pointed out by Harris[15] in his discussion on the experiments by Bekefi and Hooper[16] and by Gruber, McBee, and Shepherd[17] in which the plasma was generated by an electron beam. It is shown in this investigation that a plasma-beam model in which temperature effects are neglected can be used successfully in explaining some of the aspects of the observed harmonic emissions.

Thus, according to Ikegami,[6] the maximum intensity of the observed radiation occurs at a multiple of the electron gyrofrequency and as the discharge current is increased, the maximum intensity is progressively shifted toward higher harmonics. This observation is supported by our analysis as illustrated in the enclosed graphs. Thus when $R=0.0136$ [Fig. 1(a)] both the fundamental gyrofrequency and its harmonics are excited by the beam. However, for $R=1.36$ [Fig. 1(b)], the lowest harmonic excited by the beam is represented by $|s|=3$, whereas for $R=5.444$ [Fig. 1(c)] the lowest harmonic corresponds to $|s|=5$. The rate of growth generally decreases with increasing $|s|$ except for relatively low values of $R$. Thus, as shown in Fig. 1(a) when $\beta_\perp=v_\perp/c=0.223$ to 0.632, the rate of growth attains maximum at $|s|=2$ and then decreases when $|s|$ becomes larger.

Both Landauer[3] and Ikegami[6] observed that the harmonic emission is due primarily to the extraordinary wave ($\mathbf{E}\perp\mathbf{B}_0$). This observation is supported by our results. We have found that the extraordinary waves are the only electromagnetic waves which can be excited by the beam, i.e., the ordinary waves ($\mathbf{E}_0\|\mathbf{B}_0$) remain stationary.

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[15] E. G. Harris, General Atomic Report GA-5581, 1964 (unpublished).

[16] G. Bekefi and E. B. Hooper, Jr., Appl. Phys. Letters 4, 135 (1964).

[17] S. Gruber, W. D. McBee, and L. T. Shepherd, Appl. Phys. Letters 4, 137 (1964).

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PHYSICAL REVIEW VOLUME 148, NUMBER 1 5 AUGUST 1966

# Single-Particle Coupling to Collective Excitations; Sound in He$^3$

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(Received 8 February 1966)

In order to test the hypothesis that the observed anomalous behavior of the low-temperature specific heat of He$^3$ is due to the interaction of He$^3$ atoms with sound, the coupling of the single particles to the collective excitations must be calculated. Based on an infinite-order perturbation theory, a technique is introduced for finding the effective coupling at long wavelengths. Within this approximation scheme the coupling is found to be of the deformation-potential kind. Vertex corrections make the expression used for the single-particle self-energy exact. A singular vertex function, which is ruled out if perturbation theory is valid but which is required in order to change the coupling from deformation-potential to piezoelectric, leads to an inconsistency. All approximation schemes treated result in a deformation-potential coupling between He$^3$ atoms and sound. Treatments of this coupling have not so far led to results which can explain the observed specific heat.

## I. INTRODUCTION

OBSERVATION by Anderson[1] indicates that the recent experiments on the specific heat of He$^3$ by the Illinois group[2] appear to have a temperature- dependent density-of-states factor which becomes logarithmically singular as the temperature approaches zero. For temperatures ranging from millidegrees to tenths of degrees Kelvin, Anderson pointed out that the specific heat may be fit by a curve of the form $C=\gamma T+AT\ln B/T$.[3] In this temperature range, we may neglect the specific-heat contribution due to phonons, which is cubic in the temperature.

A suggestion proposed by Anderson was interpreted by Balian and Fredkin[4] to mean that the singular behavior of the specific heat could be accounted for by considering fermion self-energy processes of virtual emission and reabsorption of phonons of zero sound. In order to test this hypothesis, the coupling of the single particles to the collective excitations must be calculated.

In Sec. II of this paper we show (in second-order perturbation theory) how the long-wavelength coupling to collective excitations influences the specific heat. In

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* Work supported in part by a DuPont Research Grant.

[1] P. W. Anderson, Physics 2, 1 (1965).

[2] W. R. Abel, A. C. Anderson, W. C. Black, and J. C. Wheatley, Physics 1, 337 (1965).

