helium/protons in local interstellar space is constant in this interval, then the number of scattering centers, in 1963, along the cosmic-ray path exceeded 800. The lower limit to the radial extent of the cosmic-ray convection and diffusion region beyond the orbit of earth, in 1963, was 3 astronomical units. These values are not appreciably different from those used by Parker.[10]

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Quantization of Spin-2 Fields*

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A massive spin-2 field has been quantized using Schwinger's action principle. Lorentz invariance and physical positive-definiteness requirements have been verified.

I. INTRODUCTION

THE problem of quantization of massive spin-2 fields as well as other higher spin fields has been studied rather extensively in the past.[1] However, the question of whether the quantization of fields with spin 2 according to the techniques of the quantum action principle will lead to results which are consistent with Lorentz invariance as well as other physical requirements has not been touched. The recent experimental evidence on the existence of spin-2 particles arouses new interest in these problems. In this paper,[2] an attempt is made to study these problems. We limit our attention to a free, massive spin-2 field only. The quantization for massless spin-2 fields will be discussed in a separate publication.

II. CANONICAL FORMALISM

It is well known that a spin-2 tensor field should be represented by a symmetric tensor $h_{\mu\nu}$. In order to construct a Lagrange function which contains the gradient of the field variables linearly, we have to introduce additional field variables which transform like a third-rank tensor. Although the introduction of a symmetric tensor ${}_\lambda\Gamma_{\mu\nu}$ is more usual in the quantized gravitational field, we find that it is more convenient here to choose an antisymmetric tensor ${}_\mu H_{\nu\lambda}$ with the following symmetry properties[3]:

$$ {}_\mu H_{\nu\lambda} = -{}_\mu H_{\lambda\nu}\,, $$
$$ {}_\mu H_{\nu\lambda} + {}_\nu H_{\lambda\mu} + {}_\lambda H_{\mu\nu} = 0\,. $$

These two alternative descriptions are equivalent and they describe the same physical system. We first concentrate our attention on the second description only. We will show in the next section that these two descriptions are indeed equivalent. The Lagrange function of a spin-2 tensor field characterized by this antisymmetric tensor is given by[2]

$$ L = \tfrac{1}{2}(h_{\mu\nu}\cdot\partial_\lambda{}^\mu H^{\lambda\nu} - {}^\mu H^{\lambda\nu}\cdot\partial_\lambda h_{\mu\nu}) + \tfrac{1}{4}({}_\mu H_{\nu\lambda}\cdot{}^\mu H^{\nu\lambda} - H_\lambda\cdot H^\lambda) - \tfrac{1}{2}m^2(h_{\mu\nu}\cdot h^{\mu\nu} - h^2)\,. \quad (1) $$

The plus and minus signs associated with the second and third terms have physical content. They are associated with the positive-definiteness requirements of this boson system. H_λ and h are shorthand notations for

$$ {}^\mu H_{\mu\lambda}\,, \quad \text{and} \quad h^\mu{}_\mu\,, $$

respectively.

The field equations follow from the principle of stationary action:

$$ \partial_\lambda{}^{(\mu}H^{\lambda\nu)} - m^2(h^{\mu\nu} - g^{\mu\nu}h) = 0\,, \quad (2) $$

$$ 2{}_\mu H_{\nu\lambda} - (g_{\mu\nu}H_\lambda - g_{\mu\lambda}H_\nu) - 2(\partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}) = 0\,. \quad (3) $$

A symmetrization for the indices μ, ν in the parenthesis is understood. It is straightforward to show

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[1] M. Fierz, Helv. Phys. Acta **12**, 3 (1939); M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939); R. J. Rivers, Nuovo Cimento **34**, 386 (1964). A complete list of classical papers can be found in the bibliography of E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave-equations* (Blackie & Son, Ltd., Glasgow, 1953).

[2] Publication of this paper was stimulated by a recent paper of D. Adler, Can. J. Phys. **44**, 289 (1966). Throughout this paper we use the following notations: $g_{\mu\nu} = (-1, 1, 1, 1)$; all Greek indices μ, ν,$\cdots$ vary from 0 to 3 and all Latin indices i, j,$\cdots$ vary from 1 to 3. Repeated indices are to be summed over. The dots between the field operators indicate that the latter are symmetrically multiplied.

