}^{\bar{j}_1}+\bar{a}'_{i_1}\bar{b}'^{\bar{j}_1})\cdots(\bar{a}_{i_m}\bar{b}^{\bar{j}_m}+\bar{a}'_{i_m}\bar{b}'^{\bar{j}_m})|0\rangle\\&=\sum_{\text{part}}\bar{a}_{i_1}\cdots\bar{a}_{i_l}\bar{a}'_{i_{l+1}}\cdots\bar{a}'_{i_m}\bar{b}^{\bar{j}_1}\cdots\bar{b}^{\bar{j}_l}\bar{b}'^{\bar{j}_{l+1}}\cdots\bar{b}'^{\bar{j}_n}|0\rangle\,,\end{aligned} \tag{3.14}$$

where the summation extends over all partitions of $i,\cdots i_m$ into two sets. The $U(6)\times U(6)$ content of this tower is

$$(1,\bar{1})+(6,\bar{6})+\{(15,\overline{15}+(21,20)\}+\{(20,\overline{20})+2(70,\overline{70})+(56,\overline{56})\}+\cdots. \tag{3.15}$$

The expression (3.14) may be looked upon as a Clebsch-Gordan formula

$$\Psi(m)=\sum_{n+n'=m}\Phi(n)\Phi'(n')\langle n,n'|m\rangle\,, \tag{3.16}$$

where in this case, the coupling coefficients, indicated symoblically by $\langle n,n'|m\rangle$, are very simple. An explicit representation of the trilinear invariant takes the form

$$I=\sum_{n,n'}\sum_{\text{part}}c_p(n,n')\bar{\Psi}_{\bar{j}_1\cdots\bar{j}_{n+n'}}{}^{i_1\cdots i_{n+n'}}\Phi_{i_1\cdots i_n}{}^{\bar{j}_1\cdots\bar{j}_n}\Phi'_{i_{n+1}\cdots i_{n+n'}}{}^{\bar{j}_{n+1}\cdots\bar{j}_{n+n'}}\,, \tag{3.17}$$

where the summation extends over distinction partitions of $i,\cdots i_{n+n}$, into two sets $i_{p_1}\cdots i_{p_n}$ and $i_{p_{n+1}}\cdots i_{p_{n+n'}}$. Following the procedure of Ref. 6, the coefficients $c_p(n,n')$ may be determined by imposing the condition

$$M_A{}^BI=0.$$

One finds for the first few terms the explicit coefficients

$$\begin{aligned}I=\bar{\Psi}\Phi\Phi'+\tfrac{1}{2}\bar{\Psi}_{\bar{j}_1}{}^{i_1}(\Phi_{i_1}{}^{\bar{j}_1}\Phi'+\Phi\Phi'_{i_1}{}^{\bar{j}_1})+\tfrac{1}{4}\bar{\Psi}_{[\bar{j}_1\bar{j}_2]}{}^{[i_1i_2]}\Phi_{i_1}{}^{\bar{j}_1}\Phi'_{i_2}{}^{\bar{j}_2}\\+(1/24)\bar{\Psi}_{(\bar{j}_1\bar{j}_2)}{}^{(i_1i_2)}(\Phi_{i_1i_2}{}^{\bar{j}_1\bar{j}_2}\Phi'+2\Phi_{i_1}{}^{\bar{j}_1}\Phi'_{i_2}{}^{\bar{j}_2}+\Phi\Phi'_{i_1i_2}{}^{\bar{j}_1\bar{j}_2})+\cdots\,,\end{aligned} \tag{3.18}$$

where the $(\overline{15},15)$ and $(\overline{21},21)$ tensors are defined by

$$\bar{\Psi}_{\bar{j}_1\bar{j}_2}{}^{i_1i_2}=\tfrac{1}{2}\bar{\Psi}_{[\bar{j}_1\bar{j}_2]}{}^{[i_1i_2]}+\tfrac{1}{2}\bar{\Psi}_{(\bar{j}_1\bar{j}_2)}{}^{(i_1i_2)}. \tag{3.19}$$

