PHYSICAL REVIEW VOLUME 148, NUMBER 3 19 AUGUST 1966

Proton-Proton Scattering: Revision and Analysis of Experimental Measurements from 1.4 to 3.0 MeV*

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(Received 8 March 1966)

Revised experimental values are given for the differential cross section for proton-proton scattering at energies of 1.397, 1.855, 2.425, and 3.037 MeV. These result from the consolidation and re-evaluation of all published and unpublished data obtained in an experiment reported earlier by Knecht *et al.* Phase-shift analyses have been made. The preferred solutions determine the S-wave phase shift K_0 and the central-force ("effective") part γ of the P-wave phase shifts, under the assumption that the noncentral-force parts and all higher angular-momentum phase shifts are known with sufficient accuracy from one-pion-exchange theory; the values of γ derived in this way are small, with zero not excluded. Other solutions in which noncentral forces range from zero to very large values are given for comparison. The uncertainty in each of the derived phase shifts is calculated and discussed in detail; uncertainties obtained by the usual statistical method are shown to be too small. The polarization implied by each phase-shift solution has been calculated, and it is shown that reported measurements are in disagreement not only with the present data and theoretical predictions but with any combination of central, tensor, and spin-orbit forces.

INTRODUCTION

THE measurement of low-energy nucleon-nucleon scattering cross sections has been a fruitful experimental approach in studying the forces between individual nucleons. Proton-proton scattering is particularly rewarding because of the precision with which the measurements may be made. The charge and stability of the interacting particles allow one to detect them after interaction with nearly 100% efficiency, to measure accurately the number of bombarding particles, to form a highly monoenergetic and well-collimated incident beam, to define the experimental geometry precisely, etc. Gaseous hydrogen makes a good target, since its density may be determined accurately, it is easily purified, and it constitutes a uniform volume of essentially unbound protons, although the effect of the molecular electrons may not be entirely negligible.[1] On the other hand, the electrostatic interaction, which enables one to obtain precise experimental data, greatly complicates the theoretical analysis of these results. Apart from the calculational complexity resulting from the mixture of Coulomb and nuclear forces (which nowadays is accommodated easily with electronic computers), there is the difficulty that one must assume he understands the Coulomb interaction very well if he is to subtract it from the total to find the purely nuclear effects. While this assumption is now made with some degree of confidence, it was not justified in some earlier analyses; for example, the importance of the vacuum polarization effect in modifying the Coulomb potential was not appreciated[2] until the most recent experiments were in progress.

Since 1939, proton-proton scattering experiments of increasing precision have been carried out at the University of Wisconsin in the energy range accessible with electrostatic accelerators. Early data[3] were satisfactorily explained by a combination of Coulomb and nuclear S-wave interactions.[4] In 1953, Worthington, McGruer, and Findley (WMF)[5] reported measurements made with a new scattering chamber designed to yield much more precise cross-section values. The purpose of their experiment was to determine the S-wave phase shift with greater precision and to detect the possible presence of a P-wave contribution. In analyzing these data, Hall and Powell[6] found that appreciable P-wave contributions were required to explain the experimental cross sections (from 1.4 to 4.2 MeV) and that the theoretical

* The experimental measurements reported here were carried out prior to 1960 while all three authors were at the University of Wisconsin; this work was supported by the U. S. Atomic Energy Commission and by the Graduate School from funds supplied by the Wisconsin Alumni Research Foundation. The present consolidation, evaluation, and analysis was carried out while the first author was at the U. S. Air Force Weapons Laboratory and, subsequently, the U. S. Air Force Cambridge Research Laboratories; this work was supported by funds and other resources of the U. S. Air Force.

[1] G. Breit, Phys. Rev. Letters 1, 200 (1958).

[2] M. de Wit and L. Durand, III, Phys. Rev. 111, 1597 (1958).

[3] R. G. Herb, D. W. Kerst, D. B. Parkinson, and G. J. Plain, Phys. Rev. 55, 998 (1939).

[4] G. Breit, H. M. Thaxton, and L. Eisenbud, Phys. Rev. 55, 1018 (1939).

[5] H. R. Worthington, J. N. McGruer, and D. E. Findley, Phys. Rev. 90, 899 (1953). Referred to as WMF in text.

[6] H. H. Hall and J. L. Powell, Phys. Rev. 90, 912 (1953).

fit was then excellent. However, the experimenters (WMF) recommended further checks to insure against the presence of any major error, and it was indeed found later that both the experimental measurements and the theoretical analysis were in error by amounts much greater than the estimated uncertainties; the excellent agreement between theory and experiment was fortuitous. The difficulties with the theoretical treatment were pointed out by de Wit and Durand,[2] while errors in the measured cross sections were identified in subsequent experimental work at Wisconsin[7] with the same scattering chamber.

With the correction of known errors and with improvements in techniques and apparatus, a new measurement of cross-section values was undertaken. Some of these results were published in 1959 by Knecht, Messelt, Berners, and Northcliffe (KMBN),[7] who presented reasonably final values for one energy (1.855 MeV) and preliminary values for two other energies (1.397 and 2.425 MeV). Additional measurements, particularly at the latter two energies and one higher energy (3.037 MeV), were obtained too late for inclusion by KMBN. Besides, certain features of the data at energies other than 1.855 MeV suggested that there might still be present an undiscovered experimental error, a point which is clarified below. When no systematic error could be discovered and experimental efforts were finally diverted to unrelated problems, the unpublished experimental values were made available separately to several interested persons who have subsequently used them in published calculations. It is the purpose of this paper to consolidate, evaluate, and revise all of the experimental data and to amplify and extend the analysis made by KMBN.

EQUIPMENT AND PROCEDURES

The 36-in. scattering chamber originally constructed by WMF and modified by KMBN was used for all measurements reported here. A detailed description of the chamber in its present form was given by KMBN, and a vertical-section view is shown in Fig. 1 of that paper; no significant modifications were made for the subsequent measurements. Briefly the experiment was as follows. Protons of the desired energy were selected from the beam of an electrostatic accelerator by use of magnetic and electrostatic analyzers. They were well collimated by two apertures and passed into the scattering chamber without obstruction except for the scattering gas. The unscattered beam was measured after passage through an exit foil and collection in a Faraday cup. Scattered protons were detected by a proportional counter after passing through an analyzer consisting of two slits to define the scattering volume. Three interchangeable slit sizes provided the range of geometric factor required by the strong variation of the cross section with angle. The scattering angle was varied by rotating a large precision angle wheel on which the scattered-particle analyzer and counter were mounted. The chamber and its related equipment were designed to permit the precise determination of such quantities as the target-gas density, the parameters defining the scattering geometry, the total number of protons incident during the measurement, and the efficiency of scattered-particle detection. They were designed to minimize certain errors inherent in gas-scattering experiments, such as double scattering in the gas, scattering from slit edges, and contamination of the target gas.

The experimental procedure for all measurements was that described by KMBN, except that the values reported here result from a greater number of angular-distribution measurements, and a greater number of energy calibrations have been utilized.

ERRORS AND UNCERTAINTIES

A thorough discussion is presented by KMBN of the recognized experimental errors and of the uncertainties in the cross-section values which result from being unable to make exact corrections for them. Uncertainties arise in connection with the measurement of target-gas pressure and temperature, the integration of incident beam flux, the detection efficiency for scattered particles, the secondary scattering of particles by the target gas or slit edges, the contamination of the target gas, the measurement of the parameters defining the geometry, and the determination of the incident proton energy. No additions to or revisions of the corrections previously made are felt to be necessary at the present time. However, the following discussion of the energy determination is felt to be appropriate in the light of further experimental work.

Energy values for the data were based on a calibration of the cylindrical electrostatic analyzer relative to the threshold of the $Li^7(p,n)Be^7$ reaction. A target of lithium fluoride was kept permanently mounted within the chamber, where it could be rotated into the target volume. The threshold could be determined with the scattering chamber evacuated, and the energy loss in the chamber gas ahead of the target volume could be determined by remeasuring the threshold with the chamber filled with gas. The energy values for the data reported by KMBN were derived primarily from a series of six determinations (three at vacuum and three at full-chamber pressure) made during a 30-day period shortly before the reported measurements were completed. When additional measurements were begun about one year later, two things were noted. First, scattering data taken for energies above and below the calibration energy appeared to show similar but opposite systematic deviations from theoretically consistent values, suggesting that an energy error might be involved. Second, the calibration point for the electrostatic analyzer seemed to have shifted. These circumstances led to a prolonged series of investigations with this

[7] D. J. Knecht, S. Messelt, E. D. Berners, and L. C. Northcliffe, Phys. Rev. 114, 550 (1959). Referred to as KMBN in text.

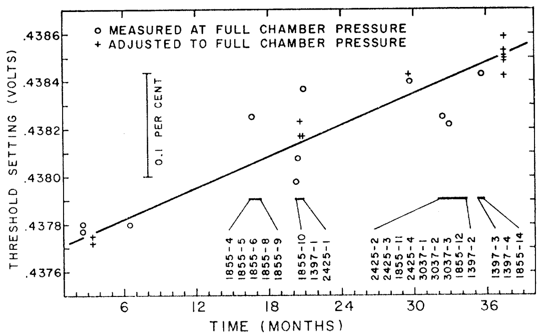

FIG. 1. History of the electrostatic analyzer calibration relative to the $Li^7(p,n)Be^7$ threshold. Also shown are the times at which the various scattering measurements were carried out.

analyzer. Several shortcomings were found, but it is not clear that these results can explain its performance during the scattering measurements. The best method for choosing energy values for the scattering data appears to be a consideration of all calibration data taken before, during, and after the scattering measurements were made. These data, spanning a three-year period, are plotted in Fig. 1, which shows the threshold value of potentiometer setting (proportional to plate voltage) as a function of time. All plotted values represent the setting required to give protons of threshold energy at the center of the chamber at full pressure; they were either measured under that condition or adjusted for whatever difference in pressure existed. (Values for the energy loss in the chamber gas have been revised downward by about 7% upon inclusion of the earlier and later data.) Two features of the data are apparent. First, the reproducibility of the data at any one time is much poorer than expected for such an analyzer. Second, there is a definite trend upward with time. No explanation for this has been proved; however, for the present purpose these data serve as the best available basis for the energy calibration. Accordingly, the solid straight line drawn in Fig. 1 has been taken as a time-dependent calibration, and all cross-section values contributing to the average values quoted below have been revised accordingly.