[3] More recent experiments which repeat and analyze the specific-heat results are in W. R. Abel, A. C. Anderson, W. C. Black, and J. C. Wheatley, Phys. Rev. Letters 15, 875 (1965).

[4] R. Balian and D. Fredkin, Phys. Rev. Letters 15, 480 (1965).

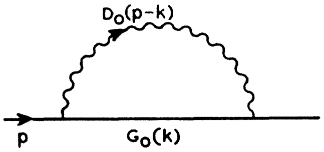

FIG. 1. Second-order self-energy diagram for single-particle–phonon interaction.

Sec. III, a technique based on an infinite-order perturbation theory is introduced for finding the effective coupling of these excitations, at long wavelengths, to the He$^3$ atoms. Vertex corrections make the expression for the single-particle self-energy exact. For a nonsingular vertex function, the coupling to zero sound is found to be deformation potential in nature. We show that a singular vertex function, which is required in order to change the coupling from deformation potential to piezoelectric, leads to an inconsistency for a class of self-energies. The inconsistency is that in this case there is no solution to the zero-sound dispersion relation.

## II. EFFECT OF THE COLLECTIVE MODES ON THE SPECIFIC HEAT

In order to show the importance of the long-wavelength coupling to collective excitations, we consider the following Hamiltonian (throughout the paper we choose units in which $\hbar=1$):

$$H_{\text{eff}}=\sum_{\mathbf{k},\sigma}\epsilon_k c_{\mathbf{k},\sigma}{}^\dagger c_{\mathbf{k},\sigma}+\sum_{\mathbf{k}}\omega_{\mathbf{k}}b_{\mathbf{k}}{}^\dagger b_{\mathbf{k}}$$
$$+\sum_{\mathbf{k},\mathbf{p},\sigma}\frac{g_k}{(2\omega_k)^{1/2}}c_{\mathbf{p}+\mathbf{k},\sigma}{}^\dagger c_{\mathbf{p},\sigma}(b_{\mathbf{k}}+b_{-\mathbf{k}}{}^\dagger). \quad (1)$$

This Hamiltonian describes free fermions, free phonons taken to behave as a boson field, and an interaction between them. The $c$'s are the fermion operators and $b$'s the phonon operators. The bare fermion single-particle energy measured relative to the Fermi energy $\mu$ is $\epsilon_k\equiv k^2/2m^*-\mu$. The bare phonon single-particle energy is $\omega_k\equiv sk$. By looking at the expression for the fermion self-energy in second order, we will show how the singular nature of this contribution is determined by the functional dependence of $g_k$ on $k$ as $k$ approaches zero. The fermion self-energy, shown in Fig. 1, to second order in $g$ is given by

$$\Sigma(p,p_0)=i\int\frac{d^4k}{(2\pi)^4}g_{p-k}{}^2D_0(\mathbf{p}-\mathbf{k})G_0(\mathbf{k})$$
$$=i\int\frac{d^4k}{(2\pi)^4}g_{p-k}{}^2\frac{1}{(p_0-k_0)^2-\omega_{p-k}{}^2+i\delta}$$
$$\times\frac{1}{k_0-\epsilon_k+i\eta\epsilon_k}\quad \delta\rightarrow 0^+,\ \eta\rightarrow 0^+, \quad (2)$$

where $D_0$ is the lowest order phonon propagator and $G_0$ is the lowest order fermion propagator. Although the integral can be done analytically, we will use a power counting argument to show how the singular part of $\Sigma$ is determined. A singularity will be obtained if we can get the denominator of the integrand to vanish at one of the limits of integration. In the language of theoreticians who determine singularities of Feynman diagrams these are called end-point singularities.

