$SU(6)_W$-Symmetric Meson Bootstrap Model*

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A bootstrap model of the pseudoscalar-meson octet and singlet and the vector-meson octet and singlet is constructed. The mesons (M) are assumed to be MM bound states, produced by one-M-exchange forces. The input MMM interaction is assumed $SU(6)_W$-symmetric, and the forces in all P-wave MM states are examined. The forces are most attractive in 36 states of the quantum numbers of the mesons; identification of these states with the M leads to output MMM interactions consistent with those assumed originally. However, the $SU(6)_W$ symmetry requires that the total coupling of the singlet vector meson should be only $\frac{2}{3}$ that of the other mesons. The method of calculation involves continuing the interaction off the mass shell in an $SU(6)_W$-symmetric fashion, and comparing the P-wave amplitudes at threshold in Born approximation. The technique can also be used to predict heavier resonances that do not correspond to $SU(6)$ representations. This fact is encouraging, since it appears that only the lighter observed hadrons form complete representations of $SU(6)$.

I. INTRODUCTION

THE approximate validity of $SU(6)$ symmetry leads to a great simplification for many problems of particle physics. One type of endeavor that has benefited from the symmetry is the construction of composite models of hadron multiplets. We consider the example of the bootstrap model of the baryon octet and $J^P=\frac{3}{2}^+$ decuplet. If only $SU(3)$ symmetry is assumed, the presence of two baryon multiplets leads to several complications. On the other hand, one need consider only the one 56-fold baryon supermultiplet B and the one 35-fold odd-parity meson supermultiplet M if $SU(6)$ symmetry is assumed.[1] Furthermore, $SU(6)$ symmetry leads to a simplification in the treatment of meson exchange forces in BB and $B\bar{B}$ S-states.[2]

Unfortunately, the techniques used in these references cannot be applied directly to processes involving one of the strongest of hadron interactions, the MMM interaction. The difficulty arises because the interaction of three odd-parity mesons is a P-wave interaction, and the coupling of orbital and spin-angular momenta is not provided for in the simplest interpretation of $SU(6)$. It was possible to circumvent a similar difficulty that arose in connection with the P-wave MBB interactions by coupling the orbital angular momentum to the meson spin, and classifying the mesons by their total angular momentum.[1] However, this technique cannot be applied to the MMM interaction, because of the requirement of permutation symmetry of the three mesons.

The prospect of achieving a dynamical understanding of the meson-meson-meson interactions received a boost from the discovery of the "collinear" group $SU(6)_W$, as this group provides a simple prescription for the trilinear interactions of the 36 meson states (π, K, η, X, ρ, K^*, φ, and ω).[3] The main purpose of this paper is to determine whether or not an $SU(6)_W$-symmetric MMM interaction may bootstrap itself in a simple one-M-exchange model. Such a bootstrap model can be successful only if two conditions are satisfied. First, the meson exchange forces must be more attractive in 36 states of the quantum numbers of the mesons than in any other P-wave MM states. Second, the assumption of bound-state poles corresponding to the mesons in these 36 states must lead to ratios of MMM coupling constants equal to those assumed originally.

A second purpose of the paper has to do with symmetry breaking. It is well known that the $SU(6)$ symmetry of nature must be broken more drastically than the purely internal symmetry of $SU(3)$. One aspect of this symmetry breaking concerns the observed meson and baryon resonances of spins greater than $\frac{3}{2}$. It is unlikely that these high-spin states are parts of complete $SU(6)$ representations, since corresponding states of large isotopic spin and hypercharge have not been discovered. This type of deviation may be placed on a quantitative basis, in the following way. Since S_z and I_z are both Hermitian generators of $SU(6)$, the ratio $R=\frac{2}{3}\sum S_z^2/\sum I_z^2$ is the same for every representation or sum of representations of $SU(6)$, if the sums are taken over all the states. The factor $\frac{2}{3}$ is included so that the predicted value is unity. However, the value of R corresponding to the experimentally verified meson and baryon resonances is much greater than one. We will call this ($R>1$) "vertical" deviation from the symmetry.

The observed vertical deviation is not surprising, since $SU(6)$ involves spin but not orbital angular momentum. It is easy to visualize composite models of hadrons in which the lighter states are S or P states that correspond to $SU(6)$ multiplets, while the heavier states involve larger angular momenta and are not described by $SU(6)$. This argument applies whether the basic building blocks are quarks or the lighter mesons and baryons themselves. Thus, the difficult problem is not to comprehend the effect, but rather to construct a technique that exploits the symmetry of the lighter

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[1] R. H. Capps, Phys. Rev. Letters 14, 31 (1965); J. G. Belinfante and R. E. Cutkosky, *ibid.* 14, 33 (1965).
[2] R. H. Capps, Phys. Rev. Letters 14, 842 (1965).
[3] H. J. Lipkin and S. Meshkov, Phys. Rev. Letters 14, 670 (1965).

hadrons, and still is useful for calculations involving the nonsymmetric multiplets.

The $SU(6)_W$ group is ideally suited for forming a connecting link between the symmetric and nonsymmetric multiplets. If the input is $SU(6)_W$-symmetric, the predicted one-particle exchange contributions to the forward and backward scattering amplitudes will preserve the symmetry. If the orbital angular momentum is large, spin-flip amplitudes (that vanish in the collinear directions) are likely to be important, so that large deviations from the symmetry are likely for predicted resonances.

In Sec. II the $SU(6)_W$-symmetric MMM interaction is constructed, and the assumptions of the model are presented. These assumptions are summarized in Sec. IID. Sections III and IV deal with the calculation of the forces and output interaction constants; the use of $SU(6)_W$ in the calculation is explained in Sec. III, while Sec. IV is concerned with the details of the calculation. Readers not interested in the calculational technique may skip from the end of Sec. II to Sec. V, where the results are presented and the input and output interaction ratios are compared. The manner in which $SU(6)_W$ symmetry may be useful for calculations involving resonance multiplets that do not correspond to $SU(6)$ multiplets is also discussed in Sec. V.

II. THE MODEL

A. The Small Coupling of the Vector Singlet

A priori, it is not obvious that $SU(6)_W$ should lead to a simple meson bootstrap model, since the generators of the group commute with the kinetic energies of all the particles only for collinear processes.[3] On the other hand, one cannot determine all the P-wave scattering amplitudes from the forward and backward amplitudes when some of the particles possess intrinsic spins, so P-wave scattering is not basically a linear process.

