Equilibrium of a Large Assembly of Particles in General Relativity

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A new approach to the problem of the equilibrium of a relativistic configuration is proposed, taking into account quantum-gravitational effects. This approach is based upon the quantization of the motion of the particles in the gravitational field. The case of an assembly of spinless particles is investigated by using the formalism of second quantization and starting from a Klein-Gordon equation which includes in the operator $\nabla_i\nabla^i$ the gravitational field. The mean values $\langle T_{ik}\rangle$ of the operator T_{ik} are computed. It is then possible to derive from the Einstein equations a system of differential equations which represents the equilibrium state for an assembly of a large number of particles. The corresponding problem for Fermi particles can be investigated in a similar way.

THE problem of terminal stellar evolution is equivalent to that of the search for a state of mechanical equilibrium (when it exists) of a large cold assembly of particles. An extensive review of this subject was recently done by Wheeler and co-workers.[1] The study of this problem requires both the general theory of relativity and quantum mechanics. Usually, the general theory of relativity is introduced to describe the macroscopic mechanical behavior of the system and the quantum theory comes into play in determining the equation of state of matter (for instance, the Pauli principle determines the properties of a superdense neutron gas).

It is our aim to present in this paper an approach to this problem based upon the quantization of motion of the particles in the gravitational field. Even if such an approach seems premature because of the present lack of knowledge about the relation between quantum theory and gravitation, a thorough treatment of it may well prove important or even essential to a satisfactory understanding of the final stellar evolution.

We shall treat the problem in a semiclassical manner because we neglect the effects of the quantized gravitational field. For the sake of simplicity, we shall consider the case of an assembly of spinless particles. However, we wish to point out that the formalism will be such that it can also be applied to the case of particles with spin.

Let us take the Lagrangian for a spinless particle:

$$L=-\frac{m^2c^3}{2\hbar}(g^{ik}\partial_i\psi\partial_k\psi+\mu^2\psi^2), \tag{1}$$

where m is the mass of the particle, $\mu=mc/\hbar$.

The corresponding equation for the ψ field is the relativistic Klein-Gordon equation[2]

$$\nabla_i\nabla^i\psi-\mu^2\psi=0. \tag{2}$$

Here the effect of the gravitational field is included in the operator $\nabla_i\nabla^i$.[2]

Using the formalism of second quantization, we shall interpret ψ as an operator. In a similar way, the tensor

$$T_{ik}=(m^2c^3/\hbar)[\partial_i\psi\partial_k\psi-\tfrac{1}{2}(g^{em}\partial_e\psi\partial_m\psi+\mu^2\psi)g_{ik}] \tag{3}$$

is the operator corresponding to the energy-momentum density.

We can write the Einstein equations

$$M_{ik}\equiv R_{ik}-\tfrac{1}{2}g_{ik}R=-\kappa\langle T_{ik}\rangle, \tag{4}$$

where $\langle T_{ik}\rangle$ is the mean value, in the quantum-mechanical sense, of the operator T_{ik}.

The line element corresponding to a time-independent spherical symmetry is given by

$$ds^2=-B(r)(dx^4)^2+A(r)dr^2+r^2(\sin^2\theta d\varphi^2+d\theta^2). \tag{5}$$

The Klein-Gordon equation becomes

$$\frac{1}{\sqrt{(-g)}}\partial_\alpha(g^{\alpha\beta}\sqrt{(-g)}\partial_\beta\psi)-B^{-1}\partial_4{}^2\psi-\mu^2\psi=0. \tag{6}$$

Here $\alpha=1, 2, 3$, $g=\|g_{ik}\|$.

We can separate the variables as follows:

$$\psi=\eta(x^4)\varphi(x^\alpha) \tag{7}$$

and we obtain from Eq. (6)

$$\frac{1}{\eta}\frac{d^2\eta}{d(x^4)^2}=-E^2,\quad \frac{1}{\sqrt{(-g)}}\partial_\alpha(g^{\alpha\beta}(-g)^{\frac{1}{2}}\partial_\beta\varphi)+(E^2B^{-1}-\mu^2)\varphi=0. \tag{8}$$

Now we put

$$\psi(x^i)=Y(\theta,\varphi)R(r)e^{ix^4E}+Y^*(\theta,\varphi)R^*(r)e^{-ix^4E}. \tag{9}$$

[1] *Gravitational Theory and Gravitational Collapse*, B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, (University of Chicago Press, Chicago, 1965).

[2] L. Bel, Rend. C. I. M. E. salice d'Ulzio, 1963 (unpublished).

This expression guarantees that the operator ψ describing a chargeless field is Hermitian. Y must be a solution of the orbital Legendre equation, and $R(r)$ is the solution of the radial equation

$$\frac{d^2R}{dr^2}+\left(\frac{2}{r}+\frac{B'}{2B}-\frac{A'}{2A}\right)\frac{dR}{dr}+\left[A\left(E^2B^{-1}-\mu^2\right)-\frac{l(l+1)}{r^2}\right]R=0. \quad (10)$$

Since we are interested in equilibrium states, we should require that R vanish at infinity and be finite for $r=0$. We consider only the bound states and, in analogy with the treatment of the H atom in quantum mechanics, we suppose that the eigenvalues E of Eq. (10) are discrete ($l=0, 1, 2, 3,\cdots$).