[3] Both descriptions are deduced from J. Schwinger, Phys. Rev. **130**, 1253 (1963). The Lagrange function for the Γ description was given by J. Schwinger to whom I am deeply indebted.

that our field equations are equivalent to the following set of equations:

$$h=0,\quad \partial_\mu h^{\mu\nu}=0, \tag{4}$$

$$_\mu H_{\nu\lambda}=\partial_\nu h_{\mu\lambda}-\partial_\lambda h_{\mu\nu}, \tag{5}$$

and

$$\partial_\lambda{}^\mu H^{\lambda\nu}=m^2h^{\mu\nu}. \tag{6}$$

These are indeed the correct equations which are satisfied by the spin-2 field. We would like to point out here that the auxiliary equation (4) is derived directly from the action principle rather than added arbitrarily as a further restriction.

Those equations which describe the time development of the system are

$$\partial_0{}^{(k}H^{0l)}=-\partial_m{}^{(k}H^{ml)}+m^2(h^{kl}-\delta^{kl}h), \tag{7}$$

$$\partial_0{}^0H^{0k}=-\partial_m{}^kH^{m0}-\partial_m{}^0H^{mk}+2m^2\,h^{0k}, \tag{8}$$

$$\partial_0h_{kl}=\partial_lh_{k0}+{}_kH_{0l}+\tfrac{1}{2}\delta_{kl}H_0, \tag{9}$$

and

$$\partial_0h_{0l}=\partial_lh_{00}+{}_0H_{0l}+\tfrac{1}{2}H_l. \tag{10}$$

The other equations which relate the field variables at the same time are

$$\partial_m{}^0H^{m0}-m^2h_{mm}=0, \tag{11}$$

$$_0H_{kl}-(\partial_kh_{0l}-\partial_lh_{0k})=0, \tag{12}$$

$$_kH_{lm}-(\partial_lh_{km}-\partial_mh_{kl})-\tfrac{1}{2}(\delta_{kl}H_m-\delta_{km}H_l)=0, \tag{13}$$

repeated Latin indices imply a summation from 1 to 3. The generator follows also from the action principle, and is of the form

$$G=\frac{1}{2}\int(h_{\nu\lambda}.\,\delta^\lambda H^{0\nu}-{}^\lambda H^{0\nu}.\,\delta h_{\lambda\nu})d^3x,$$
$$=\frac{1}{2}\int(h_{kl}.\,\delta^{(k}H^{0l)}-{}^{(k}H^{0l)}.\,\delta h_{kl}+h_{0k}.\,\delta^0H^{0k}-{}^0H^{0k}.\,\delta h_{0k})d^3x. \tag{14}$$

Note that not all the field variables appearing in the generator are independent dynamical variables. They are restricted by some further constraint equations which are obtained by manipulating the field equations. These further constraint equations introduce intrinsic complications into the theory of spin-2 fields.

Taking $k=m$ in Eq. (13) and summing from 1 to 3, we have

$$_0H_{0l}=\partial_mh_{ml}-\partial_lh_{mm}. \tag{15}$$

Likewise, one can show that

$$m^2h^{0k}-\tfrac{1}{2}\nabla^2h^{0k}+\tfrac{1}{2}\partial_k\partial_mh^{0m}=-\partial_m{}^{(m}H^{0k)}. \tag{16}$$

Then, it follows that

$$h^{0k}=(1/m^2)D_1(\partial_k\partial_l-2m^2\delta_{kl})\partial_m{}^{(m}H^{0l)}, \tag{16a}$$

where $D_1\Phi(\mathbf{x})$ is

$$\int D_1(\mathbf{x}-\mathbf{x}')\Phi(\mathbf{x}')d^3x',$$

and $D_1(\mathbf{x}-\mathbf{x}')$ is the Green's function defined by

$$-(\nabla^2-2m^2)D_1(\mathbf{x}-\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}'). \tag{17}$$