The value of 0.09% estimated by KMBN for the uncertainty in the energy still appears to be a reasonable estimate, but it is pertinent to remark that it represents primarily the possibility of a systematic error which was observed to occur in a nonrandom but unknown fashion. It is clear from the detailed data, for example, that for a given potentiometer setting, the energy of the proton beam was actually lower on some days than on others, but without appreciable drift over a period of hours (presumably long enough for an entire angular-distribution measurement to be carried out). Therefore, since most average cross-section values result from four or fewer individual measurements, it would be understandable if the data at a particular energy were to deviate from theoretical predictions in a manner and degree suggesting an energy error not exceeding the quoted uncertainty and not necessarily the same as that at some other energy.

One minor numerical error was made in the cross-section values given by KMBN; the correction required to make the energy calibration relativistically correct was applied with the wrong algebraic sign to some of the data. However, in the worst cases, cross-section values were affected by no more than a small fraction of the quoted uncertainty, since the correction itself is small, and some data were not affected at all. The mistake has been corrected in the values given here.

The energy scale used by KMBN was based on the value 1.8811 MeV, quoted for the threshold energy by Jones *et al.*[8] More recently, Marion[9] included three newer determinations and discounted one earlier one to obtain a revised weighted mean value of 1.88074 MeV. The cross-section values quoted in the following section have been adjusted to conform to the newer energy scale. At the lowest angles, they are therefore about 0.04% lower than they would be on the earlier energy scale.

VALUES OF THE CROSS SECTION

Final values of the cross section are given in Table I. They are the result of four angular-distribution runs (complete or partial) at 1.397 MeV, ten at 1.855 MeV, four at 2.425 MeV, and slightly more than one at 3.037 MeV. Whenever, during any of these runs, the cross section at some angle was measured more than once with the same or different slit systems, the resulting values have been averaged with equal weight. The final values of Table I then result from averaging, again with equal weight, the values from the several runs. This method was chosen because it was felt, in view of the nature of the uncertainties, that repeated measurement during the same run did not necessarily yield a more accurate value.

The uncertainty assigned each value is the total from all recognized sources, as discussed by KMBN, compounded quadratically. The total uncertainty reflects largely the possibility of a combination of systematic errors having a total effect of that magnitude; purely random processes contribute relatively little to the total uncertainty. This consideration is important in estimating the precision with which phase shifts are determined by the data, as discussed in a following section.

The cross-section values given in Table I are consistent in all cases with the values given by KMBN. However, although it is not reflected in the assignment of smaller uncertainties, the values given here are clearly to be preferred because of the revision of the energy values and because they result from a greater

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[8] K. W. Jones, R. A. Douglas, M. T. McEllistrem, and H. T. Richards, Phys. Rev. **94**, 947 (1954).

[9] J. B. Marion, Rev. Mod. Phys. **33**, 139 (1961). See also errata to this article, Rev. Mod. Phys. **33**, 623 (1961).

number of partially independent determinations. No internal inconsistencies or discrepancies have been observed which might cast doubt on the data from an experimental point of view.

METHOD OF ANALYSIS

It was found that the data of KMBN could be fitted well by purely central-force scattering in the S and P states; the S-wave and "effective" P-wave phase shifts (K_0 and K_1) were derived by the analysis. The P-wave phase shift was found to be small, with an uncertainty which did not exclude zero, and satisfactory fits to the data were also obtained assuming only the S-wave contribution. However, since noncentral forces are expected from theoretical considerations, the present analysis retains all three triplet P-wave phase shifts. The singlet D-wave contribution is also retained.

Under the assumptions that all collisions are elastic, that dynamic relativistic effects are negligible, and that the effect of vacuum polarization will be accounted for separately, the expression for the cross section in the center-of-mass system is found from the scattering amplitudes given by Breit and Hull.[10] To this may be added two terms to account for the effects of vacuum polarization and of scattering in states of higher L than retained in the amplitudes (including coupling between states of the same J but different L).

$$P=P_M+\Delta P_V+\Delta P_0+\Delta P_1+\Delta P_2+\Delta P_{L>2}, \qquad (1)$$

where the terms are, respectively, the ordinary Coulomb (Mott), the vacuum polarization, the S wave, the P wave, the D wave, and higher angular-momentum contributions. They are given by

$$P_M=(\eta^2/4k^2)[s^{-4}+c^{-4}-s^{-2}c^{-2}\cos(\eta\ln(s^2/c^2))], \qquad (2a)$$

$$\Delta P_0=\frac{\eta^2}{4k^2}\left[-\left(\frac{2X_0}{\eta}\right)\sin(K_0)\cos(K_0)+\left(\frac{2Y_0}{\eta}+\frac{4}{\eta^2}\right)\sin^2(K_0)\right], \qquad (2b)$$

$$\Delta P_1=\frac{\eta^2}{4k^2}\left[-\left(\frac{2X_1P_1}{\eta}\right)Z_2+\left(\frac{2Y_1P_1}{\eta}+\frac{4}{\eta^2}\right)Z_1+\left(\frac{4P_2}{\eta^2}\right)Z_3\right], \qquad (2c)$$

$$\Delta P_2=\left(\frac{\eta^2}{4k^2}\right)\left[-\left(\frac{10X_2P_2}{\eta}\right)\sin(K_2)\cos(K_2)+\left(\frac{10Y_2P_2}{\eta}+\frac{100P_2{}^2}{\eta^2}\right)\sin^2(K_2)+\left(\frac{40P_2}{\eta^2}\right)\times\sin(K_0)\sin(K_2)\cos(K_2-K_0+\zeta)\right]. \qquad (2d)$$

The terms ΔP_V and $\Delta P_{L>2}$ result from separate calculations as noted below. The following notation has been used, following Clementel and Villi,[11] for the three combinations of the triplet P-wave phase shifts which occur:

$$\begin{aligned}
Z_1&=\sin^2(\delta_0)+3\sin^2(\delta_1)+5\sin^2(\delta_2),\\
Z_2&=\sin(\delta_0)\cos(\delta_0)+3\sin(\delta_1)\cos(\delta_1)+5\sin(\delta_2)\cos(\delta_2),\\
Z_3&=\tfrac{3}{2}\sin^2(\delta_1)+\tfrac{7}{2}\sin^2(\delta_2)+4\sin(\delta_2)\sin(\delta_0)\cos(\delta_2-\delta_0)+9\sin(\delta_2)\sin(\delta_1)\cos(\delta_2-\delta_1).
\end{aligned} \qquad (3)$$

The other notation is as follows:

$$\begin{aligned}
X_0&=s^{-2}\cos(\eta\ln s^2)+c^{-2}\cos(\eta\ln c^2),\\
Y_0&=s^{-2}\sin(\eta\ln s^2)+c^{-2}\sin(\eta\ln c^2),\\
X_1&=s^{-2}\cos(\eta\ln s^2+\xi)-c^{-2}\cos(\eta\ln c^2+\xi),\\
Y_1&=s^{-2}\sin(\eta\ln s^2+\xi)-c^{-2}\sin(\eta\ln c^2+\xi),\\
X_2&=s^{-2}\cos(\eta\ln s^2+\zeta)+c^{-2}\cos(\eta\ln c^2+\zeta),\\
Y_2&=s^{-2}\sin(\eta\ln s^2+\zeta)+c^{-2}\sin(\eta\ln c^2+\zeta),\\
k&=Mv/2\hbar\ \text{(wave number)},\\
\eta&=e^2/\hbar v,\\
M&=\text{proton mass},\\
v&=\text{relative velocity},\\
e&=\text{electronic charge},\\
\hbar&=\text{Planck constant divided by }2\pi,\\
P_L&=L\text{th Legendre polynomial},\\
L&=\text{orbital angular momentum in units of }\hbar,\\
J&=\text{total angular momentum in units of }\hbar,\\
s&=\sin\theta,\\
c&=\cos\theta,\\
\theta&=\text{scattering angle in the laboratory system},\\
K_0&=S\text{-wave phase shift},\\
\delta_0,\delta_1,\delta_2&=P\text{-wave phase shifts with }J=0,1,2,\\
K_2&=D\text{-wave phase shift},\\
\xi&=2\arctan(\eta),\\
\zeta&=2\arctan(\eta)+2\arctan(\tfrac{1}{2}\eta).
\end{aligned} \qquad (4)$$

A relativistically correct calculation of parameters such as the particle velocity was employed, the importance of which was pointed out by Durand.[12]

It is well known that the differential cross section, as measured in this experiment, is insensitive to the presence of noncentral forces. First, the tensor and spin-orbit forces contribute to the triplet P-wave phase shifts in the following way:

$$\delta_0=\gamma+4\alpha-2\beta, \qquad (5a)$$

$$\delta_1=\gamma-2\alpha-\beta, \qquad (5b)$$

$$\delta_2=\gamma+\tfrac{2}{5}\alpha+\beta, \qquad (5c)$$

where γ, α, and β arise from central, tensor, and spin-orbit forces, respectively. From Eq. (2c) it is seen that, for small phase shifts, the principal contribution to the cross section comes from the Coulomb interference term

[10] G. Breit and M. H. Hull, Jr., Phys. Rev. **97**, 1047 (1955).