In the simplest type of bootstrap model of a degenerate multiplet, the sum of the squares of all the interaction constants associated with a particle is the same for all particles. However, the argument presented below shows that $SU(6)$ symmetry requires that this "total coupling" sum associated with the vector singlet is anomalous. We assume that the mesons are degenerate and define the interaction constant $\gamma(abc)$ in terms of the "decay" amplitude $M_a \rightarrow M_b + M_c$ on the mass shell. Of course the decay momentum is imaginary; the process is not physical. The Lorentz system is taken to be the rest system of the initial particle a, and the positive z axis to be in the direction of the momentum of the first final particle b. The spin components are analyzed along the z axis. The W-spin triplets and singlets of the meson multiplet $\mathbf{35}\oplus\mathbf{1}$ contain the states (V^1, P, V^{-1}) and (V^0), where V and P denote vector and pseudoscalar mesons, and the superscript is the z component of the spin.[3]

Since the mesons are all of odd parity, the M_a wave function must be odd with respect to a space reflection. Hence, the decay amplitudes (in the z direction) for a meson state of the $SU(6)_W$ representation $\mathbf{35}$ are proportional to the Clebsch-Gordan coefficients corresponding to the 35-fold antisymmetric state in the direct product $\mathbf{35}\otimes\mathbf{35}$.[4] The $SU(6)_W$ singlet (V singlet in the state $S^z=0$) decay amplitude is zero, since there is no antisymmetric singlet in the direct product $\mathbf{35}\otimes\mathbf{35}$. The total coupling of a physical particle is the decay probability, *integrated over all directions*. This probability for any spin state of a V meson must be equal to the decay probability in any direction of an unpolarized sample of the V. It follows from these arguments that the total decay probabilities of the P_1, P_8, and V_8 are equal, but that of the V_1 is only $\frac{2}{3}$ that of the others (since the $S^z=0$ state of the V_1 cannot decay along the z axis).

The fact that $SU(6)_W$ requires an anomalous input value for the total V_1 coupling suggests that one construct a model in which the masses of the different $SU(3)$ multiplets may be different, and deviations from the interaction symmetry are allowed. However, the results of such a model would depend on the detailed dynamical assumptions. Our present goal is to construct a simple first approximation to a realistic model. Thus, we will not investigate the connection between total coupling and mass. Rather, we assume degeneracy and exact $SU(6)_W$ symmetry for the vertices of the one-meson exchange diagrams.

B. Interaction on the Mass Shell

A relativistic MMM interaction I, that satisfies $SU(6)_W$ symmetry on the mass shell, may be written in the following form.

$$I = I_{VPP} + I_{VVP} + I_{VVV} + I_{VVV}', \tag{1}$$

$$I_{VPP} = C_{VPP} F_{ilm} e_i \cdot p_{lm}, \tag{2a}$$

$$I_{VVP} = (C_{VVP}/M) D_{ijl} e_i^{\mu} e_j^{\nu} p_i^{\sigma} p_j^{\lambda} \epsilon_{\mu\nu\sigma\lambda}, \tag{2b}$$

$$I_{VVV} = C_{VVV} F_{ijk} [(e_i \cdot e_j)(e_k \cdot p_{ij}) + (e_j \cdot e_k)(e_i \cdot p_{jk}) + (e_k \cdot e_i)(e_j \cdot p_{ki})], \tag{2c}$$

$$I_{VVV}' = (C_{VVV}'/M^2) F_{ijk} [(e_i \cdot p_{jk})(e_j \cdot p_{ki})(e_k \cdot p_{ij})], \tag{2d}$$

where $p_{\alpha\beta}$ is shorthand for $p_\alpha - p_\beta$. The subscripts i, j, and k are $SU(3)$ indices of V mesons, and l and m are $SU(3)$ indices of P mesons; summations over these indices are implied. The symbol e_α denotes the four-polarization vector of the V meson α, emitted or absorbed at the vertex, and p_α denotes the four-momentum of the particle α emitted at the vertex, or minus the four-momentum of the particle absorbed. The C and C' are real constants, M is the meson mass, and $\epsilon_{\mu\nu\sigma\lambda}$ is the completely antisymmetric tensor normalized by the

[4] The Clebsch-Gordan coefficients associated with the direct product $35\otimes35$ of $SU(6)$ are given by C. L. Cook and G. Murtaza, Nuovo Cimento **39**, 531 (1965). See also Lothar Schülke, Z. Physik **183**, 424 (1965).

condition $\epsilon_{0123}=1$. We use the metric $A\cdot B=\mathbf{A}\cdot\mathbf{B}-A^0B^0$. The symbols $F_{\alpha\beta\gamma}$ denote the completely antisymmetric "F-type" coupling constants referring only to the interaction of three $SU(3)$ octets. The D are the completely symmetric nonet-type coupling constants, referring to singlets and octets. We use the convention that redundant permutations of the D and F indices in the equations are not taken; i.e., each particular set of three meson states occurs no more than once in any of the four interactions.[5]

The F coefficients, and octet-octet-octet part of the D coefficients, are well known.[6] The relations between various parts of the D are given by

$$D_{\omega\omega X}=D_{\varphi\varphi X}=\sqrt{2}D_{\varphi\varphi\eta}, \qquad (3)$$

where ω, X, φ, and η denote the V_1, P_1, and isoscalar members of the V_8 and P_8, respectively. If $\Pi\{\alpha,\beta\gamma\}$ denotes the total probability of the two-particle states corresponding to the $SU(3)$ representations β and γ in the state α for the D-type coupling, the ratios of the Π are

$$\Pi\{1,88\}:\Pi\{1,11\}:\Pi\{8,88\}:\Pi\{8,(18+81)\}=16:2:5:4.$$

A simple way to calculate the various coefficients is to write the nonets as 3×3 matrices, and then identify symmetric and antisymmetric combinations of the product of two such matrices with D and F states, respectively.[7] The singlet participates only in the symmetric combination; hence the total D-type probability in the singlet state is twice that in an octet state. The relations among the constants C and C' that are implied by the symmetry may be written in the form,[5]

$$C_{VPP}F_{\rho^0\pi^+\pi^-}=(3/8)^{1/2}C_{VVP}D_{\rho^0\omega\pi^0}=C_{VVV}F_{\rho^0\rho^+\rho^-}=12C_{VVV}'F_{\rho^0\rho^+\rho^-}. \qquad (4)$$

In order to illustrate the meaning of the various terms in the interaction, we classify the meson states in three classes, the extreme helicity V states $V^{\pm1}$, the zero helicity states V^0, and the P states.[8] Four types of trilinear vertices are allowed, (V^0PP), $(V^1V^{-1}P)$, $(V^0V^0V^0)$, and $(V^1V^{-1}V^0)$. The interactions I_{VPP}, I_{VVP}, and I_{VVV}' correspond to the first three of these types, while I_{VVV} is a linear combination of the last two types.

[5] The $SU(6)_W$-symmetric MMM interaction has been derived from the hypothesis of invariance under the group $M(12)$ by B. Sakita and K. C. Wali, Phys. Rev. **139**, B1355 (1965). An alternate method of obtaining the interaction is to use the P and V universality principles discussed by R. H. Capps, Phys. Rev. **144**, 1182 (1966). The numerical relations between the VPP and VVP interactions are derived in these references.

[6] See, for example, S. L. Glashow, and J. J. Sakurai, Nuovo Cimento **25**, 337 (1962).