Perform the $k_0$ integration so that the pole of $G_0$ is not enclosed. This leads to a sum of two terms both having $\epsilon_k=0$ as one end point. One of the terms may be written as

$$\operatorname{Re}\Sigma(p,p_0)\sim\int_{\epsilon_k<0}d^3k\, g_{p-k}{}^2\frac{1}{p_0+\omega_{p-k}-\epsilon_k}\frac{1}{2\omega_{p-k}}. \quad (3)$$

Since we are interested in the self-energy at the Fermi surface $p_0\rightarrow 0$, $p\rightarrow p_f$, there is a zero of the denominator at the end-point $k\rightarrow p_f$. To continue, change variables of integration from the cosine of the angle between $\mathbf{p}$ and $\mathbf{k}$, namely $x$, to an integration over the magnitude of $\mathbf{p}-\mathbf{k}$,

$$q^2=|\mathbf{p}-\mathbf{k}|^2,\quad q\,dq=-pk\,dx.$$

With this replacement we find that

$$\operatorname{Re}\Sigma(p,p_0)\sim\int_{\epsilon=0}d\epsilon\int_{q=0}dq\, g_q{}^2[p_0+sq-\epsilon]^{-1}$$
$$\sim\int_{q=0}dq\, g_q{}^2\ln(p_0+sq). \quad (4)$$

If $g_q{}^2$ goes to a constant for $q$ going to zero, which in solid-state physics is called piezoelectric coupling, then the self-energy for $p_0$ near zero is of the form

$$\operatorname{Re}\Sigma(p,p_0)\sim p_0\ln p_0, \quad (5)$$

as Balian and Fredkin[4] obtained. The careless treatment of the regions of integration in the above counting argument leaves out the momentum dependence of the singularity.

If, on the other hand,

$$\lim_{q\rightarrow 0}g_q{}^2\sim q^2, \quad (6)$$

which is the more commonly found coupling of single particles to sound, usually called deformation-potential coupling by solid-state physicists, then the singular behavior of the fermion self-energy is[5]

$$\operatorname{Re}\Sigma(p,p_0)\sim p_0{}^3\ln p_0. \quad (7)$$

For piezoelectric coupling, Balian and Fredkin[4] point out that lowest order perturbation theory leads to a specific heat

$$C_v\sim\gamma T+AT\ln|B/T|. \quad (8)$$

For deformation-potential coupling, Eliashberg[6] has shown that the lowest order contribution is

$$C_v\sim\gamma T+A'T^3\ln|B'/T|. \quad (9)$$

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[5] A. B. Migdal, Zh. Eksperim. i. Teor. Fiz. **40**, 684 (1961) [English transl.: Soviet Phys.—JETP **13**, 478 (1961)]; S. Engelsberg and J. R. Schrieffer, Phys. Rev. **131**, 993 (1963).

[6] G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. **43**, 1005 (1962) [English transl.: Soviet Phys.—JETP **16**, 780 (1963)].

## III. COUPLING TO THE COLLECTIVE EXCITATIONS

We now turn to the basic problem of determining the coupling of He$^3$ atoms to sound. To solve this problem we must understand precisely what we mean by writing the Hamiltonian (1) for the system in terms of the single-particle excitations and the collective excitations, the phonons. Normally one would start with a Hamiltonian which takes into account the single-particle energies of the He$^3$ atoms and the interaction between them. If we take the interaction to depend only on the distance between atoms, then

$$H=\sum_{\mathbf{k},\sigma} \epsilon_k c_{\mathbf{k},\sigma}{}^\dagger c_{\mathbf{k},\sigma}+\tfrac{1}{2}\sum_{\mathbf{k}} V_k \rho_{\mathbf{k}} \rho_{-\mathbf{k}}\,, \tag{10}$$

where the density fluctuation operator

$$\rho_{\mathbf{k}}=\frac{1}{\sqrt{\Omega}}\sum_{\mathbf{p},\sigma} c_{\mathbf{p}+\mathbf{k},\sigma}{}^\dagger c_{\mathbf{p},\sigma}\,. \tag{11}$$

The Fourier transform of the interparticle potential is $V_k$, and $\Omega$ is the volume of the system. For a hard core potential, where $V_k$ is infinite, we will replace $V_k$ for small $k$ by a number $2\pi a_0/m$, a pseudopotential independent of $k$. This pseudopotential describes the low-energy helium atom–helium atom scattering and is often called the $s$-wave scattering length.[7]

Impressing a long-wavelength density fluctuation on the system implies, if there are soundlike excitations, that we have created a state having a well-defined energy which varies linearly with the momentum of the density fluctuation. For the purpose of our present discussion, we will assume that there are soundlike excitations, and we will make no distinction between zero sound and ordinary sound. Landau[8] has shown that a system of fermions interacting via short-range repulsive interactions will support such soundlike excitations. Gottfried and Picman[9] have given a precise formulation of the problem. They pointed out that such an excitation may be found as a pole in the density autocorrelation function. When we speak of the fermion self-energy due to emission and reabsorption of a phonon, we mean that starting with the particle-particle Hamiltonian (10) we sum that infinite set of diagrams which includes as a contribution to the integral a pole corresponding to the propagation of sound.