4. MATRIX ELEMENTS OF $U(6,6)$ TRANSFORMATIONS

We wish now to compute the kinematic factors $\langle m|L_p|m'\rangle$. We employ for this a graphical technique. It lends itself to a nearly closed form evaluation of matrix elements of finite transformations and we shall start with a general discussion although in fact we shall need only the Lorentz transformations. Consider the matrix element

$$\langle m'|e^{i\lambda_A{}^B M_B{}^A}|m\rangle=\langle m'|e^{i\bar{\psi}\lambda\psi}|m\rangle, \tag{4.1}$$

where $\lambda_A{}^A=0$ and $\lambda^\dagger=\gamma_0\lambda\gamma_0$. Following the method of Feynman, we treat the computation of (4.1) as if it were an S-matrix computation. Thus we attach a label t to the spinor quantities

$$\psi_A(t)=\psi_A, \qquad 0\leq t\leq 1.$$

Define the "Hamiltonian density"

$$H(t)=\bar{\psi}(t)\lambda\psi(t), \tag{4.2}$$

so that

$$\int_0^1 dt\, H(t)=\bar{\psi}\lambda\psi=H$$

and, in particular,

$$\int_0^1 dt_1\cdots dt_n T(H(t_1)\cdots H(t_n))=\int_0^1 dt_1\cdots dt_n \sum_{\text{perm}} \theta(t_{i_1}-t_{i_2})\cdots\theta(t_{i_{n-1}}-t_{i_n})H(t_{i_1})\cdots H(t_{i_n})$$

$$=\int_0^1 dt_1\cdots dt_n \sum_{\text{perm}} \theta(t_{i_1}-t_{i_2})\cdots\theta(t_{i_{n-1}}-t_{i_n})H^n$$

$$=H^n.$$

This enables us to write

$$e^{iH}=T\left\{\exp\left[i\int_0^1 dt\, H(t)\right]\right\}. \tag{4.3}$$

One can go further: let $\lambda_A{}^B$ be considered a function of t and define the functional

$$\Phi(\lambda)=\langle 0|T\left\{\exp\left[i\int_0^1 dt\,\bar{\psi}(t)\lambda(t)\psi(t)\right]\right\}|0\rangle. \tag{4.4}$$

Its derivatives

$$\frac{\delta}{\delta\lambda_{A_1}{}^{B_1}(t_1)}\cdots\Phi(\lambda)=\langle 0|T\left[\bar{\psi}^{A_1}(t_1)\psi_{B_1}(t_1)\cdots\exp\left(i\int_0^1 dt\,\bar{\psi}\lambda\psi\right)\right]|0\rangle$$

include, in particular, the desired matrix elements

$$\left[\frac{\delta}{\delta\lambda_{i_1}{}^{\bar{j}_1}(1)}\cdots\frac{\delta}{\delta\lambda_{\bar{k}}{}^{l}(0)}\cdots\Phi(\lambda)\right]_{\lambda=\text{const}}=\langle 0|a^{i_1}b_{\bar{j}_1}\cdots e^{iH}\cdots\bar{a}_{l_1}\bar{b}^{\bar{k}_1}|0\rangle. \tag{4.5}$$

For the evaluation of $\Phi(\lambda)$ we can set up a graphical prescription. The lines in a Feynman graph correspond to

$$\langle 0|T(\psi_A(t_1)\bar{\psi}^B(t_2))|0\rangle=\Delta_A{}^B(t_1-t_2) \tag{4.6}$$

and the vertices to $i\lambda_A{}^B(t)$. The vacuum diagrams must consist entirely of simple closed loops corresponding to the terms

$$i^n\,\mathrm{Tr}(\lambda(t_1)\Delta(t_1-t_2)\lambda(t_2)\cdots\Delta(t_n-t_1)). \tag{4.7}$$

The propagators appearing here are very simple. Since

$$T(\psi_A(t_1)\bar{\psi}^B(t_2))=\theta(t_1-t_2)\psi_A(t_1)\bar{\psi}^B(t_2)+\theta(t_2-t_1)\bar{\psi}^B(t_2)\psi_A(t_1),$$

$$\Delta_A{}^B(t_1-t_2)=\begin{bmatrix}-\theta_-(t_1-t_2)\delta_i{}^j & 0\\ 0 & \theta_+(t_1-t_2)\delta_{\bar{i}}{}^{\bar{j}}\end{bmatrix}. \tag{4.8}$$

For the particular case of a pure Lorentz transformation we take

$$\lambda_A{}^B = \tfrac{1}{2}\chi\begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix}. \tag{4.9}$$