[11] E. Clementel and C. Villi, Nuovo Cimento **2**, 1165 (1955).

[12] L. Durand, III, Phys. Rev. **108**, 1597 (1957).

TABLE I. Experimental values of the cross section.

Center-of-mass scattering angle Θ	Experimental values of the center-of-mass cross section			
	T (barns) at 1.397 MeV	T (barns) at 1.855 MeV	T (barns) at 2.425 MeV	T (barns) at 3.037 MeV
12°	20.148 ±0.34%	11.180 ±0.30%	6.4487 ±0.29%	4.0652 ±0.29%
14°	10.535 ±0.31%	5.8265 ±0.26%	3.3584 ±0.25%	2.1238 ±0.29%
16°	5.9572 ±0.29%	3.2899 ±0.25%	1.9032 ±0.25%	1.2117 ±0.27%
20°	2.2718 ±0.26%	1.2664 ±0.23%	0.75202±0.24%	0.49592±0.25%
24°	1.0334 ±0.25%	0.59919±0.22%	0.37851±0.22%	0.26613±0.24%
30°	0.42021±0.22%	0.27810±0.19%	0.20396±0.18%	0.16159±0.19%
35°	0.25556±0.18%	0.19691±0.18%	0.16204±0.17%	0.13815±0.17%
40°	0.19133±0.15%	0.16759±0.13%	0.14879±0.14%	0.13161±0.16%
50°	0.15730±0.13%	0.15632±0.11%	0.14640±0.13%	0.13235±0.16%
60°	0.15646±0.12%	0.15994±0.11%	0.15064±0.13%	0.13574±0.16%
70°	0.16064±0.12%	0.16447±0.11%	0.15424±0.12%	0.13871±0.15%
80°		0.16732±0.11%	0.15649±0.12%	0.14041±0.16%
90°		0.16779±0.14%	0.15715±0.12%	0.14116±0.18%
100°			0.15626±0.12%	

containing Z_2, proportional to the first power of the phase shifts, and not from the terms in Z_1 and Z_3, proportional to the square of the phase shifts. In the approximation that the phase shifts are small, $Z_2 \approx 9\gamma$, and the cross section is independent of tensor and spin-orbit forces. Second, the detection of much stronger noncentral forces is prevented by a further ambiguity which is discussed quantitatively in a following section.

In the present analysis, therefore, it was not expected to determine α or β from the data, and these two parameters were specified arbitrarily on the basis of other arguments; at most, only the parameters K_0, γ, and K_2 were adjustable in the least-squares procedure, while the option of fixing K_2 or both K_2 and γ was also provided in the computer program. The quantity E minimized in the analysis was the weighted sum of the squares of the differences between the experimental and calculated cross-section values; all results presented here were obtained by weighting with the reciprocal square of the experimental uncertainty. The equations which result from setting to zero the derivatives of E with respect to the adjustable parameters were linearized by the following approximation:

$$P \approx P^* + A\frac{\partial P}{\partial K_0} + B\frac{\partial P}{\partial \gamma} + C\frac{\partial P}{\partial K_2}. \qquad (6)$$

The estimated cross section P^* was calculated from Eqs. (1) and (2) using trial values of the adjustable phase shifts specified on the basis of prior knowledge or by the following rule: Set K_2^* and γ^* equal to zero and take K_0^* equal to the average of values found by solving the cross-section expression (as approximated by only the Coulomb, vacuum-polarization, and S-wave terms) for K_0 and evaluating by use of the measured cross sections for angles near 90 deg. Expressions for the derivatives appearing in Eq. (6) were obtained by differentiating Eqs. (2b) through (2d). Solutions for A, B, and C were obtained from the linearized set of equations, and the adjusted phase shifts were given by

$$K_0 = K_0^* + A\,, \qquad (7a)$$

$$\delta_0 = \gamma^* + 4\alpha - 2\beta + B\,, \qquad (7b)$$

$$\delta_1 = \gamma^* - 2\alpha - \beta + B\,, \qquad (7c)$$

$$\delta_2 = \gamma^* + \tfrac{2}{5}\alpha + \beta + B\,, \qquad (7d)$$

$$K_2 = K_2^* + C\,. \qquad (7e)$$

Since Eq. (6) is an approximation, the computer program was designed to make successive iterations, using adjusted phase shifts obtained in one iteration as trial values for the next, until the adjustments were all less than 10^{-5} deg.

Two quantities which serve as figures of merit for the goodness of fit were calculated from the departures R of the experimental values from the calculated values of the cross section. The quantity E, the weighted sum of the square of R, is the quantity minimized in the analysis. The quantity Φ is the root-mean-square value of R when R is expressed as a fraction of the quoted uncertainty; Φ therefore has a special usefulness, and it is simply related to E.

$$E = \sum (R^2/\epsilon^2)\,, \qquad (8a)$$

$$\Phi = [G^{-1} \sum (R/\epsilon)^2]^{1/2}\,, \qquad (8b)$$

where ϵ is the uncertainty in the measured cross section and G is the number of angles at which measurements were made.

A detailed description of the method of analysis, including derivations and Fortran programs, has been given in a separate technical report.[13]

RESULTS OF PHASE-SHIFT ANALYSIS

The values obtained for the parameters which are adjustable in the analysis depend to some degree on

[13] D. J. Knecht, U. S. Air Force Weapons Laboratory Technical Documentary Report Number WL TDR-64-78, 1964 (unpublished).

TABLE II. Vacuum-polarization and higher angular-momentum contributions to the cross section.

Center-of-mass scattering angle Θ	Calculated values of contributions to the cross section from vacuum polarization (ΔP_V) and from scattering in higher angular-momentum states (ΔP_L), in millibarns at 1.397 MeV ΔP_V	at 1.397 MeV ΔP_L	at 1.855 MeV ΔP_V	at 1.855 MeV ΔP_L	at 2.425 MeV ΔP_V	at 2.425 MeV ΔP_L	at 3.037 MeV ΔP_V	at 3.037 MeV ΔP_L
12°	90.96	−0.46	56.31	−0.56	35.55	−0.68	24.06	−0.78
14°	52.39	−0.30	32.11	−0.37	20.12	−0.43	13.53	−0.47
16°	32.07	−0.20	19.50	−0.23	12.13	−0.27	8.123	−0.283
20°	13.75	−0.09	8.258	−0.088	5.090	−0.083	3.385	−0.067
24°	6.714	−0.030	4.008	−0.015	2.460	+0.006	1.632	+0.033
30°	2.752	+0.008	1.651	+0.032	1.022	+0.060	0.6838	+0.093
35°	1.515	+0.020	0.932	+0.0420	0.5901	+0.0695	0.4034	+0.1001
40°	0.9541	+0.0214	0.6114	+0.0400	0.4022	+0.0621	0.2833	+0.0865
50°	0.5560	+0.0087	0.3944	+0.0150	0.2800	+0.0222	0.2080	+0.0299
60°	0.4648	−0.0113	0.3514	−0.0189	0.2595	−0.0280	0.1975	−0.0382
70°	0.4507	−0.0302	0.3492	−0.0500	0.2613	−0.0732	0.2004	−0.0983
80°	0.455	−0.0439	0.356	−0.0735	0.2659	−0.1038	0.2042	−0.1388
90°	0.459	−0.0490	0.357	−0.0830	0.2678	−0.1145	0.2057	−0.1530

several quantities which must be specified on the basis of knowledge not acquired in the experiment; such separately calculated or estimated quantities are the vacuum-polarization contribution, the contribution of higher angular-momentum states, and the size of noncentral forces.

The vacuum polarization contribution ΔP_V was originally defined and computed by Durand[12] for the energies and angles of this experiment. A more precise calculation by Heller[14] has yielded the values listed in Table II. These have been used in all of the present analyses; they do not differ appreciably from those of Durand. The values at 1.855 MeV have been plotted in Fig. 2 as fractions of the cross section. It is important to remember that these values of ΔP_V do not include the S-wave vacuum-polarization contribution; therefore, all higher phase shifts, but not K_0, will have had the vacuum-polarization portion removed.

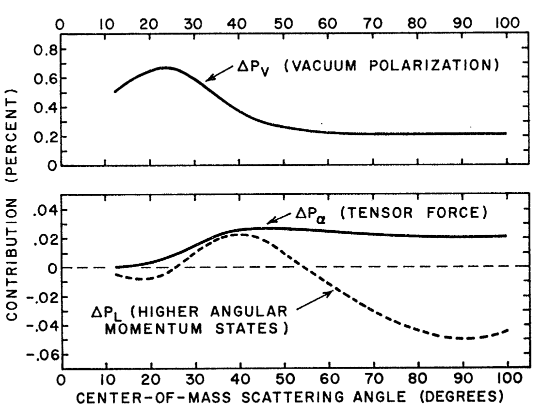

FIG. 2. Small contributions to the cross section at 1.855 MeV which must be calculated on a basis other than the experimental data. Values of ΔP_V are those calculated by Heller; values of ΔP_α result from the OPE value for α (0.1017°) given by Noyes; values of ΔP_L are those calculated by Noyes for scattering in states with $L \geq 2$.

[14] L. Heller (private communication).