This indicates that $_0H_{0l}$ and h^{0l} are not independent dynamical variables. A further reduction of dynamical variables is possible by substituting Eq. (15) into Eq. (11), giving

$$(\nabla^2-\tfrac{3}{2}m^2)h_{mm}-\tfrac{3}{2}(\partial\partial h^T)=0, \tag{18}$$

with

$$h_{kl}{}^T=h_{kl}-\tfrac{1}{3}\delta_{kl}h_{mm},$$
$$(\partial h^T)_k=\partial_lh_{kl}{}^T,$$
$$(\partial\partial h^T)=\partial_k\partial_lh_{kl}{}^T.$$

Then, we have

$$h_{mm}=-\tfrac{3}{2}D_2(\partial\partial h^T), \tag{19}$$

where D_2 is the Green's function defined by

$$-(\nabla^2-\tfrac{3}{2}m^2)D_2(\mathbf{x}-\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}'). \tag{20}$$

Then the generator can be expressed in terms of the dynamical variables $h_{kl}{}^T$ and their conjugate variables π_{kl}

$$G=\frac{1}{2}\int h_{kl}.\,\delta({}^kH^{0l}-\partial_kh^{0l}+\delta_{kl}\partial_mh^{0m})-({}^kH^{0l}-\partial_kh^{0l}+\delta_{kl}\partial_mh^{0m}).\,\delta h_{kl}d^3x,$$
$$=\frac{1}{2}\int(h_{kl}{}^T.\,\delta\pi_{kl}-\pi_{kl}.\,\delta h_{kl}{}^T)d^3x, \tag{21}$$

where

$$\pi_{kl}={}_{(k}H^0{}_{l)}+D_1[\partial_k\partial_{m(l}H^0{}_{m)}+\partial_l\partial_{m(k}H^0{}_{m)}]-(1/m^2)D_1\partial_k\partial_l\partial_p\partial_{q(p}H^0{}_{q)}+(1/m^2)D_2(\partial_k\partial_l-\tfrac{1}{2}\delta_{kl}m^2)\partial_p\partial_{q(p}H^0{}_{q)}. \tag{22}$$

Both $h_{kl}{}^T$ and π_{kl} are symmetric and traceless in spatial indices. They form a possible set of dynamical variables. The $h_{kl}{}^T$ and π_{kl} satisfy the following equations of motion:

$$\begin{aligned}\partial_0\pi_{kl}&=m^2h_{kl}{}^T+[\partial_k(\partial h^T)_l+\partial_l(\partial h^T)_k-\tfrac{2}{3}\delta_{kl}(\partial\partial h^T)-\nabla^2h_{kl}{}^T]-\tfrac{1}{2}D_2D_2(\partial_k\partial_l-\tfrac{1}{3}\delta_{kl}\nabla^2)\nabla^2(\partial\partial h^T),\\ \partial_0h_{kl}{}^T&=-\pi_{kl}+m^{-2}[\partial_k(\partial\pi)_l+\partial_l(\partial\pi)_k-\tfrac{2}{3}\delta_{kl}(\partial\partial\pi)]-(2/3m^4)(\partial_k\partial_l-\tfrac{1}{3}\delta_{kl}\nabla^2)(\partial\partial\pi).\end{aligned} \tag{23}$$

Quantization follows from the identification of the operator G, which is associated with boundary variations, with the infinitesimal generator of unitary transformations on a quantum-mechanical system.[4] This im-

[4] J. Schwinger, Phys. Rev. 82, 914 (1951); 91, 713 (1953).

plies the fundamental equal-time commutation relations among the dynamical variables

$$i[h_{kl}{}^T(\mathbf{x}),\pi_{pq}(\mathbf{x}')]=\delta_{kl,pq}{}^T\delta(\mathbf{x}-\mathbf{x}')\,,\quad [h_{kl}{}^T(\mathbf{x}),h_{pq}{}^T(\mathbf{x}')]=[\pi_{kl}(\mathbf{x}),\pi_{qp}(\mathbf{x}')]=0\,, \tag{24}$$

with

$$\delta_{kl,pq}{}^T=\tfrac{1}{2}(\delta_{kp}\delta_{lq}+\delta_{kq}\delta_{lp}-\tfrac{2}{3}\delta_{kl}\delta_{pq})\,.$$