In evaluating and discussing the analytical properties of the He$^3$ single-particle self-energy, we will first sum the set of diagrams shown in Fig. 2. The shaded regions are the irreducible polarization parts $P(k)$, and the dashed lines represent the interaction $V_k$. The solid line internal to the self-energy diagram is the dressed He$^3$ propagator. Within the framework of this approximation the self-energy of the He$^3$ atom is

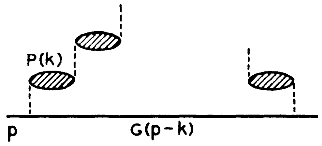

FIG. 2. Self-energy diagram for particle-particle interactions, which includes a contribution from sound-like excitations. The irreducible polarization part $P(k)$ is represented by the shaded regions and recurs an infinite number of times between the dashed lines which represent the interaction $V_k$.

$$\Sigma(p)=-i\int\frac{d^4k}{(2\pi)^4}G(p-k)\frac{V_k}{1-V_kP(k)}\,, \tag{12}$$

where

$$G(p)=[p_0-\epsilon(p)-\Sigma(p)+i\delta\,\mathrm{sgn}p_0]^{-1}. \tag{13}$$

The quantity

$$\frac{V_k}{1-V_kP(k)}\equiv V_k{}^2\left[\frac{P(k)}{1-V_kP(k)}\right]+V_k\,, \tag{14}$$

where

$$S(k)\equiv\frac{P(k)}{1-V_kP(k)}=\int_{-\infty}^{\infty}dt\,e^{+ik_0t}[-i\langle T(\rho_{\mathbf{k}}(t)\rho_{-\mathbf{k}}(0))\rangle - \langle\rho_{\mathbf{k}}(t)\rangle\langle\rho_{-\mathbf{k}}(0)\rangle], \tag{15}$$

defines the irreducible polarization part $P(k)$. This may be seen either diagrammatically or by use of functional derivative techniques, outlined in the Appendix. The poles of $S(k)$, the collective excitations, occur in Eq. (12) for the self-energy. This is true in the exact expression for the self-energy as well.

At this point, the diagram summation we have chosen is only an approximation. We have not included all possible diagrams. However, those we have chosen contain as a contribution to the self-energy integral a pole which corresponds to a coupling to a collective mode of the system.

It is a simple matter to determine the effective coupling for long wavelengths to the collective modes. We close the contour in the energy integration of (12) so as to avoid the poles of $G$ and pick up only the contribution from the poles of $V_k/[1-V_kP(k)]$. This term should have a pole corresponding to a collective mode at long wavelengths if we are to be correct in our assumption that soundlike excitations exist. The coupling constant given in (2) is related to the residue at the pole by

$$g_k{}^2/2\omega_k=-1/[(\partial/\partial k_0)P(\mathbf{k},k_0)]. \tag{16}$$

An explicit approximation for the polarizability $P$ will be given later. However, we will show that one need only have the functional form of $P$ in order to determine the $k\rightarrow 0$ limit of $g_k{}^2$. The polarizability $P(\mathbf{k},k_0)$ may be rewritten as

$$P(k,k_0)\equiv\Pi(\mathbf{k}/k_0,\,k/2p_f)\,. \tag{17}$$

The Fermi momentum $p_f$ is used here just as a scaling factor. In the limit $\mathbf{k}/2p_f\rightarrow 0$, we expect $\Pi(\mathbf{k}/k_0,\,\mathbf{k}/2p_f)$

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[7] J. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), pp. 60–61. Reference 9 also uses this convention.

[8] L. D. Landau, Zh. Eksperim. i Teor. Fiz. **32**, 59 (1957) [English transl.: Soviet Phys.—JETP **5**, 101 (1957)].

[9] K. Gottfried and L. Picman, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **32**, No. 13 (1960).