The quantity ΔP_L, the contribution to the cross section from scattering in states with $L>1$ (including coupling between states of the same J but different L) has been computed by Noyes[15] as predicted by one-pion-exchange theory (OPE); values of this quantity for the energies and angles of the present experiment are also included in Table II, and those for 1.855 MeV are plotted in Fig. 2 as fractions of the cross section. It should be noted that this ΔP_L includes both ΔP_2 and $\Delta P_{L>2}$ of Eq. (1). This contribution is seen to be almost negligibly small and arises principally from the OPE-predicted D-wave phase shift. Therefore, it was included in all analyses reported here except those of Tables V and VI, in which the entire D-wave phase shift was determined by the analysis.

The size of the tensor and spin-orbit parts of the triplet P-wave phase shifts, as predicted by OPE, have also been calculated by Noyes,[16] who finds that the OPE values for α differ from values predicted by the Yale and Hamada-Johnston models by less than 10% and that all three models predict the spin-orbit contribution to be less than 10% of the tensor contribution. Therefore, in the principal ones of the analyses which follow, α was taken to be the value computed by Noyes from OPE, and β was taken to be zero. Fortunately, some uncertainty in these values does not preclude an accurate determination of the central-force phase shifts K_0 and γ, a circumstance which is explored quantitatively below. The OPE-predicted tensor-force contribution to the cross section at 1.855 MeV is also shown in Fig. 2.

Phase-shift solutions have been obtained under the following sets of assumptions: (1) that all of the above-mentioned contributions are valid, (2) that all are valid except noncentral forces are zero, and (3) that all are valid except noncentral forces are much larger than predicted.

[15] H. P. Noyes (private communication).

[16] H. P. Noyes, Phys. Rev. Letters 12, 171 (1964).

TABLE III. Preferred phase-shift solutions (obtained under the most plausible assumptions).

Energy (MeV) E	K_0	Phase shifts[a] δ_0	δ_1	δ_2	Phase-shift contributions γ	α	β	Z_2	Quality of the fit E	Φ
1.397	39.236°	+0.251°	−0.136°	+0.019°	−0.0072°	+0.0645°	0	−0.00114	3.383	0.555
1.855	44.249°	+0.399°	−0.211°	+0.033°	−0.0078°	+0.1017°	0	−0.00123	6.591	0.712
2.425	48.280°	+0.606°	−0.317°	+0.052°	−0.0093°	+0.1539°	0	−0.00146	2.842	0.451
3.037	50.952°	+0.865°	−0.429°	+0.089°	+0.0024°	+0.2156°	0	+0.00037	4.971	0.618

[a] Except for K_0, the phase shifts given in this and following tables have had the vacuum-polarization contribution removed as noted in the text.

Preferred Solutions

In Table III are given the phase shifts derived as the result of specifying ΔP_L, α, and β according to the calculations referred to above and permitting K_0 and γ to be adjustable. These are considered to be the preferred solutions in that they result from specifying arbitrary parameters in the analysis on the basis of the best present knowledge. The closeness with which the experimental data are fitted is shown in Fig. 3. The fits are satisfactory at all energies, as can be seen from the plots and from the values of E and Φ given in Table III. The differences between the experimental and calculated cross-section values do not in all respects appear to be random, but are somewhat systematic with angle; this is not unexpected since the uncertainties largely reflect the possibility of systematic experimental errors.

It is of some interest to know how much, if any, improvement in the fits can be obtained by adjusting K_2 as well as K_0 and γ. Table IV shows the way in which the phase shifts and the quality of fit are changed if the D-wave contribution is not limited to the OPE value. The fit is only slightly improved. At all energies but the lowest, the data favor values of K_2 which are positive and more than double the OPE values (which range from +0.002° at 1.397 MeV to +0.013° at 3.037 MeV). However, the additional D-wave contribution favored by the data is still a small fraction of the experimental uncertainty and cannot be considered significant.

If one chooses to question the reliability of the OPE values for the tensor and spin-orbit parts of the P-wave phase shifts, it is important to know how an uncertainty in these affects the accuracy with which K_0 and γ are determined. A series of analyses was made in which α was varied from zero to twice the OPE-predicted value. The effect on K_0 and γ is shown in Fig. 4; curves have also been obtained for nonzero values of β; they are similar to those for α, but the effect on K_0 and γ of a given value of β is about half of that produced by an α of the same magnitude.

TABLE IV. Changes to the solutions of Table III if the D-wave contribution is also adjustable.

Energy (MeV) E	Change in phase shift ΔK_0	$\Delta\gamma$	ΔK_2	Quality of the fit E	Φ
1.397	−0.010°	−0.004°	−0.010°	2.926	0.516
1.855	+0.020°	+0.008°	+0.013°	4.797	0.607
2.425	+0.015°	+0.006°	+0.009°	2.054	0.383
3.037	+0.017°	+0.008°	+0.015°	3.516	0.520

Solutions with Purely Central Forces

It may be useful, in order to compare with the KMBN analyses and to assess further the importance of using the above-mentioned OPE predictions, to present the results of analyses in which the scattering

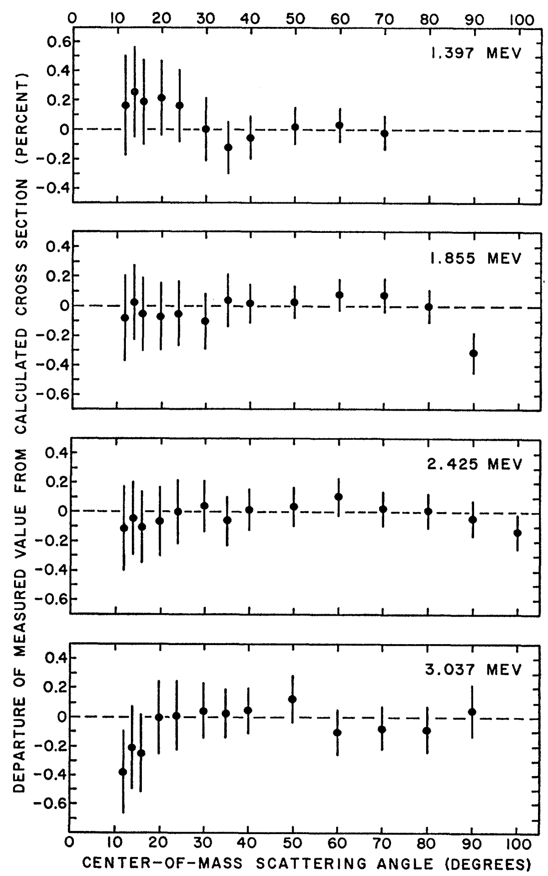

FIG. 3. Quality of fit obtained for the preferred phase-shift solutions, in which noncentral P wave and all higher angular-momentum scattering is predicted from OPE theory and S-wave and central P-wave scattering is determined from fitting the experimental data.

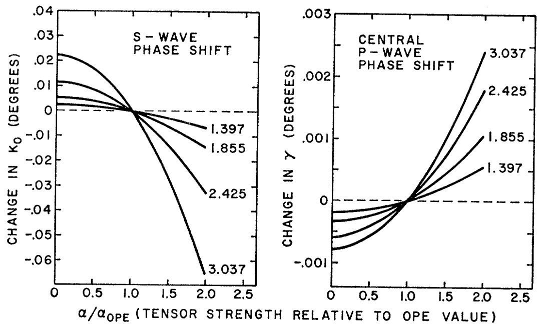

FIG. 4. Sensitivity of K_0 and γ derived in the analysis to the assumed tensor strength α. Positive and negative values of α produce the same effect.

force was required to be entirely central and where the D-wave phase shift, if not constrained to zero, was determined by the least-squares procedure. Three types of such analyses were made: (1) K_0, γ, and K_2 all adjustable, (2) K_0 and γ adjustable with K_2 held to zero, and (3) only K_0 adjustable with γ and K_2 held to zero. The vacuum polarization contribution was included, of course, while α, β, and ΔP_L were set equal to zero; solutions of types 2 and 3 were therefore obtained by identically the same method as, respectively, the solutions S, P, $V\cdots 1$ and S, $V\cdots 1$ shown in Table IV of KMBN. The present solutions are given in Table V. The pure S-wave fits (type 3 solutions) are shown in Fig. 5. It is clear that satisfactory fits to the data can be obtained without the inclusion of any P-wave or D-wave contributions.

Solutions with Large Noncentral Forces

It is well known that the angular dependence of noncentral P-wave cross-section contributions (the terms in Z_1 and Z_3) may be duplicated quite well by an appropriate combination of central S-, P-, and D-wave contributions. It should therefore be possible to specify values of α and β much larger than predicted by OPE and maintain satisfactory fits to the data by making compensating adjustments of K_0, γ, and K_2. In searching for fits to the WMF data which would imply measurable polarizations, Hull and Shapiro[17] found acceptable phase-shift solutions with β as large as 5 deg. (They did not look for solutions with tensor-force contributions.)

While extremely large values of α or β may not be acceptable on theoretical grounds, it is useful to note which values, if any, are ruled out by the present data and to confirm that the conclusions which Hull and Shapiro obtained from the incorrect WMF data are nevertheless valid. Accordingly, the present data were analyzed with successively increasing values specified for α and β, both alone and in combination. Typical phase-shift solutions for the present data are given in Table VI; solutions are given only for positive values of α and β, since corresponding solutions are obtained for negative values (i.e., identical values for K_0, γ, and K_2

TABLE V. Phase-shift solutions with purely central forces.