Noting that

$$\int_0^1 dt_1 \cdots dt_{2m}\theta_+(t_1-t_2)\theta_-(t_2-t_3)\cdots\theta_-(t_{2m}-t_1) = 2^{m-1}\frac{(m-1)!m!}{(2m)!}, \tag{4.10}$$

we get the result

$$\begin{aligned}\Phi(\chi) &= \langle 0|e^{i\chi J_{03}}|0\rangle \\ &= \exp\left[\sum_{m=1}^{\infty}(-)^m 2^{m-1}\frac{(m-1)!m!}{2m!}(\tfrac{1}{2}\chi)^{2m}\right] \\ &= \exp\left[1-\frac{1}{(1+x^2)^{1/2}}\{x\ln(x+(1+x^2)^{1/2})+1\}\right],\end{aligned} \tag{4.11}$$

where $x=\chi/2\sqrt{2}$. For large χ, this matrix element behaves like $1/\chi$. The graphical procedure needed to obtain the general functional $\Phi(\lambda(t))$ and thus any other desired matrix element is straightforward. All other matrix elements will contain a factor like $\Phi(\chi)$ multiplied into a series of the type which appears in the exponent of the exponential in (4.2). Altogether one may therefore expect that the kinematic factors arising from matrix elements of $(m_2|L_p|m_1)$ fall with increasing momenta.

5. THE OUTLOOK

There are two major problems which need further consideration:

(1) *The determination of appropriate representations for baryons and mesons.* As is well known, the simplest $U(6,6)$ Feynman towers $B=(56,1)$, $(126,\bar{6})$, $\cdots$ and $M=(1,\bar{1})$, $(6,\bar{6})$, 21, $\bar{2}\bar{1})$, $\cdots$ do not couple. One must find more sophisticated (less degenerate) representations in which to place the (56,1) (and any other known) baryon $U(6)\times U(6)$ multiplet and similarly for the mesons. Assuming that one does succeed in finding towers which allow, e.g., for the requisite coupling, one would still need to verify that all the successful predictions of the previous $U(6)\times U(6)|_p$ theory—at least for the vertex function—survive.

It is indeed possible that $U(6,6)$ is not the right symmetry group. There are several other possibilities one may consider; e.g., to accommodate kinetic supermultiplets considered by Gatto and others[6] one may need the group $O(3,1)\times U(6,6)$—or in analogy with the hydrogen atom case, the more attractive possibility $O(4,1)\times U(6,6)$.

(2) The second unresolved problem concerns mass-breaking for the noncompact symmetry and the effect this would have on the unitarity of the S matrix. Experimentally the mass differences between incipient rungs of possible towers suggested so far appear quite considerable and the symmetric S matrix can therefore be unitary only in the sight of the Lord. One of the attractive features of the higher symmetry-breaking provided by Bargmann-Wigner equations in the earlier $\tilde{U}(12)$ theory was the automatic mass-split which occurred between particles of different spin if the unitarity corrections were taken into account.[14] The Bargmann-Wigner equations obviated any need for the introduction of separate spin-splitting terms in any $U(6)\times U(6)$ mass formula. One may find this remark of value also in connection with the coupling problem mentioned above. What we have in mind is the possibility that though the Feynman towers for $U(6,6)$ theory do not couple for the vertex function, the corresponding boosted towers for $U(6,6)\times U(6,6)$ (see Ref. 6) (where the extra degrees of freedom are cut down by the use of Bargmann-Wigner equations) do so. The levels of such towers are indeed representations of the covariant subgroup $U(6)\times U(6)|_p$ and, what is extremely important, the meson-baryon coupling is exactly the coupling written down in Ref. 1, with its merits and demerits. This type of mixed approach which exploits both the ideas of this paper and of the previous work of Ref. 1 would perhaps be more in keeping with the attitude that the origin of noncompact groups is to be attributed to dynamical accidents which may occur in special dynamical situations for special values of physical parameters[15] and is not something in the nature of a fundamental characteristic of elementary-particle physics.

(3) There is one other possibility which one may explore in connection with the coupling and the mass-symmetry-breaking problem. This is to assume that the S matrix is not a scalar in the $U(6,6)$ space but a (Lorentz-scalar) part of $U(6,6)$ tensors.

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[14] The unitarity relations like $\mathrm{Re}T=\int[\mathrm{Im}T(k^2)dk^2/(k^2-m^2)]$ for a two-point function are in fact the Lehmann mass formulas. Even if $\mathrm{Im}T$ is independent of spin for any $U(6)\times U(6)$ multiplet, the Lehmann formulas have different forms for each spin value and therfore $\mathrm{Re}T$ (i.e., the physical mass) will vary with spin.

[15] For example, the Cutkosky hydrogen-spectrum symmetry $O(5,1)$ makes its appearance only for the case when the total energy-momentum vector P_μ for the atom is identically zero.