[7] A detailed derivation of the octet-octet-octet interactions by this method is given by J. J. Sakurai, *Theoretical Physics, Lectures Presented at a Seminar, Trieste, 16 July–25 August, 1962* (International Atomic Energy Agency, Vienna, 1963), pp. 227–249.

[8] This method of analysis has been used to illustrate the meaning of $SU(6)_W$ by H. Lipkin, *Symmetry Principles at High Energy, Third Conference, January, 1966* (W. H. Freeman and Company, San Francisco, California, 1966), pp. 97–106.

The choices of F and nonet-type D coupling, together with the conditions of Eq. (4), are sufficient to guarantee $SU(6)_W$ symmetry.

One may verify the symmetry of the interaction by checking the permutation symmetry of the vertices, and by comparing the coupling constants with the Clebsch-Gordan coefficients of the representation **35** contained antisymmetrically in the direct product **35**⊗**35**.[4] We will discuss here only the subtle question of the permutation symmetry. It is convenient to choose both the spin and internal symmetry representations so that each of the 36 meson states corresponds to a Hermitian field. The plane polarization states V^x and V^y are considered, rather than the linear combinations $V^{\pm1}$. The state V^0 is denoted V^z. We use $\gamma(abc)$ to denote the dimensionless (nonphysical) "decay" amplitude $M_a\to M_b+M_c$ on the mass shell, the positive z direction being that of the M_b momentum. The indices a, b, and c denote helicity and internal quantum numbers. For each of the four interaction types of Eqs. (2a)–(2d), we consider one term in the sum. It is sufficient for the required permutation symmetry if the phases of the particle states may be chosen so that the $\gamma(abc)$ in each of these terms is completely antisymmetric. This symmetry condition may be satisfied if the phases of all the V^z states are increased by $\frac{1}{2}\pi$, i.e., if initial and final states V^z are replaced by iV^z and $(-iV^z)$, respectively. With this phase choice, all the amplitudes $\gamma(abc)$ are real, since the F and D coefficients are imaginary and real, respectively, in a Hermitian representation.

We demonstrate this permutation symmetry for the VPP interaction term $F_{123}e_1\cdot(p_2-p_3)$. (The constant C is suppressed.) The longitudinal and scalar components of the V four-polarization vector e_1 in the Lorentz system characterized by the four-momentum $p_1=(\mathbf{p}_1,\omega_1)$ are related to the unit three-polarization vector $\mathbf{V}_1$ of the V rest system by the equations,

$$\mathbf{e}_1\cdot\mathbf{p}_1=\mathbf{V}_1\cdot\mathbf{p}_1(\omega_1/M), \quad e_1{}^0=(\mathbf{V}_1\cdot\mathbf{p}_1)/M. \qquad (5)$$

We consider the amplitude $M_a\to M_b+M_c$ in the rest system of the M_a. The mass shell energy relation is then $\omega_b+\omega_c=M$. If the above phase convention is used, the amplitude $V_1{}^z\to P_2+P_3$ is proportional to the quantity $[iF_{123}(p_2{}^z-p_3{}^z)]=2iF_{123}p_2{}^z$. On the other hand, the amplitude $p_2\to V_1{}^z+P_3$ is proportional to the quantity, $-iF_{123}(\mathbf{V}_1\cdot\mathbf{p}_1)(\omega_1+\omega_3+M)/M=-2iF_{123}p_1{}^z$. An extension of this argument may be used to show that all the amplitudes are antisymmetric, so the permutation condition is satisfied.

The convention of changing the phase of the states V^z by $\frac{1}{2}\pi$ is unusual, but perfectly proper. If one is considering scattering processes at arbitrary angles, a more appropriate procedure would be to make such a phase change in all V states, as was done by the author in Ref. 5. Of course $SU(6)_W$ symmetry is not present for the nonlinear processes.

Any $SU(6)_W$-symmetric MMM interaction may be written in the form of Eqs. (1), (2), and (4) on the mass shell. This follows because the ratios of all the coupling constants are specified unambiguously by the symmetry requirement.

C. Continuation Off the Mass Shell

The various P-wave MM scattering amplitudes τ may be expressed in terms of elements of the unitary S matrix by the equation,

$$\tau_{ij}=(S_{ij}-\delta_{ij})\rho(s)/(2i|\mathbf{p}|^3)\,, \tag{6}$$

where $s=4\omega^2$ is the square of the total energy in the center-of-mass system, and the function $\rho(s)$ contains no zeros or poles at threshold. In principle, one may write partial-wave dispersion relations for these amplitudes, including only one-meson exchange contributions on the left-hand cut. This procedure involves the ambiguity always present in such treatments of partial waves, resulting from the fact that a polynomial in s may be added to an amplitude without changing the discontinuity across the left-hand cut. If one uses the N/D method, the ambiguity concerns the choice of the function $\rho(s)$ of Eq. (6) that is assumed to lead to an N function with a simple pole in the direct-pole process. Many prescriptions that are commonly used treat the P, longitudinal V, and transverse V states differently. Such a prescription would lead to a loss of $SU(6)_W$ symmetry, and must be avoided here.

Since we are interested only in relative forces and in relative values of coupling constants, we need not write actual dispersion relations. We will base our conclusions on a comparison of one-meson-exchange amplitudes in Born approximation. This procedure circumvents the ambiguity discussed above, *provided the interaction remains $SU(6)_W$-symmetric when one particle is taken off the mass shell.* One may study the off-mass-shell symmetry conveniently by consider the direct pole amplitudes $M_a+M_b\to M_c\to M_{a'}+M_{b'}$ in the z direction. We exclude $V^zV^zV^z$ vertices from the discussion temporarily. It may be shown that if Eqs. (1) through (2d) are applied off the mass shell, the amplitudes referring to V^z direct poles do not contain a multiplicative factor of s/M^2 that is contained in the other amplitudes. For example, the $P+P\to V^z\to P+P$ Feynman amplitudes are proportional to $(s-M^2)^{-1}(\mathbf{p}\cdot\mathbf{p}')$, while the $V^z+P\to P\to V^z+P$ amplitudes are proportional to

$$(s-M^2)^{-1}(\mathbf{e}\cdot\mathbf{p})(\mathbf{e}'\cdot\mathbf{p}')=(s-M^2)^{-1}(\mathbf{V}\cdot\mathbf{p})(\mathbf{V}'\cdot\mathbf{p}')[s/(4M^2)].$$

(A prime denotes a final-state variable.) Thus, the interaction must be modified; in order that the results be reasonable, it is necessary that the modification does not introduce threshold zeros or poles into the one-meson exchange contributions to the P-wave amplitudes.