In order to calculate the commutation between field variables with definite tensor transformation properties, we express these field variables in terms of the dynamical variables. With the help of Eqs. (4)–(6), field variables with definite tensor transformation properties can be expressed in terms of the dynamical variables as

$${}_{(k}H^0{}_{l)}=\pi_{kl}-(1/2m^2)[\partial_k(\partial\pi)_l+\partial_l(\partial\pi)_k]+(1/3m^2)\delta_{kl}(\partial\partial\pi)\,, \tag{25}$$

$${}^0H_{kl}=(1/m^2)[\partial_l(\partial\pi)_k-\partial_k(\partial\pi)_l]\,, \tag{26}$$

$$h_{kl}=h_{kl}{}^T-\tfrac{1}{2}\delta_{kl}D_2(\partial\partial h^T)\,, \tag{27}$$

$$h^0{}_k=-(1/m^2)(\partial\pi)_k+(2/3m^4)\partial_k(\partial\partial\pi),\ \text{etc.}, \tag{28}$$

where

$$(\partial\pi)_k=\partial_l\pi_{kl}$$
$$(\partial\partial\pi)=\partial_k\partial_l\pi_{kl}\,.$$

An equivalent set of equal-time commutation relations among those field variables is

$$\begin{aligned} i[h_{kl}(\mathbf{x}),{}_{(p}H^0{}_{q)}(\mathbf{x}')]=&\;\delta_{kl,pq}{}^T\delta(\mathbf{x}-\mathbf{x}')\\ &+(1/3m^2)\delta_{pq}\partial_k\partial_l\delta(\mathbf{x}-\mathbf{x})\\ &-(1/4m^2)(\delta_{qk}\partial_p\partial_l+\delta_{ql}\partial_p\partial_k\\ &+\delta_{pk}\partial_q\partial_l+\delta_{pl}\partial_q\partial_k)\delta(\mathbf{x}-\mathbf{x}')\,, \end{aligned} \tag{29}$$

$$[h_{kl}(\mathbf{x}),h_{qp}(\mathbf{x}')]=[{}_{(k}H^0{}_{l)}(\mathbf{x}),{}_{(p}H^0{}_{q)}(\mathbf{x}')]=0\,.$$

All other equal-time commutation relations, such as

$$i[h_{kl}(\mathbf{x}),{}^0H_{pq}(\mathbf{x}')]=(1/2m^2)(\delta_{pl}\partial_q\partial_k+\delta_{pk}\partial_q\partial_l-\delta_{ql}\partial_p\partial_k-\delta_{qk}\partial_p\partial_l)\delta(\mathbf{x}-\mathbf{x}')\,, \tag{30}$$

$$\begin{aligned} i[h_{kl}(\mathbf{x}),h^0{}_p(\mathbf{x}')]=&\;(1/2m^2)\\ &\times(\delta_{pk}\partial_l+\delta_{pl}\partial_k-\tfrac{2}{3}\delta_{kl}\partial_p)\delta(\mathbf{x}-\mathbf{x}')\\ &-(2/3m^4)\partial_k\partial_l\partial_p\delta(\mathbf{x}-\mathbf{x}')\,, \end{aligned} \tag{31}$$

follow from the equations of constraint. The commutation relations can also be expressed covariantly as

$$\begin{aligned} i[h_{\mu\nu}(x),h_{\lambda\sigma}(x')]=&\;\{\tfrac{1}{2}(g_{\mu\lambda}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\lambda}-\tfrac{2}{3}g_{\mu\nu}g_{\lambda\sigma})\\ &-(1/2m^2)(g_{\mu\lambda}\partial_\nu\partial_\sigma+g_{\mu\sigma}\partial_\nu\partial_\lambda+g_{\nu\lambda}\partial_\mu\partial_\sigma+g_{\nu\sigma}\partial_\mu\partial_\lambda)\\ &+(1/3m^2)(g_{\mu\nu}\partial_\lambda\partial_\sigma+g_{\lambda\sigma}\partial_\mu\partial_\nu)\\ &+(2/3m^4)\partial_\mu\partial_\nu\partial_\lambda\partial_\sigma\}\Delta(x-x')\,, \end{aligned} \tag{32}$$

where $\Delta(x-x')$ is the invariant function introduced by Schwinger. An analogous form for $i[h_{\mu\nu}(x),{}_\lambda H_{\sigma k}(k')]$ can be obtained easily from the field equations. We will not reproduce the calculation here.