Energy (MeV) E	Phase shifts K_0	$\delta_0=\delta_1=\delta_2$	K_2	Quality of the fit E	Φ
Fits with K_0, γ, and K_2 all adjustable:					
1.397	39.228°	−0.011°	−0.008°	2.926	0.516
1.855	44.274°	0.000°	+0.017°	4.851	0.611
2.425	48.307°	−0.004°	+0.016°	2.052	0.382
3.037	50.992°	+0.012°	+0.028°	3.462	0.516
Fits with K_0 and γ adjustable, $K_2=0$:					
1.397	39.236°	−0.008°	0	3.217	0.541
1.855	44.248°	−0.011°	0	7.945	0.782
2.425	48.279°	−0.015°	0	4.703	0.580
3.037	50.961°	−0.004°	0	8.251	0.797
Fits with K_0 adjustable, $\gamma=0$, $K_2=0$:					
1.397	39.250°	0	0	5.178	0.686
1.855	44.260°	0	0	11.110	0.924
2.425	48.294°	0	0	9.400	0.819
3.037	50.966°	0	0	8.460	0.807

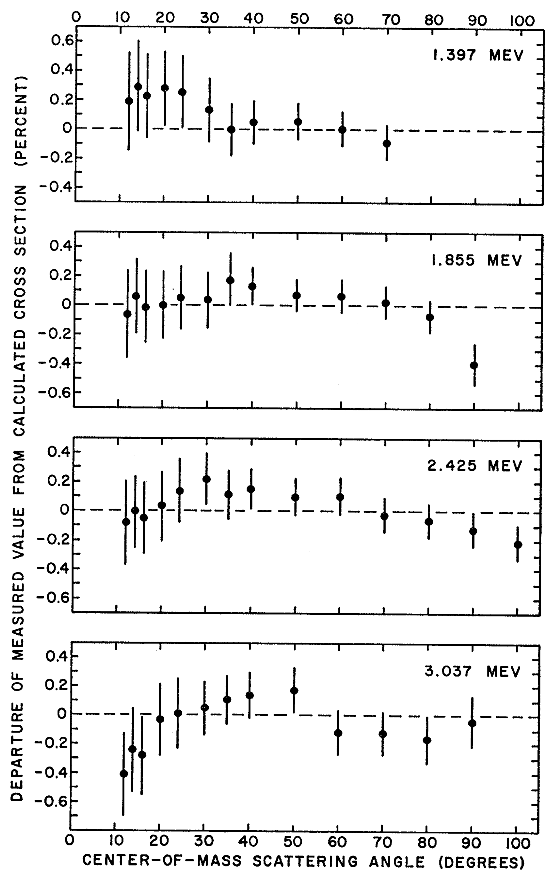

FIG. 5. Quality of fit obtained for pure S-wave solutions. (All phase shifts except K_0 are zero.) These are the last four solutions given in Table V.

[17] M. H. Hull, Jr., and J. Shapiro, Phys. Rev. 109, 846 (1958).

TABLE VI. Phase-shift solutions with large noncentral forces.

Energy (MeV) E	Phase shifts K_0	δ_0	δ_1	δ_2	K_2	Phase-shift contributions γ	α	β	Maximum polarization neg.	pos.	Quality of fit Φ
1.397	39.228°	−0.011°	−0.011°	−0.011°	−0.008°	−0.011°	0	0	0	0	0.516
	38.679°	+4.028°	−1.972°	+0.428°	−0.026°	+0.028°	+1.000°	0	−0.006	0	0.520
	37.001°	+8.161°	−3.839°	+0.961°	−0.081°	+0.161°	+2.000°	0	−0.022	0	0.533
	34.078°	+12.429°	−5.571°	+1.629°	−0.187°	+0.429°	+3.000°	0	−0.043	0	0.564
	29.574°	+16.933°	−7.067°	+2.533°	−0.385°	+0.933°	+4.000°	0	−0.063	0	0.640
	22.232°	+22.041°	−7.959°	+4.041°	−0.919°	+2.041°	+5.000°	0	−0.069	+0.023	1.007
	38.969°	−2.017°	−1.017°	+0.983°	+0.026°	−0.017°	0	+1.000°	−0.009	+0.001	0.518
	38.183°	−4.031°	−2.031°	+1.969°	+0.128°	−0.031°	0	+2.000°	−0.018	+0.003	0.531
	36.847°	−6.052°	−3.052°	+2.948°	+0.305°	−0.052°	0	+3.000°	−0.026	+0.009	0.577
	34.904°	−8.071°	−4.071°	+3.929°	+0.567°	−0.071°	0	+4.000°	−0.033	+0.023	0.700
	32.243°	−10.079°	−5.079°	+4.921°	+0.943°	−0.079°	0	+5.000°	−0.036	+0.046	0.996
	38.417°	+2.021°	−2.979°	+1.421°	+0.008°	+0.021°	+1.000°	+1.000°	−0.011	+0.001	0.521
	35.906°	+4.135°	−5.865°	+2.935°	+0.054°	+0.135°	+2.000°	+2.000°	−0.021	+0.006	0.540
	31.355°	+6.397°	−8.603°	+4.597°	+0.121°	+0.397°	+3.000°	+3.000°	−0.028	+0.027	0.573
	23.347°	+9.053°	−10.947°	+6.653°	+0.160°	+1.053°	+4.000°	+4.000°	−0.026	+0.081	0.616
1.855	44.274°	0.000°	0.000°	0.000°	+0.017°	0.000°	0	0	0	0	0.611
	43.766°	+4.025°	−1.975°	+0.425°	+0.001°	+0.025°	+1.000°	0	−0.005	0	0.612
	42.229°	+8.113°	−3.887°	+0.913°	−0.050°	+0.113°	+2.000°	0	−0.017	0	0.617
	39.622°	+12.293°	−5.707°	+1.493°	−0.138°	+0.293°	+3.000°	0	−0.035	0	0.630
	35.814°	+16.623°	−7.377°	+2.223°	−0.282°	+0.623°	+4.000°	0	−0.054	0	0.665
	30.425°	+21.232°	−8.768°	+3.232°	−0.537°	+1.232°	+5.000°	0	−0.068	+0.011	0.783
	44.039°	−2.011°	−1.011°	+0.989°	+0.050°	−0.011°	0	+1.000°	−0.008	+0.001	0.612
	43.330°	−4.043°	−2.043°	+1.957°	+0.148°	−0.043°	0	+2.000°	−0.015	+0.002	0.621
	42.137°	−6.092°	−3.092°	+2.908°	+0.314°	−0.092°	0	+3.000°	−0.022	+0.007	0.654
	40.438°	−8.152°	−4.152°	+3.848°	+0.554°	−0.152°	0	+4.000°	−0.027	+0.017	0.757
	38.190°	−10.214°	−5.214°	+4.786°	+0.881°	−0.214°	0	+5.000°	−0.031	+0.034	1.039
	43.529°	+2.013°	−2.987°	+1.413°	+0.033°	+0.013°	+1.000°	+1.000°	−0.009	+0.001	0.613
	41.264°	+4.062°	−5.938°	+2.862°	+0.079°	+0.062°	+2.000°	+2.000°	−0.019	+0.004	0.621
	37.341°	+6.189°	−8.811°	+4.389°	+0.147°	+0.189°	+3.000°	+3.000°	−0.024	+0.020	0.639
	31.290°	+8.504°	−11.496°	+6.104°	+0.223°	+0.504°	+4.000°	+4.000°	−0.028	+0.056	0.669
	20.864°	+11.416°	−13.584°	+8.416°	+0.197°	+1.416°	+5.000°	+5.000°	−0.023	+0.135	0.690
2.425	48.307°	−0.004°	−0.004°	−0.004°	+0.016°	−0.004°	0	0	0	0	0.382
	47.812°	+4.012°	−1.988°	+0.412°	0.000°	+0.012°	+1.000°	0	−0.004	0	0.383
	46.328°	+8.069°	−3.931°	+0.869°	−0.049°	+0.069°	+2.000°	0	−0.014	0	0.386
	43.844°	+12.193°	−5.807°	+1.393°	−0.132°	+0.193°	+3.000°	0	−0.029	0	0.394
	40.311°	+16.427°	−7.573°	+2.027°	−0.257°	+0.427°	+4.000°	0	−0.046	<0.001	0.417
	35.556°	+20.854°	−9.146°	+2.854°	−0.449°	+0.854°	+5.000°	0	−0.060	+0.008	0.484
	48.081°	−2.019°	−1.019°	+0.981°	+0.049°	+0.019°	0	+1.000°	−0.006	<0.001	0.384
	47.401°	−4.064°	−2.064°	+1.936°	+0.145°	+0.064°	0	+2.000°	−0.012	+0.002	0.390
	46.265°	−6.135°	−3.135°	+2.865°	+0.307°	+0.135°	0	+3.000°	−0.018	+0.005	0.415
	44.662°	−8.225°	−4.225°	+3.775°	+0.536°	+0.225°	0	+4.000°	−0.024	+0.013	0.498
	42.573°	−10.325°	−5.325°	+4.675°	+0.837°	+0.325°	0	+5.000°	−0.028	+0.027	0.725
	47.586°	+1.995°	−3.005°	+1.395°	+0.032°	−0.005°	+1.000°	+1.000°	−0.007	<0.001	0.384
	45.415°	+3.999°	−6.001°	+2.799°	+0.077°	−0.001°	+2.000°	+2.000°	−0.015	+0.004	0.388
	41.742°	+6.044°	−8.956°	+4.244°	+0.144°	+0.044°	+3.000°	+3.000°	−0.022	+0.016	0.399
	36.351°	+8.206°	−11.794°	+5.806°	+0.220°	+0.206°	+4.000°	+4.000°	−0.026	+0.044	0.420
	28.417°	+10.667°	−14.333°	+7.667°	+0.273°	+0.667°	+5.000°	+5.000°	−0.024	+0.104	0.449
3.037	50.992°	+0.012°	+0.012°	+0.012°	+0.028°	+0.012°	0	0	0	0	0.516
	50.498°	+4.021°	−1.979°	+0.421°	+0.012°	+0.021°	+1.000°	0	−0.003	0	0.515
	49.023°	+8.058°	−3.942°	+0.858°	−0.037°	+0.058°	+2.000°	0	−0.012	0	0.512
	46.575°	+12.148°	−5.852°	+1.348°	−0.119°	+0.148°	+3.000°	0	−0.025	0	0.504
	43.141°	+16.329°	−7.671°	+1.929°	−0.237°	+0.329°	+4.000°	0	−0.040	+0.001	0.491
	38.623°	+20.666°	−9.334°	+2.666°	−0.409°	+0.666°	+5.000°	0	−0.053	+0.008	0.472
	50.768°	−2.007°	−1.007°	+0.993°	+0.061°	+0.007°	0	+1.000°	−0.005	<0.001	0.515
	50.097°	−4.061°	−2.061°	+1.939°	+0.158°	+0.061°	0	+2.000°	−0.011	+0.001	0.509
	48.978°	−6.147°	−3.147°	+2.853°	+0.320°	+0.147°	0	+3.000°	−0.017	+0.005	0.493
	47.408°	−8.257°	−4.257°	+3.743°	+0.544°	+0.257°	0	+4.000°	−0.021	+0.012	0.473
	45.377°	−10.383°	−5.383°	+4.617°	+0.834°	+0.383°	0	+5.000°	−0.025	+0.024	0.511
	50.275°	+2.001°	−2.999°	+1.401°	+0.044°	+0.001°	+1.000°	+1.000°	−0.006	<0.001	0.514
	48.128°	+3.976°	−6.024°	+2.776°	+0.089°	−0.024°	+2.000°	+2.000°	−0.014	+0.003	0.508
	44.542°	+5.971°	−9.029°	+4.171°	+0.155°	−0.029°	+3.000°	+3.000°	−0.020	+0.015	0.499
	39.399°	+8.050°	−11.950°	+5.650°	+0.230°	+0.050°	+4.000°	+4.000°	−0.023	+0.040	0.486
	32.216°	+10.347°	−14.653°	+7.347°	+0.292°	+0.347°	+5.000°	+5.000°	−0.024	+0.089	0.473