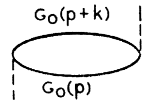

FIG. 3. Lowest order contribution to the irreducible polarization part, often denoted as the random-phase-approximation contribution.

$-\Pi(\mathbf{k}/k_0, 0)$ to approach zero. The correction terms, although small, need not necessarily be analytic in $\mathbf{k}/2p_f$. If $\Pi$ did not exist at long wavelengths, that is, if there were a pole of some kind at the origin, then the sound mode would not exist. The dispersion relation $1-V_k\Pi=0$ could not be satisfied. Infinite-order perturbation calculations which have been performed do not lead to such singularities.[10] With these restrictions, $\Pi$ may be written as

$$\lim_{k/2p_f\to 0}\Pi(\mathbf{k}/k_0, k/2p_f)=\Pi^{(0)}(\mathbf{k}/k_0)+O[(\mathbf{k}/2p_f)^{\alpha}], \quad (18)$$

with $\alpha>0$. Given the form of (18), the residue at the pole in the long-wavelength limit is easily computed using (16)

$$\lim_{k\to 0} g_k^2 = 2sk^2/\Pi^{(0)\prime}(1/s). \quad (19)$$

The coupling at long wavelengths is deformation potential in character as long as $\Pi^{(0)\prime}(1/s)$ does not vanish. The value of the sound velocity may be varied over a finite range by varying the coupling constant. The derivative $\Pi^{(0)\prime}(x)$ is not related to the velocity of sound. As we have seen, the velocity of sound is determined by the equation

$$\Pi^{(0)}(1/s)=m/2\pi a_0.$$

Therefore, it is impossible for the derivative $\Pi^{(0)\prime}(1/s)$ to vanish for a finite range of $s$.

The above argument does not rule out an accidental zero of $\Pi^{(0)\prime}$ for $k_0/k$ at the sound velocity of the given system. If this were the case, however, we could change the experimental results completely by altering the velocity of sound very slightly. This may be done by changing the pressure of the He$^3$ by a very small amount. The experiments[2,3] included such a variation, but no significant change in the form of the specific heat was observed.

To illustrate the above ideas in lowest order perturbation theory, we compute the random-phase-approximation contribution to $P$ shown in Fig. 3 and generated from (A6), (A8), and (A9) by the lowest order approximation $\Gamma(p,k)\approx -1$, $G\approx G_0$. This approximation gives the Lindhardt dielectric function[11]

$$P(k,k_0)\approx -2i\int\frac{d^4p}{(2\pi)^4}G_0(p+k)G_0(p) = \Pi^{(0)}(k/k_0)+O[(k/2p_f)^2], \quad (20)$$

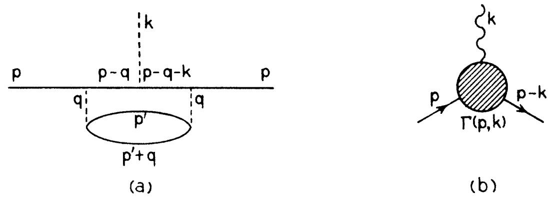

FIG. 4(a). Low-order vertex correction; (b) most general vertex correction which makes the expression for the self-energy exact. The wavy line labeled $k$ represents the effective potential $A(k)$ given by the Fourier transform of Eq. (A4).

where

$$\Pi^{(0)}\left(\frac{k}{k_0}\right)=-\frac{mp_f}{2\pi^2}\left(2-x\ln\left|\frac{x+1}{x-1}\right|\right), \quad (21)$$

with

$$x=k_0m/p_fk.$$

Thus, the sum of bubbles alone satisfies (18) and leads to deformation-potential coupling.

In summing the diagrams shown in Fig. 2, we have omitted all the vertex corrections of the form shown in Fig. 4(a) or, more generally, those of Fig. 4(b). The inclusion of such vertex corrections modifies (12), so that

$$\Sigma(p)=+i\int\frac{d^4k}{(2\pi)^4}V_kG(p-k)\frac{\Gamma(p,k)}{1-V_kP(k)}. \quad (22)$$

This expression for the self-energy is exact [see (A7)]. If the vertex function is singular as $k\to 0$, with $k_0/k=s$, then our conclusions in (19) are incorrect. In order to have the coupling to phonons $g_k$, go to a constant, we must have

$$\lim_{k\to 0,\,k_0/k=s}\Gamma(p,k)\sim 1/k^2. \quad (23)$$

Perturbation-theory calculations do not lead to this singular behavior.