III. THE ALTERNATIVE DESCRIPTION

The Lagrange function for the alternative description is given by

$$\begin{aligned} L=&-(h'^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}h')\\ &\times(2\partial_\lambda{}^\lambda\Gamma_{\mu\nu}-\partial_\mu\Gamma_\nu-\partial_\nu\Gamma_\mu)+2(\Gamma_\nu.{}^\nu\Gamma-{}^\lambda\Gamma_{\mu\nu}.{}^\nu\Gamma^\mu{}_\lambda)\\ &-\tfrac{1}{2}m^2(h'_{\mu\nu}.h'^{\mu\nu}-h'^2)\,, \end{aligned} \tag{33}$$

with

$${}^\lambda\Gamma_{\mu\nu}={}^\lambda\Gamma_{\nu\mu}$$
$${}^\lambda\Gamma={}^\lambda\Gamma^\alpha{}_\alpha$$
$$\Gamma^\lambda={}_\alpha\Gamma^{\lambda\alpha}\,.$$

The field equations are

$$-(\partial_\lambda{}^\lambda\Gamma_{\mu\nu}-\partial_{(\mu}\Gamma_{\nu)})+\tfrac{1}{2}g_{\mu\nu}(\partial_\lambda{}^\lambda\Gamma-\partial_\lambda\Gamma^\lambda)-\tfrac{1}{2}m^2(h'_{\mu\nu}-g_{\mu\nu}h')=0\,, \tag{34}$$

and

$${}^\nu\Gamma^\mu{}_\lambda+{}^\mu\Gamma^\nu{}_\lambda=\partial_\lambda(h'^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}h')-\delta^{(\mu}{}_\lambda\partial_\sigma(h'^{\nu)\sigma}-\tfrac{1}{2}g^{\nu)\sigma}h')+\delta^{(\mu}{}_\lambda{}^{\nu)}\Gamma+g^{\mu\nu}\Gamma_\lambda \tag{35}$$

which are equivalent to

$$h'=0\,,\quad \partial_\mu h'^{\mu\nu}=0\,, \tag{36}$$

$${}_\lambda\Gamma_{\mu\nu}=\tfrac{1}{2}(\partial_\mu h'_{\lambda\nu}+\partial_\nu h'_{\mu\lambda}-\partial_\lambda h'_{\mu\nu})\,, \tag{37}$$

$$\partial_\mu{}^\lambda\Gamma^{\mu\nu}=\tfrac{1}{2}m^2h'^{\lambda\nu}\,. \tag{38}$$

The ${}_\lambda\Gamma_{\mu\nu}$ are anologous to the Christoffel symbols of the gravitational field. The generator can be expressed as

$$\begin{aligned} G=&-\int[(h'^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}h').\,\delta({}^0\Gamma_{\mu\nu}-\delta^0{}_{(\mu}\Gamma_{\nu)})\\ &-({}^0\Gamma_{\mu\nu}-\delta^0{}_{(\mu}\Gamma_{\nu)}.\,\delta(h'^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}h')]d^3x\,,\\ =&\frac{1}{2}\int[h'_{kl}{}^T.\,\delta\pi_{kl}'-\pi_{kl}'.\,\delta h'_{kl}{}^T]d^3x\,, \end{aligned} \tag{39}$$

with

$$h'_{kl}{}^T=h'_{kl}-\tfrac{1}{3}\delta_{kl}h'_{mm}\,,\quad \pi'_{kl}=-2{}^0\Gamma_{kl}-2D_2(\partial_k\partial_l-\tfrac{1}{2}\delta_{kl}m^2){}^0\Gamma_{mm}\,. \tag{40}$$

It is straightforward to show that $h'_{kl}{}^T$ and π'_{kl} satisfy the same equations of motion and the same commutation relations as $h_{kl}{}^T$ and π_{kl}. Therefore, they must describe the same physical system. The equal-time commutation relations among these tensor fields can be transcribed easily from the other descritpion. We will not reproduce the results here.