The necessary modification factors may be applied to the terms of type $e_1\cdot(p_2-p_3)$ in Eqs. (2a) and (2c). For each such term we make the replacement,

$$e_1\cdot(p_2-p_3)\to e_1\cdot(p_2-p_3)[-(p_2+p_3)^2/M^2]^{1/2}. \tag{7}$$

In order to complete the modification, it is necessary to multiply I_{VVV}' by a factor chosen so that the energy dependence of the amplitudes involving $V^zV^zV^z$ vertices is the same as that of the other amplitudes. This may be done. However, it is not necessary for us to determine this modification factor, for reasons given in Sec. IID.

It must be emphasized that this modification procedure is only a substitute for the use of dispersion relations to determine the energy dependence of the P-wave amplitudes. The procedure introduces a threshold zero into the S-wave amplitudes, so that modification of the P and transverse V exchange amplitudes would be more appropriate in a treatment of S waves.[9] The effect of the modification on the present calculation is discussed further in Sec. IID.

D. Criteria for Bound States and Output Coupling Constants

The threshold energy is particularly convenient for projecting out the P-wave parts of the Born approximation amplitudes. The threshold P-wave amplitude is obtained by dividing by $\mathbf{p}^2$ those terms in the Feynman amplitude that are linear in both the initial and final three-momenta. For each amplitude, both types of exchange processes (t-channel and u-channel forces) must be considered.

The amplitudes are matrices, since many channels are involved. It is assumed that bound states correspond to the largest positive eigenvalues of the threshold amplitude matrices. It is assumed further that if these Born amplitudes were used as N functions in partial-wave dispersion relations, the right-hand cuts would provide a common energy denominator for the different components of the matrix amplitude. Hence, the ratio of the interaction constants coupling two different two-particle channels to a particular bound state is equal to the ratio of the components of the channels in the appropriate eigenamplitude of the threshold Born approximation.

It is seen from Eq. (2d) that if the interaction I_{VVV}' were unmodified, the amplitudes associated with this interaction would vanish at threshold. This situation is maintained when the interaction is modified. One may verify this statement by showing that the relative contributions of the $V^xV^xV^z$ and $V^zV^zV^z$ parts of the interaction I_{VVV} to the one-meson exchange amplitudes at threshold are in accordance with the requirements of $SU(6)_W$ symmetry. Hence, the interaction I' may be neglected.

The V meson propagator is $(\delta_{\mu\nu}+p_\mu p_\nu/M^2)/(-p^2-M^2)$ where p is the four-momentum of the virtual V meson. The second term in the numerator does not contribute to the threshold P-wave amplitudes, and may be neglected. Furthermore, it can be shown that

[9] Such a procedure is used in Ref. 2.

all the contributions from exchanged V mesons in the zero helicity state (zero spin in the direction of the momentum transfer) result from the δ_{00} term of the propagator. These facts simplify the calculation.

We now study the effect of the modification of Eq. (7) on the P-wave amplitudes. The contributions to the unmodified Feynman amplitudes resulting from the exchange of P mesons, and V mesons polarized transverse to the momentum transfer, are proportional to $\mathbf{A}\cdot\mathbf{\Delta}\mathbf{B}\cdot\mathbf{\Delta}$, where $\mathbf{A}$ and $\mathbf{B}$ are spin operators in the V-meson spin space, and $\mathbf{\Delta}$ is the three-momentum transfer. These amplitudes vanish in the forward direction; however, the P-wave amplitudes, defined as in Eq. (6), do not vanish at threshold. The modification of Eq. (7) introduces a multiplicative factor of $(-\Delta^2/M^2)$ into the contribution from the exchange of zero-helicity V mesons, preserving the $SU(6)_W$ symmetry. It is instructive to compare these zero-helicity V contributions with those that would result if no modification were made. The virtual V mesons concerned are coupled at each vertex in the manner $Ge\cdot(p+p')$, where p and p' are initial and final four-momenta and the factors contained in G depend on whether the real mesons are V or P mesons. If Eqs. (2a) and (2c) are not modified, and 1 and 2 denote the two vertices, the Feynman amplitude may be written

$$G_1G_2\frac{(p_1+p_1')\cdot(p_2+p_2')}{M^2+(p_1-p_1')^2}.$$

Near threshold, the P-wave part of this amplitude in the center-of-mass system is $(G_1G_2/M^2)(-8\mathbf{p}_1\cdot\mathbf{p}_1'-2\mathbf{p}_1\cdot\mathbf{p}_1')$ where the first and second terms result from the time-like component, and space components, of the virtual V meson. If the modification of Eq. (7) is applied, the P-wave part of this amplitude near threshold is $(G_1G_2/M^2)(-8\mathbf{p}_1\cdot\mathbf{p}_1')$. We conclude that the use of Eqs. (2a) and (2c) without modification would destroy the $SU(6)_W$ symmetry, but would lead to calculated numbers that do not differ much from those calculated in Sec. IV from the symmetric procedure.

Readers uninterested in the details of the calculations may skip Secs. III and IV, and proceed directly to Sec. V, where the results are summarized.

III. THE USE OF $SU(6)_W$ SYMMETRY

The one-meson exchange amplitudes are zero in the forward direction and satisfy $SU(6)_W$ in the backward direction. We digress briefly from the description of the calculational method to show how the symmetry of the backward amplitudes may be demonstrated. As in Sec. IIB, we start with a Hermitian representation for the mesons, and then multiply the initial and final V^z states by the phase factors (i) and $(-i)$, respectively. It is shown in Sec. IIC that the Feynman amplitudes for the direct pole processes $a+b\to c\to a'+b'$ satisfy $SU(6)_W$ symmetry in the forward and backward directions. (These indices denote helicity and internal quantum numbers.) In the center-of-mass system, the amplitude for such a process may be written

$$-K\gamma_{cab}\gamma_{ca'b'}s(M^2-s)^{-1}p_a{}^zp_{a'}{}^z, \qquad (8)$$

where K is a constant. The crossed process for this amplitude is the c-exchange contribution to the amplitude $a+a'\to b+b'$. It may be shown that the threshold value of this amplitude term in the z direction is

$$K\gamma_{cab}\gamma_{ca'b'}(\mathbf{p}_b{}^z-\mathbf{p}_a{}^z)^2. \qquad (9)$$

The proportionality of the coefficients of Eqs. (8) and (9) guarantees the $SU(6)_W$ symmetry of the exchange contributions of Eq. (9).

We now return to the relation of $SU(6)_W$ to the calculation of the P-wave amplitudes. The $\cos\theta$ terms of the P-wave amplitudes are given by one-half the difference between the forward and backward amplitudes. The coefficients of these terms satisfy $SU(6)_W$ symmetry. These terms are not sufficient for determining all the P-wave amplitudes, but the symmetry is useful for determining many coefficients, and for understanding the results.