We can write the general solution of Eq. (6) as follows:

$$\psi=\sum_{l,m,n}\left[U_{lmn}{}^{+}Y_m{}^lR_n{}^le^{ix^4E_n}/\sqrt{2}+U_{lmn}{}^{-}Y_m{}^{*l}R_n{}^{*l}e^{-ix^4E_n}/\sqrt{2}\right). \quad (11)$$

Let us write the operator ψ as the sum of two operators

$$\psi=\psi^{+}+\psi^{-}, \quad (12)$$

where

$$\psi^{+}=\sum_{l,m,n}\left(U_{lmn}{}^{+}Y_m{}^lR_n{}^le^{ix^4E_n}/\sqrt{2}\right), \quad (13)$$

$$\psi^{-}=\sum_{l,m,n}\left(U_{lmn}{}^{-}Y_m{}^{*l}R_n{}^{*l}e^{-ix^4E_n}/\sqrt{2}\right). \quad (14)$$

$U_{lmn}{}^{+}$ and $U_{lmn}{}^{-}$ must be interpreted, respectively, as the creation and annihilation operators of a particle in the state with angular momentum $\hbar l$ (z component $\hbar m$) and energy E_{ln}.

By writing T_{ik} as a quadratic form of the creation and annihilation operators, we can compute the mean values of the operator T_{ik},

$$\langle T_{ik}\rangle=\langle\Phi|T_{ik}|\Phi\rangle, \quad (15)$$

where $\langle\Phi|$ is the state vector describing N particles in the state of lower energy. The components of the energy-momentum tensor turn out to be

$$\langle\Phi|T_4{}^4|\Phi\rangle=-\tfrac{1}{2}(m^2/c^3)N\times\left[(B^{-1}E_{01}{}^2+\mu^2)R_{01}{}^2+A^{-1}(R_{01}')^2\right],$$

$$\langle\Phi|T_{4\alpha}|\Phi\rangle=0,$$

$$\langle\Phi|T_1{}^1|\Phi\rangle=\tfrac{1}{2}(m^2c^3/\hbar)N\times\left[(R_{01}')^2A^{-1}+(B^{-1}E_{01}{}^2-\mu^2)R_{01}{}^2\right],$$

$$\langle\Phi|T_2{}^2|\Phi\rangle=\langle\Phi|T_3{}^3|\Phi\rangle=\tfrac{1}{2}(m^2/c^3)N\times\left[(B^{-1}E_{01}{}^2-\mu^2)R_{01}{}^2-A(R_{01}')^2\right],$$

where a prime denotes differention with respect to r.

The Einstein equations with Eq. (10) give the following system:

$$M_1{}^1\equiv\frac{B'}{ABr}-\frac{1}{r^2}\left(1-\frac{1}{A}\right)=\tfrac{1}{2}\epsilon\left[-A^{-1}(R_{01}')^2+(\mu^2-B^{-1}E_{01}{}^2)R_{01}{}^2\right],$$

$$M_4{}^4\equiv\frac{A'}{A^2r}+\frac{1}{r^2}\left(1-\frac{1}{A}\right)=\tfrac{1}{2}\epsilon\left[(B^{-1}E_{01}{}^2+\mu^2)R_{01}{}^2+A^{-1}(R_{01}')^2\right], \quad (16)$$

$$R_{01}''+R_{01}'\left(\frac{2}{r}+\frac{B'}{2B}-\frac{A'}{2A}\right)+\left[A^{-1}(E_{01}{}^2B^{-1}-\mu^2)\right]R=0,$$

where $\epsilon=(m^2c^3/\hbar)\kappa N$. E_{01} represents the first eigenvalue of Eq. (10) ($l=0$) with the boundary conditions

$$4\pi\mu^3\int_0^\infty A^{1/2}R_{01}{}^2r^2dr^2=1, \quad R_{01}(0)=\text{finite value},$$

$$R_{01}(\infty)\leq\frac{1}{r^{1+\gamma^2}}, \quad (17)$$

where γ^2 is an arbitrary positive quantity. The other equations

$$M_2{}^2=-\kappa T_2{}^2,$$
$$M_3{}^3=-\kappa T_3{}^3,$$

are consequences of Eq. (16) because of the Bianchi identities

$$\nabla_iM_k{}^i=0 \quad (18)$$

and of the relation

$$\nabla_iT_k{}^i=0. \quad (19)$$

Actually, if we give the value N, the solution (if it exists) of the system (16) under the aforesaid boundary conditions will represent the equilibrium state for an assembly of N particles. Of course, for astrophysical purposes, it is more interesting to consider the corresponding problem for Fermi particles. This can be done in a similar way, and we consider briefly the case of particles with spin $\frac{1}{2}$.

We start from the spinorial equation in the general theory of relativity[3]

$$\nabla_i\nabla^i\Gamma^A-\tfrac{1}{4}R\Gamma^A-\mu^2\Gamma^A=0. \quad (20)$$

It is possible to write the expressions for the energy-momentum operators of the spinor field and to compute their values. From these, we obtain a system equivalent to the system (4) valid for an assembly of particles with spin $\frac{1}{2}$.

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[3] See, for example, A. Lichnerowicz, Bull. Soc. Math. France **92**, 11 (1964).