and for the calculated cross section, but with inverted splitting of δ_0, δ_1, and δ_2). Good solutions are seen to exist for very large values of α and β, but large compensating changes are required, particularly in the S-wave phase shift. Table VI also lists the maximum negative and positive values found for the polarization; some very large values are found. Figure 6 summarizes, for the 1.855-MeV data, the compensation required and the quality of fit maintained as α alone ($\beta=0$) or β alone ($\alpha=0$) is increased. A continuum of satisfactory solu-

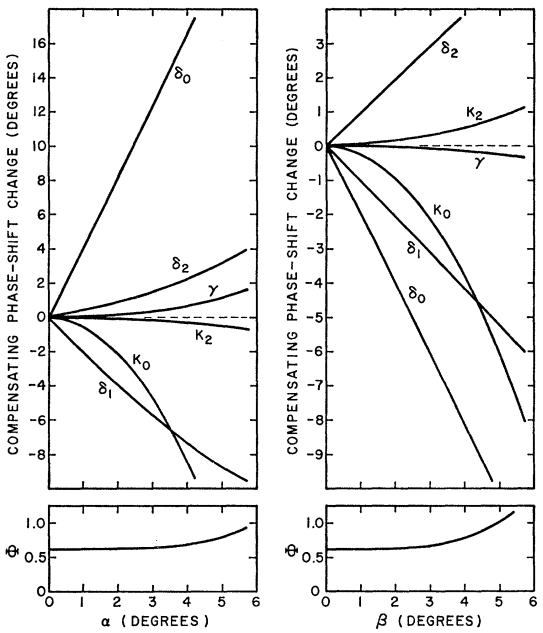

FIG. 6. Curves showing the way in which K_0, γ, and K_2 must be varied to maintain a good fit to the 1.855-MeV data as the tensor coefficient α (left) and the spin-orbit coefficient β (right) are varied from zero to very large positive values. Corresponding values of δ_0, δ_1, and δ_2 are also shown. Variation through negative values of α and β yields identical curves for K_0, γ, and K_2, and identical calculated cross sections, but not the same polarizations. In the lower portion is shown the quality of fit maintained, in terms of the parameter Φ.

tions exists; in Fig. 7 is shown, for each energy, the quality of fit which can be maintained for any combination of α and β. If neither α nor β lies outside the range -5 to $+5$ degrees, any combination allows a satisfactory fit ($\Phi<1$).

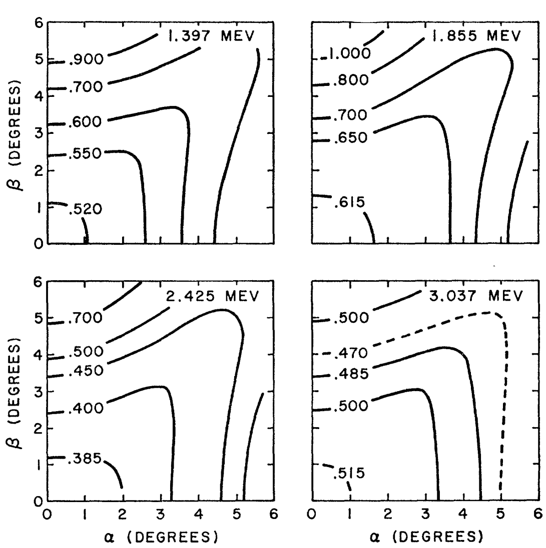

FIG. 7. Approximate contours in the α-β plane on which equally good fits to the data are obtained if K_0, γ, and K_2 can be chosen without restriction. Each is labeled with the appropriate value of Φ. Other quadrants are not shown since contours are symmetric about the α and β axes.

In Fig. 8 is shown, for 1.855 MeV, the angular dependence exhibited by the contribution to the cross section from each of the various phase shifts and phase-shift parts; the quantity actually plotted is the increment (to the cross section calculated with $K_0=44.27°$ and all other phase shifts zero) which results from the indicated increment in phase shift. (Note: the D-wave angular dependence was plotted incorrectly by KMBN.) In Fig. 9 are shown the central S-, P-, and D-wave contributions chosen by the least-squares procedure to compensate for a large tensor contribution arbitrarily specified in a typical analysis; the result shown is for the analysis at 1.855 MeV with $\alpha=4°$ and $\beta=0$, as listed in Table VI.

UNCERTAINTIES IN THE PHASE SHIFTS

The assignment of uncertainties to the phase shifts derived in the foregoing analyses cannot be done with complete rigor. If the experimental cross-section values had uncertainties resulting from purely random processes, an uncertainty could be calculated for each derived phase shift by the usual procedure, which defines it in terms of the phase-shift increments which produce unit increase in E. (The quality E, defined in Eq. (8a) is more commonly denoted χ^2.) Specifically, if δA and δB denote variables which are increments of the phase shifts K_0 and γ, respectively, over the values derived in a least-squares analysis in which they alone are adjustable, then a plot in the δA-δB plane of combinations of δA and δB which produce unit increase in E will be, to first order, an ellipse. The major and minor axes will not, in general, coincide with the δA and δB axes, and the uncertainties to be quoted for K_0 and γ are the maximum values of δA and δB on the ellipse. Uncertainties derived by the use of this rule are denoted σ_A

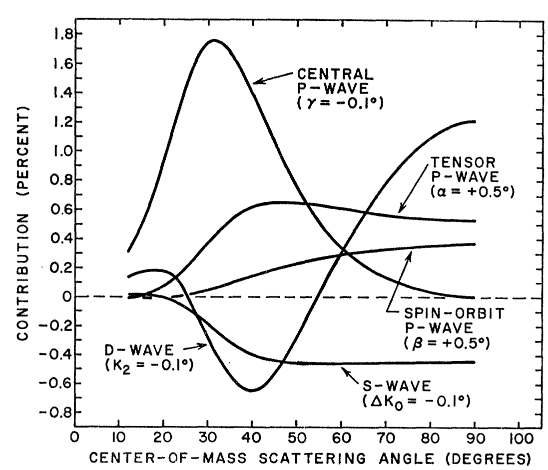

FIG. 8. Angular dependence of small contributions to the cross section at 1.855 MeV from the various phase shifts. Phase-shift magnitudes have been chosen arbitrarily to yield contributions which can be compared on the same scale.

and σ_B; they are shown, along with their related ellipse, for a typical analysis at 1.855 MeV, in Fig. 10. Because of the linearization effected by Eq. (6), these uncertainties can be calculated from a simple formula. The method applies similarly with one adjustable phase shift (where the ellipse is replaced by two points) or with three adjustable phase shifts (where the ellipse is replaced by an ellipsoid).

The assumption made above (that the uncertainties arise entirely from random processes) defines one extreme case which can be treated as noted. There also exists an opposite extreme case, defined by the assumption that the quoted uncertainties reflect only the probability of one or more systematic errors whose total effect on the cross section exactly duplicates, in angular dependence, that of one of the adjustable phase shifts. In the first case, repeated experimental measurements at the same or different angles serve to increase the precision of the results; in the second case, no improvement at all can be expected. It is easily shown that the uncertainties in the second extreme case are found in exactly the same way as for the first, except that the criterion of unit increase in E is replaced by that of unit increase in Φ. Uncertainties obtained from this rule, denoted τ_A and τ_B, are larger than σ_A and σ_B by a factor of $(4EG)^{1/4}$, which is about four for the present data; τ_A and τ_B, with their related ellipse, are also shown in Fig. 10 for the same analysis.