The allowed P-wave amplitudes are antisymmetric in the combined spin and internal indices. The coefficients of $\cos\theta$ in these amplitudes are linear combinations of amplitudes referring to the various antisymmetric terms in the direct product $\mathbf{35}\otimes\mathbf{35}$ of $SU(6)_W$. It is helpful to draw an analogy with an "internal $SU(6)$" model, in which V mesons interact with the VVV interaction discussed by Cutkosky, and $SU(6)$ is an internal symmetry.[10] The reduction of the antisymmetric representations in the direct product $\mathbf{35}\otimes\mathbf{35}$ is $\mathbf{35}\oplus\mathbf{280}\oplus\mathbf{280}^*$. The one-meson exchange force in the representation $\mathbf{280}\oplus\mathbf{280}^*$ is zero.[11] Hence, the $\cos\theta$ term in the P-wave amplitude of the present model corresponds *entirely* to the $SU(6)_W$ representation **35**. We determine the sign of this $\cos\theta$ amplitude by noting that the only contribution to $\pi\pi$ elastic scattering results from ρ exchange. The ρ exchange amplitude is positive, corresponding to an attractive force.

We denote the triplets and singlets that correspond to the spin-like quantum number of the internal-symmetry model by T and S. The structure of the 35-fold multiplet is (T_1, T_8, S_8). The wave functions for these particles correspond to the antisymmetric 35-35-35 Clebsch-Gordan coefficients of $SU(6)$, and are given by,[4]

$$\begin{aligned}\psi(T_1)&=(T\times T)_D\,,\\ \psi(T_8)&=(\tfrac{1}{2})^{1/2}(T\times T)_D+(\tfrac{1}{2})^{1/2}(TS)_F\,,\\ \psi(S_8)&=(\tfrac{3}{4})^{1/2}(T\cdot T)_F+(\tfrac{1}{4})^{1/2}(SS)_F\,,\end{aligned} \qquad (10)$$

[10] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963).

[11] It follows from the argument of Ref. 10 that for any $SU(n)$, the only nonzero forces in this type of model are those associated with the regular representation. A more direct way to demonstrate that the force in the state $\mathbf{280}\oplus\mathbf{280}^*$ is zero is to use Eq. (3) of Ref. 2.

where all the quantities in parenthesis are normalized to unity, the cross and dot products refer to the way in which the spin-like indices of the T are combined, and F and D refer to the F and nonet D-coupling described in Sec. IIB.

One may use these equations to write analogous equations for the components of the W-spin triplet $(V^{x,y}, P)$ and singlet (V^z) that occur in the $\cos\theta$ terms of the P-wave amplitudes. The W-spin equations may be written in the shorthand form

$$\psi(P_1)=(V^{x,y}V^{y,x})_D\,, \tag{11a}$$

$$\psi(V_1^{x,y})=(V^{y,x}P)_D\,, \tag{11b}$$

$$\psi(V_1^z)=\text{uncoupled}\,, \tag{11c}$$

$$\psi(P_8)=(\tfrac{1}{2})^{1/2}(V^{x,y}V^{y,x})_D+(\tfrac{1}{2})^{1/2}(V^zP)_F\,, \tag{11d}$$

$$\psi(V_8^{x,y})=(\tfrac{1}{2})^{1/2}(V^{y,x}P)_D+(\tfrac{1}{2})^{1/2}(V^{x,y}V^z)_F\,, \tag{11e}$$

$$\psi(V_8^z)=(\tfrac{1}{2})^{1/2}(V^{x,y}V^{x,y})_F+(\tfrac{1}{4})^{1/2}(PP)_F+(\tfrac{1}{4})^{1/2}(V^zV^z)_F\,, \tag{11f}$$

where the states have been defined so that the relative phases are positive. The relative phase of the two (VV) terms of $\psi(V_8^z)$ is given in conventional notation by $V^xV^x+V^yV^y-V^zV^z$, i.e., one may write

$$\psi(V_8^z)=(\tfrac{2}{3})^{1/2}(VV)_2+(\tfrac{1}{12})^{1/2}(VV)_0+(\tfrac{1}{4})^{1/2}(PP)\,, \tag{12}$$

where the subscript denotes the total spin angular momentum.

Since the force in the representation **280⊕280*** vanishes, the $\cos\theta$ terms of the general P-wave amplitudes are proportional to the expression

$$\sum_{i=1}^{35}\psi(i)\psi'(i)\,, \tag{13}$$

where the prime denotes the final state, and the sum is over the 35 states of Eqs. (11a)–(11f).

Since we are interested only in comparing different amplitudes, we will normalize in a manner that is convenient. The crossing matrix coefficient for scattering in the state **35** of the internal symmetry model is unity, if both t and u channel forces are included.[10] Hence, we normalize the elastic $SU(6)_W$ amplitude in the state **35** to unity. The $SU(6)_W$ amplitude is the coefficient of $p^zp'^z/\mathbf{p}^2$ in the general P-wave amplitude. The trace of any P-wave amplitude depends only on this $\cos\theta$ term. We normalize the P-wave amplitudes so that the trace of any term (with each state multiplied by the weight $2J+1$) is equal to three times the trace of the spin part of the coefficient of $p^zp'^z/\mathbf{p}^2$ (with each state multiplied by the spin weight). These dimensionless P-wave amplitudes and their eigenvalues are denoted by U; they are essentially crossing matrix elements multiplied by three. The factor of three is included so that U and the W-spin amplitude are equal for elastic PP scattering.

The octet and singlet states may be characterized by the charge conjugation parity of the $I_z=Y=0$ members. Transitions between states of opposite C parity are forbidden.

IV. THE CALCULATIONS

A. States of the Representations 27, 10, and 10*

The $SU(6)_W$ representation **35** contains no states of the $SU(3)$ representations **27**, **10**, and **10***. It follows from the argument of Sec. III that the $\cos\theta$ terms of all Born amplitudes corresponding to these $SU(3)$ representations vanish. It may be shown that in fact, all terms of these amplitudes vanish. The techniques used in the rest of this section may be used to demonstrate this fact, but we omit the demonstration.

B. The 1^+ States

We now turn to the $SU(3)$ singlet states of positive C parity. There are no singlet P-wave PP states, so these states are all of the type VV. The $SU(3)$ singlet state is symmetric in the direct products **8⊗8** and **1⊗1**; hence the spin state must be the antisymmetric spin-one state. We denote the unit VV spin vector by $\mathbf{S}$, i.e., $\mathbf{S}=(2)^{-1/2}(\mathbf{V}_a\times\mathbf{V}_b)$, where a and b label the two mesons. One may use Eqs. (2) and (7) to compute the forms of the V and P exchange contributions to the dimensionless amplitudes U for the $VV\to VV$ process. That part of the V exchange contribution that refers to unit VV spin is proportional to the corresponding part of the P exchange contribution. The form is,

$$U\sim\tfrac{1}{2}[(\mathbf{S}\cdot\mathbf{k})(\mathbf{S}'\cdot\mathbf{k}')+(\mathbf{S}\cdot\mathbf{k}')(\mathbf{S}'\cdot\mathbf{k})]\,, \tag{14}$$

where the primes denote final-state variables, and $\mathbf{k}$ and $\mathbf{k}'$ are unit three-vectors in the direction of the momenta in the center-of-mass system.