IV. THE STRESS TENSOR

We now use the action principle to find the stress tensor from which the ten generators of the Lorentz transformation can be constructed. The stress tensor $T^{\mu\nu}$ is defined by[4]

$$\delta W=\int(d^4x)\partial_\mu\delta x_\nu T^{\mu\nu}\,, \tag{41}$$

as a measure of the response of the system to the space-time displacement

$$x^\mu \to x^\mu + \delta x^\mu.$$

Making use of

$$W=\int d^4x\, L\,,$$

$$\partial_\mu \to \partial_\mu - (\partial_\mu \delta x^\nu)\partial_\nu\,,$$

$$d^4x \to (d^4x)(1+\partial_\mu \delta x^\mu)\,,$$

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\lambda(h_{\mu\nu})\delta x^\lambda + \tfrac{1}{2}(\partial_\lambda \delta x_\mu - \partial_\mu \delta x_\lambda)h^\lambda{}_\nu + \tfrac{1}{2}(\partial_\lambda \delta x_\nu - \partial_\nu \delta x_\lambda)h_\mu{}^\lambda \cdots\,,$$

we obtain, for the ${}^\mu H^{\lambda\nu}$ description, that

$$T_{\mu\nu}{}^{(H)} = T_{\nu\mu}{}^{(H)} = -\tfrac{1}{2}g_{\mu\nu}\partial_\lambda({}^\alpha H^{\lambda\beta}.\,h_{\alpha\beta}) + {}_\lambda H_{(\mu\sigma}.\,\partial_{\nu)}h^{\lambda\sigma} + \partial_\lambda[{}_{(\mu}H_{\nu)\sigma}.\,h^{\lambda\sigma} - {}^\lambda H_{(\mu\sigma}.\,h_{\nu)}{}^\sigma - {}^\sigma H_{(\mu}{}^\lambda.\,h_{\nu)\sigma}]\,, \tag{42}$$

and, for the ${}^\mu\Gamma^{\lambda\nu}$ description, that

$$T_{\mu\nu}{}^{(\Gamma)} = T_{\nu\mu}{}^{(\Gamma)} = g_{\mu\nu}\partial_\lambda({}^\lambda\Gamma^{\alpha\beta}.\,h^{\alpha\beta}) + 2(2{}_\mu\Gamma_{\lambda\sigma}.\,{}_\nu\Gamma^{\lambda\sigma} + m^2 h_{\lambda\mu}.\,h^\lambda{}_\nu - h^{\lambda\sigma}.\,\partial_\lambda\partial_\sigma h_{\mu\nu})\,. \tag{43}$$

It is easy to verify that both stress tensors are conserved and are related through

$$T^{\mu\nu(H)} = T^{\mu\nu(\Gamma)} + \partial_\lambda\partial_\sigma \Lambda^{\mu\nu,\lambda\sigma}\,, \tag{44}$$

with

$$\Lambda^{\mu\nu,\lambda\sigma} = \Lambda^{\lambda\sigma,\mu\nu} = -\Lambda^{\mu\sigma,\lambda\nu} = -\Lambda^{\lambda\nu,\mu\sigma} = \tfrac{1}{2}(-g^{\mu\nu}h^{\lambda\beta}.\,h^\sigma{}_\beta + g^{\mu\sigma}h^{\nu\beta}.\,h^\lambda{}_\beta - g^{\lambda\sigma}h^{\mu\beta}.\,h^\nu{}_\beta + g^{\lambda\nu}h^{\mu\beta}.\,h^\sigma{}_\beta + 2h^{\mu\nu}.\,h^{\lambda\sigma} - 2h^{\mu\sigma}.\,h^{\nu\lambda})\,. \tag{45}$$