The circumstances of the present experiment justify neither of the extreme assumptions just considered. On one hand, only a small fraction of the total quoted uncertainty arises from such purely random processes as counting statistics and scale-reading accuracy. On the other hand, while the uncertainties largely reflect systematic errors, their total effect is unlikely to duplicate, in angular dependence, that of any one phase shift.

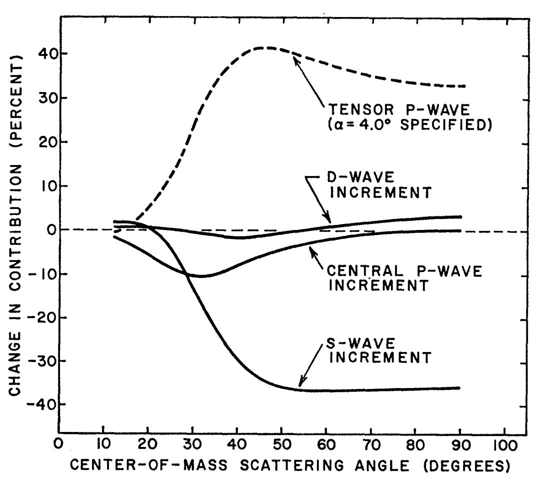

FIG. 9. Adjustment of central-force contributions to the cross section required in a typical analysis of the 1.855-MeV data by the specification of a large tensor contribution ($\alpha=+4°$). Although individual contributions are as large as 40%, cancellation of the tensor by the central contributions is total to within 0.1%.

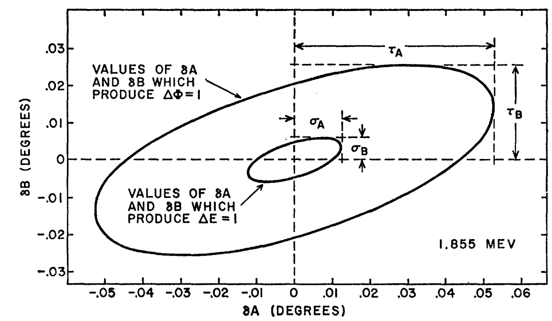

FIG. 10. Demonstration of the procedure for determining uncertainties in the phase shifts K_0 and γ, as specified by the criteria of unit increase in E and unit increase in Φ. This example is taken from the preferred analysis of the 1.855-MeV data.

Moreover, some evidence suggests that at least one systematic error (arising from the energy determination) occurred with a magnitude which varied during the experiment in an essentially random way. Thus, it is clear that the true uncertainty must be bounded by the values yielded by these two extreme rules, but it is unfortunately also clear that there is no method based on rigorous argument which can be made to yield the correct value.

It is reasonable to expect that a more detailed examination of the experimental data could aid in making a fairly reliable estimate. Specifically, if a satisfactory fit is obtained, the uncertainty in a particular phase shift may be equated roughly to the variation required to raise or lower the calculated curve by one error bar over an angular region within which the change cannot be expected to be compensated by a concomitant adjustment of the other phase shift(s). This is discussed in greater detail in Ref. 13, which outlines how the computer program for phase-shift analysis can be used to estimate the uncertainties on this basis.

The best estimate $\mathcal{E}$ for the phase-shift uncertainty has been taken to be the geometric mean of σ and τ, on the basis of a somewhat subjective assessment of the data and the methods described. The uncertainties in K_0 and γ for the preferred solutions of Table III are given in Table VII. It is well to emphasize that these

TABLE VII. Estimated uncertainties in the derived phase shifts.

Energy (MeV) E	Uncertainty in K_0 $\mathcal{E}_A$ (σ_A; τ_A)	Uncertainty in γ $\mathcal{E}_B$ (σ_B; τ_B)
1.397	±0.029° (0.015°; 0.057°)	±0.011° (0.006°; 0.022°)
1.855	±0.025° (0.012°; 0.053°)	±0.012° (0.006°; 0.025°)
2.425	±0.026° (0.014°; 0.047°)	±0.013° (0.007°; 0.023°)
3.037	±0.041° (0.021°; 0.081°)	±0.017° (0.009°; 0.033°)

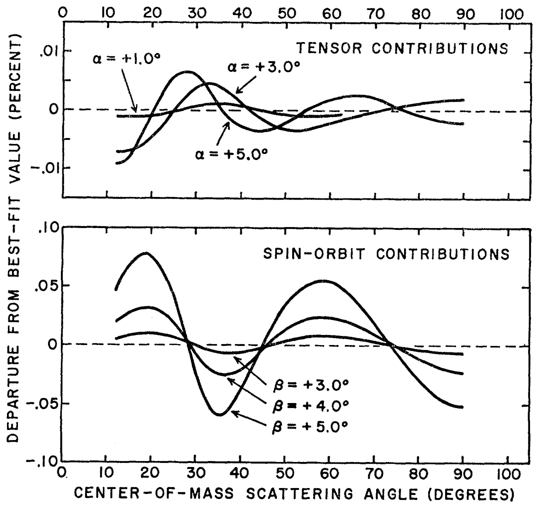

FIG. 11. Accuracy with which hypothetical cross sections which include large noncentral-force contributions can be fitted by central-force contributions alone.

reflect solely the limited precision of the experimental data; an additional uncertainty in each phase shift is contributed by the limited accuracy of assumptions made in the analysis, such as the magnitudes of the tensor, spin-orbit, and higher angular-momentum contributions. No evaluation of the correctness or precision of these assumptions is intended here; however, the data presented in Figs. 4 and 6, Table V, etc., allow such an evaluation to be translated into terms of corresponding uncertainties in the derived phase shifts. For example, it is pertinent to note from Fig. 4 that while an error of a factor of 2 in the assumed value of α would not seriously affect the accuracy of γ, the resulting error in K_0 could exceed the experimental uncertainty.

EFFECT OF NONCENTRAL FORCES ON THE CROSS SECTION

While Fig. 8 shows that tensor and spin-orbit forces, if present, can contribute appreciably to the cross section, it has also been noted that their contribution cannot easily be recognized in the experimental data, since it may be duplicated quite accurately by a combination of central forces. This fact is illustrated quantitatively by Fig. 11, which shows how well cross sections calculated from several hypothetical sets of phase shifts which include large noncentral parts can be fitted by sets of phase shifts which include only central forces; the quantity plotted is that portion of the hypothetical cross section (with the indicated amount of tensor or spin-orbit contribution) which is left over after making the best fit with purely central forces. Clearly, even the strongest of these noncentral forces (an α or β of 5 deg) results in a discrepancy in the fit of only a small fraction of the experimental uncertainty; to detect noncentral forces of a size expected on theoretical grounds would require an improvement in precision by several orders of magnitude.

The effects of noncentral P-wave contributions on the cross-section data and their analysis may be summarized as follows: (1) Even if such contributions were very large, they would not be distinguishable on the basis of their angular dependence. (2) If such contributions were appreciable and not well known, they would preclude the accurate determination of the other phase shifts from the experimental data. (3) If such contributions are predicted for the present experiment by OPE with an uncertainty not greater than 50%, the resulting additional uncertainty in γ is completely negligible and in K_0 is less than the experimental uncertainty.

EFFECT OF NONCENTRAL FORCES ON THE POLARIZATION

Noncentral forces cause protons scattered from an unpolarized beam by an unpolarized target to be polarized after scattering, and experiments to measure this polarization have been proposed and performed for the purpose of identifying noncentral forces. The polarization as a function of angle may be calculated from the following expression:

$$\mathcal{P}=\frac{\eta}{4k^2}\left(\frac{SX_1}{P}J_1+\frac{SY_1}{P}J_2-\frac{4CS}{\eta P}J_3\right), \tag{9}$$

where the combinations of the P-wave phase shifts which occur are given by the following:

$$J_1=-2\sin^2(\delta_0)-3\sin^2(\delta_1)+5\sin^2(\delta_2), \tag{10a}$$

$$J_2=-2\sin(\delta_0)\cos(\delta_0)-3\sin(\delta_1)\cos(\delta_1)+5\sin(\delta_2)\cos(\delta_2), \tag{10b}$$

$$J_3=9\sin(\delta_2)\sin(\delta_1)\sin(\delta_2-\delta_1)+6\sin(\delta_2)\sin(\delta_0)\sin(\delta_2-\delta_0). \tag{10c}$$

The sum of the terms in J_1 and J_2 is the Coulomb interference polarization, while the term in J_3 is the nuclear polarization. For small phase shifts, J_1, J_2, and J_3 involve the second, first, and third powers of the phase shifts, respectively, and the term in J_2 might be

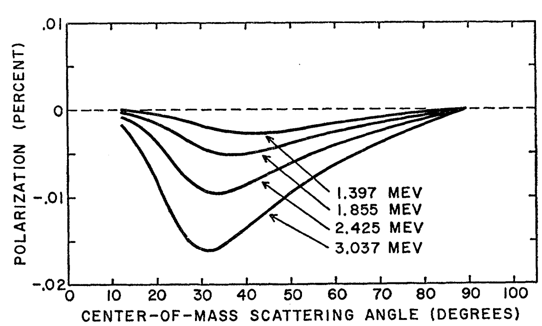

FIG. 12. Polarization implied by the preferred phase-shift solutions for the present experimental data. These are the solutions of Table III. (Note the units; polarizations are all less than 2 parts in 10 000.)

expected to be dominant. But if J_1, J_2, and J_3 are expressed in terms of γ, α, and β, it is found for small phase shifts that J_2 depends only on β; J_1 depends strongly on α and is multiplied by an angle-dependent factor an order of magnitude larger than that which multiplies J_2. Thus, pure tensor and pure spin-orbit forces produce Coulomb interference polarizations of comparable magnitude but of different angular dependence. The nuclear polarization term becomes appreciable only when the magnitude of α or β reaches a few degrees.