We now make use of the $SU(6)_W$ equations of Sec. III by considering the coefficient of the $k^zk'^z$ term of U; this coefficient is proportional to the expression,

$$S^zS'^z=\tfrac{1}{2}(\mathbf{V}_a\times\mathbf{V}_b)^z(\mathbf{V}_a'\times\mathbf{V}_b')^z. \tag{15}$$

The total VV spin is one in all the D-type VV states of Eqs. (11a) and (11d). The expression of Eq. (15) corresponds to the $\psi(P_1)\psi'(P_1)$ contribution of Eqs. (13) and (11a). The relation of the V_8V_8 and V_1V_1 contributions to Eq. (15) must be the same as in Eq. (11a), i.e., the coupling is of the nonet D-type. The normalization convention of Sec. III, together with the unit coefficient of the $V^{x,y}V^{y,x}$ term of Eq. (11a), implies the trace condition $U_1+3U_3+5U_5=3$ on the eigenvalues of the general P-wave amplitudes. The subscript is the multiplicity $(2J+1)$.

Finally, one applies a partial-wave analysis to the general P-wave amplitude, Eq. (14), and uses the above trace condition. The results are

$$U_1=2\,,\quad U_3=-\tfrac{1}{2}\,,\quad U_5=\tfrac{1}{2}. \tag{16}$$

The most positive eigenvalue corresponds to $J=0$, and may be identified with the P_1 meson.

C. The 1^- States

The only $SU(3)$ singlet states of negative C parity are of the type (VP). There are three types of contributions to the $VP\to VP$ amplitudes, denoted here by $U(V_F)$, $U(P_F)$, and $U(V_D)$. The first represents F-type V exchange associated with the crossed process $VV\to V\to PP$, and the second and third represent F-type P exchange and D-type V exchange processes associated with the crossed processes $VP\to P$ (or $V)\to VP$. If $\mathbf{V}$ denotes the V spin vector, these contributions are proportional to the expressions

$$U(V_F)\sim-(\mathbf{V}\cdot\mathbf{V}')(\mathbf{k}\cdot\mathbf{k}'),\tag{17a}$$

$$U(P_F)\sim\tfrac{1}{2}[(\mathbf{V}\cdot\mathbf{k})(\mathbf{V}'\cdot\mathbf{k}')+(\mathbf{V}\cdot\mathbf{k}')(\mathbf{V}'\cdot\mathbf{k})],\tag{17b}$$

$$U(V_D)\sim(\mathbf{V}\cdot\mathbf{V}')(\mathbf{k}\cdot\mathbf{k}')-\tfrac{1}{2}[(\mathbf{V}\cdot\mathbf{k})(\mathbf{V}'\cdot\mathbf{k}')+(\mathbf{V}\cdot\mathbf{k}')(\mathbf{V}'\cdot\mathbf{k})].\tag{17c}$$

It can be shown that the contributions of the $U(V_F)$ and $U(P_F)$ terms to representations contained symmetrically in the direct product $(8\otimes8)$ are additive, so that the sum is proportional to $U(V_D)$. The $U(V_D)$ term is the only contribution to the elastic and inelastic amplitudes involving the $(1\otimes1)$ state. Thus, all contributions to the singlet (VP) states are of the form of Eq. (17c).

In the forward direction, the contribution is of the form $V^xV'^x+V^yV'^y$. This corresponds to the $\psi(V_1)\psi'(V_1)$ terms of Eqs. (11b) and (13). If we consider the full amplitude at all angles, the V spin and orbital angular momentum may couple to form states of $J=0$, 1, and 2. A partial wave analysis of Eq. (17c), normalized with the convention of Sec. III, leads to the results

$$U_1=-1,\quad U_3=\tfrac{3}{2},\quad U_5=\tfrac{1}{2}.\tag{18}$$

The triplet state may be identified with the V_1 particle.

D. The 8^+ States

The octet MM states, whose $I_z=Y=0$ members are of even C parity, are of several types. The V_8V_8 and V_8V_1 states that are symmetric in internal indices and are of total internal spin one, and antisymmetric V_8P_8 states contribute to the scattering. Actually, P_8P_1 states and V_8V_1 states of total internal spins 0 and 2 are also of the correct C parity, but it may be shown that all elastic and inelastic amplitudes involving these latter states vanish.

The Born amplitudes for the VV states are of the form of Eq. (14). The ratios of the Clebsch-Gordan coefficients for these states to those of the 1^+ states treated in Sec. B are equal to the corresponding ratios in the forward direction. These may be determined from Eqs. (11a) and (11d).

We now consider the $VP\to VP$ elastic amplitudes. In the antisymmetric octet state, the $U(V_F)$ and $U(P_F)$ terms of Eqs. (17a) and (17b) are subtractive, rather than additive. It may be shown that these terms combine with the $U(V_D)$ term in such a manner that the total result is proportional to the expression

$$\tfrac{1}{2}[(\mathbf{V}\cdot\mathbf{k})(\mathbf{V}'\cdot\mathbf{k}')+(\mathbf{V}\cdot\mathbf{k}')(\mathbf{V}'\cdot\mathbf{k})].\tag{19}$$

In the forward direction, only the $V^zV'^z$ term contributes. This is consistent with the $SU(6)_W$ expressions of Eqs. (11d) and (13).

It may be shown that the contributions to the inelastic amplitudes $VV\to VP$ corresponding to unit VV spin $(\mathbf{S})$ that result from V and P exchange are of the respective forms

$$U(V)\sim(\mathbf{V}'\cdot\mathbf{S})(\mathbf{k}\cdot\mathbf{k}')+\tfrac{1}{2}[(\mathbf{V}'\cdot\mathbf{k})(\mathbf{S}\cdot\mathbf{k}')+(\mathbf{V}'\cdot\mathbf{k}')(\mathbf{S}\cdot\mathbf{k})],$$

$$U(P)\sim-(\mathbf{V}'\cdot\mathbf{S})(\mathbf{k}\cdot\mathbf{k}')+\tfrac{1}{2}[(\mathbf{V}'\cdot\mathbf{k})(\mathbf{S}\cdot\mathbf{k}')+(\mathbf{V}'\cdot\mathbf{k}')(\mathbf{S}\cdot\mathbf{k})].$$

The coefficients are such that these expressions are additive, so that the total $VV\to VP$ amplitude is proportional to the expression

$$\tfrac{1}{2}[(\mathbf{V}'\cdot\mathbf{k})(\mathbf{S}\cdot\mathbf{k}')+(\mathbf{V}'\cdot\mathbf{k}')(\mathbf{S}\cdot\mathbf{k})].\tag{20}$$

It is seen from Eqs. (14), (19), and (20) that all the elastic and inelastic amplitudes involving the VV and VP states are of the same form, when expressed in terms of the total (unit) intrinsic spin of the mesons. Hence, the relative contributions to states of total $J=0$, 1, and 2 for each of these amplitudes is given by Eq. (16). The amplitude matrix U corresponding to $J=0$ is twice the matrix that corresponds to the $\cos\theta$ terms. This may be determined from Eq. (11d). The result is

$$\begin{array}{c} \\ (VV)_D \\ (VP)_F \end{array}\begin{array}{c} (VV)_D \quad (VP)_F \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{array}.\tag{21}$$

The $(VV)_D$ states involve both V_8V_8 and V_8V_1 states, with the relative coupling prescribed by the nonet D-coupling rules of Sec. IIB. The eigenvalues are 2 and 0. The state of eigenvalue 2 may be identified with the P meson octet.