These imply that the energy-momentum and angular-momentum operators which are defined through

$$P^\mu = \int T^{\mu 0} d^3x\,, \tag{46}$$

$$J^{\mu\nu} = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu})d^3x\,, \tag{47}$$

are the same for both systems, as are constants of motion. These operators can be expressed in terms of dynamical variables as

$$P^\mu = \frac{1}{2}\int (\pi_{lm}.\,\partial^\mu h_{lm}{}^T - h_{lm}{}^T.\,\partial^\mu \pi_{lm})d^3x\,, \tag{48}$$

$$J_{kl} = \int [\pi_{mp}(x_k\partial_l - x_l\partial_k).\,h_{mp}{}^T + 2(\pi_{km}.\,h_{lm}{}^T - \pi_{lm}.\,h_{km}{}^T)d^3x\,, \tag{49}$$

$$J^0{}_k = \frac{1}{2}\int [\pi_{pm}.\,(x^0\partial_k - x_k\partial^0)h_{mp}{}^T - h_{mp}{}^T.\,(x^0\partial_k - x_k\partial^0)\pi_{pm} + 2\pi^0{}_m.\,h_{km}{}^T - 2\pi_{km}.\,h^0{}_m]d^3x\,, \tag{50}$$

with

$$\pi^0{}_m = (\partial h^T)_m - \tfrac{3}{2}D_2D_2(\nabla^2 - 2m^2)\partial_m(\partial\partial h^T)\,.$$

We will next verify that the integral quantities P_μ and $J_{\mu\nu}$ behave correctly in the sense that they are generators of Lorentz transformations. With the help of the fundamental field commutation relations, it is straightforward, although tedious, to verify that

$$i[h_{kl}(x),P^\mu] = \partial^\mu h_{kl}(x)\,,$$

$$i[{}_kH^0{}_l(x),P^\mu] = \partial^\mu {}_kH^0{}_l(x)\,,$$

$$i[h_{kl}(x),J_{\mu\nu}] = (x_\mu\partial_\nu - x_\nu\partial_\mu)h_{kl}(x) + (g_{k\mu}h_{\nu l} - g_{k\nu}h_{\mu l} + g_{l\mu}h_{k\nu} - g_{l\nu}h_{k\mu})\,,$$

$$i[{}_kH^0{}_l(x),J_{\mu\nu}] = (x_\mu\partial_\nu - x_\nu\partial_\mu)\,{}_kH^0{}_l(x) + (g_{k\mu}\,{}_\nu H^0{}_l - g_{\nu k}\,{}_\mu H^0{}_l + g_{\mu l}\,{}_kH^0{}_\nu - g_{l\nu}\,{}_kH^0{}_\mu + \delta^0{}_\mu\,{}_kH_{\nu l} - \delta^0{}_\nu\,{}_kH_{\mu l})\,,$$

which are the correct relations to be satisfied by P_μ and $J_{\mu\nu}$ in accordance with their interpretation as infinitesimal generators of the Lorentz group. Making use of these results, it is rather straightforward to verify that the generators P_μ and $J_{\mu\nu}$ do satisfy the usual commutation relations with each other as well as with the field variables $h_{\alpha\beta}$ and ${}_\lambda H_{\alpha\beta}$

$$i[P_\mu,P_\nu] = 0\,,$$

$$i[P_\mu,J_{\lambda\nu}] = g_{\lambda\mu}P_\nu - g_{\mu\nu}P_\lambda\,,$$

$$i[J_{\mu\nu},J_{\lambda\sigma}] = g_{\mu\lambda}J_{\sigma\nu} - g_{\mu\sigma}J_{\lambda\nu} + g_{\nu\lambda}J_{\mu\sigma} - g_{\nu\sigma}J_{\mu\lambda}\,.$$

These observations show that our quantum-mechanical system is consistent with the requirement of Lorentz invariance.

We now examine some positive-definiteness requirements. The requirement that the energy must be positive is indeed satisfied, which can be verified by direct computation

$$P^0 = \int \{\tfrac{1}{2}(\pi_{kl})^2 + (1/m^2)[(\partial\pi)_k]^2 + (1/3m^4)(\partial\partial\pi)^2 + \tfrac{1}{2}m^2(q_{kl})^2 + \tfrac{1}{2}(\partial_m q_{kl})^2 + (1/8m^2)[\partial_k(\partial q)_l - \partial_l(\partial q)_k]^2\}d^3x > 0\,,$$

where

$$q_{kl} = h_{kl}{}^T + D_1[\partial_k(\partial h^T)_l + \partial_l(\partial h^T)_k] - (1/m^2)D_1\partial_k\partial_l(\partial\partial h^T) + (1/m^2)D_2(\partial_k\partial_l - \tfrac{1}{2}\delta_{kl}m^2)(\partial\partial h^T)\,.$$

However, the energy density itself is not positive-definite.