The polarization was calculated as a function of angle for all phase-shift solutions discussed above. The polarizations implied by the preferred solutions of Table III (i.e., predicted by OPE) are shown in Fig. 12; they are seen to be very small.

One experiment has been carried out to measure the polarization at an energy near those of the present experiment; Alexeff and Haeberli[18] performed a double-scattering experiment near 3.3 MeV which measured the polarization at the three scattering angles of 30°, 45°, and 53°. These values with the quoted uncertainties are shown as the three data points in Fig. 13. They are clearly inconsistent, both in magnitude and sign, with the OPE prediction shown in Fig. 12. To detect polarizations of the OPE-predicted magnitude, experimental

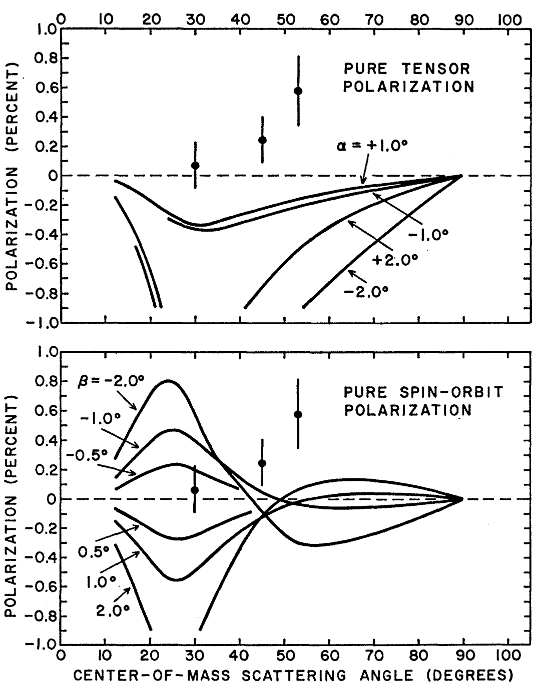

FIG. 13. Polarization produced by pure tensor and pure spin-orbit splitting of the P-wave phase shifts. Curves result from solutions to the 3.037-MeV data, some of which are listed in Table VI. The three data points are the measurements of Alexeff and Haeberli near 3.3 MeV. Curves drawn for 3.3 MeV are not significantly different from the 3.0-MeV curves shown here, being only slightly less in magnitude.

[18] I. Alexeff and W. Haeberli, Nucl. Phys. **15**, 609 (1960).

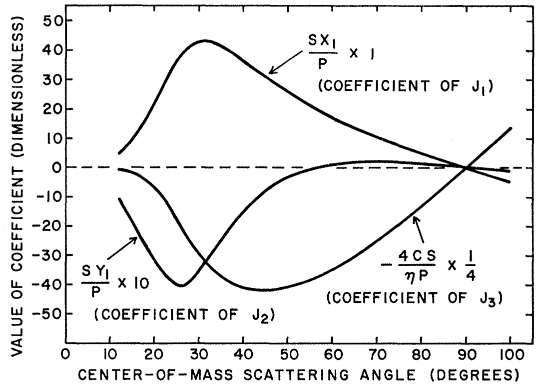

FIG. 14. Angular dependence exhibited by the three terms in the expression for the polarization. The coefficients multiplying J_2 and J_3 have been adjusted in magnitude by factors of 10 and $\frac{1}{4}$, respectively, to allow plotting all three curves on the same scale.

uncertainties would have to be reduced by more than one order of magnitude. The polarization resulting from various amounts of pure tensor and pure spin-orbit splitting has been plotted in the upper and lower parts, respectively, of Fig. 13, and it is seen that none of these curves is consistent with the data. (The solid curve shown in Fig. 6 of Ref. 18 was not claimed to, and indeed does not, represent a possible fit to the data; it shows an angular dependence found only for pure spin-orbit splitting when the nuclear polarization is dominant and has a magnitude of several percent.) The critical characteristic of the measured polarizations is a positive value at 53° in the absence of a negative value at 30°; this is inconsistent with both pure tensor splitting (which produces only negative polarization at all angles) and pure spin-orbit splitting (which can produce positive polarization at 53° only in combination with a much larger negative polarization at 30°). It remains to ask whether some combination of tensor and spin-orbit contributions can be found which is consistent with both the polarization measurements and the present cross-section measurements. Alexeff and Haeberli did not analyze their results, and a complete analysis has not been carried out here, but the following examination has been made.

Several simple arguments can be made to yield a qualitative understanding. The angular dependences of the three terms of Eq. (9) are shown in Fig. 14. In the approximation that $\sin\delta_J=\delta_J$ and $\cos\delta_J=1$ and that γ is much smaller than the larger of α and β, the following expressions result for the dependence of J_1, J_2, and J_3 on α and β:

$$J_1=24\alpha\beta-43.2\alpha^2-6\beta^2, \tag{11a}$$

$$J_2=12\beta, \tag{11b}$$

$$J_3=-106.56\alpha^2\beta+36\alpha\beta^2-51.84\alpha^3-54\beta^3. \tag{11c}$$

Several points are apparent concerning these relations and the curves of Fig. 14. First, it is not possible for J_1

to be positive, and, since the factor multiplying it is positive for all angles, this term always leads to negative polarizations. Second, the factor multiplying J_2 has a zero near 55°. Therefore, this term cannot produce any appreciable polarization at 53° without producing near 30° a polarization an order of magnitude greater, which could not effectively be cancelled by another term and could not have escaped detection in the experiment. Third, while the term in J_3 can produce appreciable positive polarization at 53° if α and/or β are large and positive, it seems unlikely that the other terms could cancel the contribution at 30°. One is thus led to suspect that no combination of tensor and spin-orbit contributions can be found which yield the polarizations measured.

Two further calculations were carried out which tend to confirm this suspicion. First, the polarization was expressed in terms of the two variables α and β and evaluated at 30° and 53°, where the polarization was assumed to be zero and 0.5%, respectively, yielding two equations in the two unknowns. It was found that no combination of α and β exists which yields exactly these polarization values. Second, since it might still be possible to fit the data, not exactly but within the error bars, the polarization was mapped over the entire α-β plane, using increments in α and β which appear to have been suitably small. No satisfactory combination of α and β was found; in confirmation of the foregoing qualitative argument, one region of the plane (β approximately $+2.5°$ and α between $+1.0°$ and $+2.0°$) yielded good fits to the measured values at 45° and 53° but yielded at 30° negative polarizations which failed to fit the measured value by about five standard deviations. Thus, while they are not entirely rigorous, these consistency arguments alone lead one to believe that the measured polarizations must somehow be in error. It may be noted, however, that if actual uncertainties were about twice the quoted values, these measurements would not be inconsistent with the expected nearly zero polarization and would serve to rule out values of α or β greater than about 1° in magnitude.

CONCLUSIONS

The cross-section values reported here result from extensive experimental measurement, checkwork, evaluation, and analysis and are believed, with some confidence, to be correct to within the quoted uncertainty. The preferred phase-shift values are regarded with equal confidence, subject to the reliability of assuming that the OPE-predicted noncentral force is accurate to 50% and of making a good choice for their quoted uncertainty.

With regard to uses already made of preliminary data from this experiment, it is pertinent to remark that wherever the reliability of theoretical conclusions has been limited by the uncertainties in the data, these should be reexamined on the basis of the final data reported here. First, the accuracy attributed to the derived phase shifts is important; KMBN quoted a reasonably large rough estimate, but others, in making their own phase-shift analyses, have usually taken for the uncertainty the too small value given by the χ^2 test. Second, the phase-shift values themselves are important; those reported here can differ from those derived from preliminary data[16] by as much as 40% of the quoted uncertainty (or 80% of the too small uncertainty given by the χ^2 test). Both of these considerations can greatly affect the force of any conclusions drawn from the data with marginal confidence.

A final comment concerns the likelihood of further experimental clarification at these energies in the near future. It was shown above that measurement of the cross section cannot be done accurately enough to yield information on noncentral forces. While a remeasurement of these cross sections would be valuable as an independent check and particularly useful if extended to higher energy, it is not likely at present that the experimental accuracy could be bettered by a factor of more than 2 or 3, even with great care and effort. Whether doubly accurate cross sections could be analyzed to yield doubly accurate phase shifts may be questioned, and comment on the value to theoretical work of such an improvement is left to others. It was also noted above that the present polarization measurement does not seem credible except as a gross upper limit. While a remeasurement would serve to clarify this discrepancy and might be done with greater accuracy now that polarized proton beams are available, the experiment is still very difficult and the expected polarizations are far smaller than one can hope to measure. It appears more likely that experimental clarification will result from the extension of precise cross-section measurements to a greater energy range, the measurement of one or more of the larger triple-scattering parameters, and possibly the conduct of certain experimental tests which have yet to be noted as critical to current theory and delineated for the experimentalist.

ACKNOWLEDGMENTS

The authors wish to thank Professor R. G. Herb, in whose laboratory the experimental measurements were made, for his advice, encouragement, and support. They are grateful to Professor H. P. Noyes and Dr. L. Heller for helpful discussions and for making available the several numerical calculations noted in the text. The continued interest of Professor G. Breit and the individuals just mentioned encouraged this consolidation and evaluation of the data. They are indebted to Harry Jenkins and Ray Morgan for valuable assistance with the numerical calculations.