The corresponding matrices for total angular momenta 1 and 2 may be obtained by multiplying Eq. (21) by $(-\frac{1}{4})$ and $(\frac{1}{4})$, respectively.

E. The 8^- States

The most complicated case is that of the octet states whose $I_z=Y=0$ members are of odd C parity. Hence, we present only a brief outline of the calculation in this case.

The MM states that may contribute are antisymmetric (F-type) V_8V_8 states of total internal spins 0 and 2, symmetric V_8P_8, V_1P_8, and V_8P_1 states, and antisymmetric P_8P_8 states. It may be shown that the elastic $PP \to PP$ amplitudes are of the simple form, $\mathbf{k}\cdot\mathbf{k}'$, the elastic $VP \to VP$ amplitudes are of the form of Eq. (17c), and the $PP \to VP$ amplitudes vanish. The remaining amplitudes are the elastic and inelastic amplitudes involving VV states. The forms of these amplitudes are listed below. The VV states are not analyzed into spin 2 and spin 0 components in these equations. The symbols $\mathbf{A}$ and $\mathbf{B}$ denote the spin vectors of the two V mesons in the VV states, while $\mathbf{V}$ is the spin vector for VP states. The amplitudes are

$$
\begin{aligned}
&U_{VV\to VV} \sim U(V)+U(P)\,,\\
&\quad U(V)=2\mathcal{P}_{AB}[(\mathbf{A}\cdot\mathbf{A}')(\mathbf{B}\cdot\mathbf{B}')(\mathbf{k}\cdot\mathbf{k}')]\\
&\qquad\quad -2\mathcal{P}_{kk'}[(\mathbf{A}\cdot\mathbf{B})(\mathbf{A}'\cdot\mathbf{k})(\mathbf{B}'\cdot\mathbf{k}')\\
&\qquad\quad +(\mathbf{A}'\cdot\mathbf{B}')(\mathbf{A}\cdot\mathbf{k})(\mathbf{B}\cdot\mathbf{k}')]\\
&\qquad\quad +\mathcal{P}_{AB,A'B',kk'}[(\mathbf{A}\cdot\mathbf{A}')(\mathbf{B}\cdot\mathbf{k})(\mathbf{B}'\cdot\mathbf{k}')]\,,\\
&\quad U(P)=\mathcal{P}_{kk',AB}[\mathbf{k}\cdot(\mathbf{A}\times\mathbf{A}')\mathbf{k}'\cdot(\mathbf{B}\times\mathbf{B}')]\,,\\
&U_{VV\to VP} \sim \mathcal{P}_{kk',AB}[(\mathbf{k}\cdot\mathbf{A})(\mathbf{k}'\cdot\mathbf{V}'\times\mathbf{B})]\,,\\
&U_{VV\to PP} \sim (\mathbf{A}\cdot\mathbf{B})(\mathbf{k}\cdot\mathbf{k}')-\mathcal{P}_{kk'}[(\mathbf{A}\cdot\mathbf{k})(\mathbf{B}\cdot\mathbf{k}')]\,,
\end{aligned}
\tag{22}
$$

where the symbol $\mathcal{P}_{ii',jj',\cdots}f(i,i',\cdots)$ is shorthand for $(1+X_{ii'})(1+X_{jj'})\cdots f$, and the operator $X_{ii'}$ exchanges the variables i and i' in the function f. The quantities $U(V)$ and $U(P)$ are the V and P exchange contributions to the VV elastic amplitude; the two types of exchange lead to proportional contributions to the $VV \to VP$ amplitude. The contributions of V and P exchange to the $VV \to PP$ amplitude are proportional, respectively, to

$$2(\mathbf{A}\cdot\mathbf{B})(\mathbf{k}\cdot\mathbf{k}')-\mathcal{P}_{kk'}[(\mathbf{A}\cdot\mathbf{k})(\mathbf{B}\cdot\mathbf{k}')]$$

and

$$\{-\mathcal{P}_{kk'}[(\mathbf{A}\cdot\mathbf{k})(\mathbf{B}\cdot\mathbf{k}')]\}\,.$$

One can determine the coefficients of the various terms by comparing the coefficients of $k^zk'^z$ with the amplitudes determined from Eqs. (11e), (11f), and (13). The results are listed below for the different values of J.

$$J=3,\quad U(VV)_2=\tfrac{1}{2}\,, \tag{23a}$$

$$J=0,\quad U(VP)=-\tfrac{1}{2}\,, \tag{23b}$$

$$
J=2,\qquad
U=\begin{array}{c} (VV)_2 \\ (VP) \end{array}
\overset{\begin{array}{cc} (VV)_2 & (VP) \end{array}}{\begin{pmatrix} -\tfrac{1}{4} & (3/16)^{1/2} \\ & \tfrac{1}{4} \end{pmatrix}}
\tag{23c}
$$

$$
J=1,\qquad
U=\begin{array}{c} (VV)_2 \\ (VV)_0 \\ (VP) \\ (PP) \end{array}
\overset{\begin{array}{cccc} (VV)_2 & (VV)_0 & (VP) & (PP) \end{array}}{\begin{bmatrix} 11/12 & (5/36)^{1/2} & (15/16)^{1/2} & (5/12)^{1/2} \\ & 1/12 & 0 & (1/48)^{1/2} \\ & & 3/4 & 0 \\ & & & \tfrac{1}{4} \end{bmatrix}}
\tag{23d}
$$

TABLE I. Quantum numbers of multiplets with nonzero eigenvalues of the amplitude U. The superscript is the charge conjugation parity of the $I_z=Y=0$ states.

Eigenvalue	States					
2	$(1,1)^+$	$(8,1)^+$	$(8,3)^-$			
$\frac{3}{2}$	$(1,3)^-$					
$\frac{1}{2}$	$(1,5)^+$	$(1,5)^-$	$(8,5)^+$	$(8,7)^-$	$(8,5)^-$	$(8,3)^-$
$-\frac{1}{2}$	$(1,3)^+$	$(8,3)^+$	$(8,5)^-$	$(8,1)^-$	$(8,3)^-$	
-1	$(1,1)^-$					

where the subscripts on the symbols (VV) denote the total spin angular momentum. The relations between the V_8P_8, V_8P_1 and V_1P_8 parts of the VP states are given by the nonet D coupling rules. The matrices are symmetric, so redundant elements are omitted. The eigenvalues corresponding to $J=2$ are $\pm\frac{1}{2}$, while those corresponding to $J=1$ are 2, $\pm\frac{1}{2}$, and 0. The eigenvalue 2 may be identified with the V octet.