The second requirement is that the vacuum expectation value of the commutator $[i\partial_0 A,A]$, for an arbitrary A, must be positive-definite.[5] We can verify this relation for those A which are linear in the dynamical variables. Assume that

$$A = \int f_{kl}(x)h_{kl}{}^T(x)d^3x\,,$$

[5] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959); Phys. Rev. **130**, 800 (1963).

where $f_{kl}(x)$ is a numerical function and may be chosen to be symmetric and traceless. Then

$$[i\partial_0 A, A] = \int [f_{kl}(x)]^2 + (2/m^2)[(\partial f)_k]^2 + (2/3m^4)(\partial\partial f)^2 d^3x > 0$$

which is an operator equation. A similar result can be obtained for those A which are linear in π_{kl}, but the expression is much more complicated.

V. THE GREEN'S FUNCTION

The Green's function of a free spin-2 tensor field can be introduced easily with the aid of external sources. It is defined by[6]

$$G_{\mu\nu,\lambda\sigma}(x,x') = \frac{\delta\langle h_{\mu\nu}(x)\rangle^j}{\delta j^{\lambda\sigma}(x')}\bigg|_{j=0}, \tag{51}$$

as a measure of the response of the system to the variation of the external source. It is easy to prove that it satisfies the following differential equation:

$$\begin{aligned}(\partial^2 - m^2)G_{\mu\nu,\lambda\sigma}(x,x') = \{ & -\tfrac{1}{2}(g_{\mu\lambda}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\lambda} - \tfrac{2}{3}g_{\mu\nu}g_{\lambda\sigma}) \\ & + (1/2m^2)(g_{\nu\sigma}\partial_\mu\partial_\lambda + g_{\nu\lambda}\partial_\mu\partial_\sigma + g_{\mu\sigma}\partial_\nu\partial_\lambda + g_{\mu\lambda}\partial_\nu\partial_\sigma) \\ & - (1/3m^2)(g_{\mu\nu}\partial_\lambda\partial_\sigma + g_{\lambda\sigma}\partial_\mu\partial_\nu) \\ & - (2/3m^4)\partial_\mu\partial_\nu\partial_\lambda\partial_\sigma\}\delta(x-x').\end{aligned} \tag{52}$$

[6] J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452 (1951).

Note that both the covariant commutator $i[h_{\mu\nu}, h_{\lambda\sigma}]$ and the Green's function $G_{\mu\nu,\lambda\sigma}(x,x')$ have the same differential structure. This serves as another test of our quantization procedures. Under appropriate boundary condition, the Fourier transform of the Green's function can be represented by

$$\begin{aligned}(2\pi)^{-4}(p^2 + m^2 - i\epsilon)^{-1}\{ & \tfrac{1}{2}(g_{\mu\lambda}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\lambda} - \tfrac{2}{3}g_{\mu\nu}g_{\lambda\sigma}) \\ & + (1/2m^2)(g_{\sigma\nu}p_\mu p_\lambda + g_{\nu\lambda}p_\mu p_\sigma + g_{\sigma\mu}p_\lambda p_\nu + g_{\mu\lambda}p_\nu p_\sigma) \\ & - (1/3m^2)(g_{\mu\nu}p_\lambda p_\sigma + g_{\lambda\sigma}p_\mu p_\nu) + (2/3m^4)p_\mu p_\nu p_\lambda p_\sigma\},\end{aligned}$$

which is sometimes called the propagation function. It is easy to verify that the expression in the curly brackets is just equal to the sum of possible polarization tensors

$$\sum_{a=1}^{5} \epsilon_{\mu\nu}{}^{(a)}\epsilon_{\lambda\sigma}{}^{(a)},$$

where $\epsilon_{\mu\nu}{}^{(a)}$ are traceless and symmetric, and satisfy

$$p^\mu \epsilon_{\mu\nu}{}^{(a)} = 0,$$
$$\epsilon_{\mu\nu}{}^{(a)}\epsilon_{\mu\nu}{}^{(b)} = \delta_{ab}, \quad a, b = 1, 2, \cdots, 5.$$

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