V. RESULTS

The quantum numbers of the P-wave Born-approximation amplitudes corresponding to nonzero eigenvalues U are listed in Table I. The normalization of the U is defined in Sec. III. The states with eigenvalues 2 and $\frac{3}{2}$ may be identified with the 36 meson states assumed originally. The eigenvalue associated with the V_1 particle is smaller than the others; it is pointed out in Sec. IIA that a similar condition applies to the direct pole residues.

We next check the consistency of the output and input values of the ratios of constants coupling different two-particle states to the same meson. As discussed in Sec. IID, the output values are assumed proportional to the components of the two-particle states in the appropriate eigenamplitude. In the cases of the (1,1), (1,3), and (8,1) amplitudes, each off-diagonal element of the U matrix is the positive geometric mean of the related diagonal elements, provided the phases are chosen correctly. This condition implies that these matrices have only the one nonzero eigenvalue. Hence, the relative components of different MM states in the eigenamplitude are the same as the ratios of the Born-approximation amplitudes. The ratios are in accordance with $SU(6)_W$ and agree exactly with the input ratios.

On the other hand, the matrix of Eq. (23d), corresponding to the multiplet V_8, has more than one nonzero eigenvalue. (This extra complication results from the fact that both the $V^{x,y}$ type and V^z type of helicity state contribute to the $\cos\theta$ terms of the amplitudes.) One can use the matrix of Eq. (23d) to find the output eigenstate $\psi_{\rm out}(V_8)$. The result is,

$$(36)^{1/2}\psi_{\rm out}(V_8)=(20)^{1/2}(VV)_2+(VV)_0+(12)^{1/2}(VP)+(3)^{1/2}(PP)\,. \tag{24}$$

The output and input coupling constants are equal in the internal symmetry model discussed in Sec. III.

Therefore, we may determine the input ratios from the $SU(6)_W$ equations, using the fact that the probability of a component of $\psi(V_8)$ that is characterized by a particular internal symmetry and internal spin is $\frac{2}{3}$ the corresponding probability of Eq. (11e), plus $\frac{1}{3}$ the corresponding probability of Eq. (12). The result is that the input wave function is identical to the output wave function, Eq. (24).

Since the eigenvalues U corresponding to the P_1, P_8, and V_8 are equal, it is reasonable to assume that the sums of the squares of the output coupling constants associated with each of these states are equal. This is consistent with the input assumption. The output value of the ratio of the sum of the squares of the V_1 coupling constants to the sums corresponding to the other states cannot be determined without more detailed assumptions concerning the energy dependence of the bound-state amplitudes. Thus, the output results are consistent with the input assumptions.

It is interesting to note that the probability of the $\pi\pi$ state in the wave function for the ρ is only 1/18, so this model differs greatly from the simple $\rho-\pi\pi$ bootstrap model.

We now turn to the weakly attractive states that correspond to the eigenvalue $U=\frac{1}{2}$. It may be shown from Table I that the ratio $R=\frac{2}{3}\sum S_z^2/\sum I_z^2$ for these states is 14/3. It is pointed out in Sec. I that the value $R=1$ is required for a sum of $SU(6)$ multiplets. Thus a large "vertical" symmetry breaking occurs. In the present case, the force in these states is sufficiently weak that resonances may not occur. However, the results domonstrate how $SU(6)_W$-symmetric forces may lead to large vertical symmetry breaking.

VI. CONCLUSIONS

The reason that the results of this model are simpler than the method of calculation may be seen from the following considerations. The P-wave amplitude connecting each pair of MM states is a scalar formed from S, S', L, and L', where S and L are the total intrinsic spin and the orbital angular momentum, and the prime denotes a final-state variable. The usual angular momentum analysis involves coupling S and L together, and contracting with a similar combination of S' and L'. In order to understand the influence of $SU(6)_W$ symmetry, it is useful to consider an alternate coupling order, in which various combinations of L and L' are contracted with combinations of S and S'. Since only P waves are involved, the vectors $\mathbf{L}$ and $\mathbf{L}'$ may form a scalar, vector, or second rank tensor. The scalar and tensor amplitudes include $L^zL'^z$ terms, and are thus related to the $SU(6)_W$ symmetry. The structure of the intrinsic spin parts of the $L^zL'^z$ terms could be used to separate the scalar and tensor parts. On the other hand, there is no $L^zL'^z$ term in the amplitude involving the vector $\mathbf{L}\times\mathbf{L}'$. These vector amplitudes are the only amplitudes that may not be determined from $SU(6)_W$ and the internal symmetry model of Sec. III. The calculation of the one-meson exchange amplitudes at threshold is necessary only to determine the $\mathbf{L}\times\mathbf{L}'$ amplitudes. However, it is easy to demonstrate that the vertices involving virtual P and transverse V mesons are odd in the exchange of the initial and final three-momenta of the real particles, while the vertices involving the exchange of longitudinal V mesons are even in this exchange. Hence, *all* contributions to the amplitudes are symmetric in the exchange of L and L'; the $\mathbf{L}\times\mathbf{L}'$ terms vanish. This leads to simple results.

If one were to compute the meson exchange amplitudes at an energy higher than threshold, the $\mathbf{L}\times\mathbf{L}'$ terms would not all vanish. However, if one evaluated the vertices at threshold, terms arising from higher powers of $(\mathbf{p}-\mathbf{p}')^2$ in the expansion of the function $[M^2+(\mathbf{p}-\mathbf{p}')^2]^{-1}$ of the propagator would not destroy the symmetry with respect to the exchange of $\mathbf{L}$ and $\mathbf{L}'$. Thus, the static assumption used in this model need not be very severe. One may use relativistic interactions throughout; it is sufficient if the vertices are evaluated when the real particles are nearly at rest.

The main conclusion of the paper is that an $SU(6)_W$-symmetric MMM interaction can bootstrap itself. Previous references lead to a similar conclusion regarding the meson-baryon-baryon interaction.[1] If the most important force diagrams responsible for binding the various mesons and baryons are the simplest diagrams involving only the meson and baryon supermultiplets $35\oplus 1$ and 56, the interactions may bootstrap themselves and, simultaneously, lead to heavier predicted multiplets that deviate from $SU(6)$ in the vertical (large spin) fashion observed in nature. Clearly, the technique of imposing $SU(6)_W$ symmetry on the input forces should be applied to various MM, MB, BB, and $B\bar{B}$ states, in order to test the general model. This procedure does not involve arbitrary parameters, so that the predictions concerning the relative magnitudes of forces and interaction constants will be definite.

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