However, let us assume that anomalous behavior given in (23); it is possible to show that such an assumption leads to a contradiction. We assume piezoelectric coupling due to the form (23) of the vertex function and obtain a singular self-energy [see (5)].

To calculate $P(k)$, given (A9), in the limit $k\to 0$, $k_0/k=s$ we factor out the $1/k^2$ the vertex function contributes. Thus

$$P(k)\sim\frac{2i}{k^2}\int\frac{d^4p}{(2\pi)^4}G(p+k)G(p). \quad (24)$$

No additional anomalous behavior of $\Gamma(p,k)$ as a function of $p$ in the neighborhood of the Fermi surface is included. For self-energies of the form

$$\Sigma(\mathbf{p},p_0)=-p_0\varphi(\epsilon(p)),$$

where $\varphi$ is an arbitrary function, the result is of the form

$$\lim_{k\to 0,\,k_0/k=s}P(k)\sim(1/k^2)f(s), \quad (25)$$

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[10] A. J. Glick, Phys. Rev. **129**, 1399 (1963); D. F. DuBois, Ann. Phys. (N.Y.) **7**, 174 (1959); **8**, 24 (1959).

[11] J. Lindhardt, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **28**, No. 8 (1954).

where $f(s)$ is a function which is not identically zero and represents the value of the integral.[12] Thus we cannot obtain a solution to the dispersion relation for soundlike excitations

$$\lim_{k\to 0,k_0/k=s}(1-V_kP(k))\sim 1/k^2,$$

and the contradiction is apparent.

As before, we may be at a particular value of $s$ for which $f(s)$ in (25) accidentally vanishes. In this case we need even better luck to have that value of $s$ satisfy

$$1-V_kP(k)=0.$$

Once again these extraordinary results would change drastically under the slight application of pressure.

The above analysis prescribes some of the conditions that the Green's function and vertex function must satisfy in order to obtain piezoelectric coupling without an inconsistency. Perturbation theory and the proposal of Balian and Fredkin[4] do not satisfy these conditions; on the other hand, we have not shown that it is impossible.

From a qualitative point of view it is clear that the coupling to soundlike excitations must go to zero as the wavelength of sound approaches infinity.[13] The excitation of phonons with wave vectors approaching zero corresponds to a uniform translation of the system. The scattering of He$^3$ atoms from these phonons must go to zero since the force between He$^3$ atoms is not of infinite range. A He$^3$ atom cannot distinguish a small uniform translation from no translation at all. Hence the coupling to the phonon displacement field vanishes.[14]

## IV. CONCLUSIONS AND DISCUSSION

We have presented a method by which the coupling of collective excitations to single particles may be calculated within an infinite-order perturbation approximation. Such a technique may often be useful for choosing an approximate Hamiltonian which describes the effects of collective excitations on the single-particle behavior. An example which may be calculated using this technique is the coupling of electrons to plasmons.

We have used this technique to calculate the coupling of soundlike excitations to He$^3$ atoms and found that the coupling $g_k{}^2\sim k^2$ at long wavelengths. The conclusion is that the coupling to zero sound in lowest order perturbation theory cannot lead to a contribution singular enough to correspond to Anderson's suggestion that the specific heat of He$^3$ has a $T\ln T$ term.

## ACKNOWLEDGMENTS

We would like to thank Dr. P. W. Anderson for several very helpful and stimulating discussions. The interest and knowledgeable comments of Dr. S. J. Buchsbaum, Dr. P. Hohenberg, Professor J. Luttinger, and Professor G. Tiktopoulos have also been very much appreciated.

## APPENDIX A

Using the functional derivative techniques developed by Schwinger,[15] we outline here the derivation of the relations between the functions introduced in this paper. These relations are implicit in a paper by Martin and Schwinger.[16] We include this Appendix as a guide for the definition of the various functions. To the Hamiltonian (10) we add a driving term

$$\sum_\sigma \int d^3r\, U(\mathbf{r})\psi_\sigma{}^\dagger(\mathbf{r})\psi_\sigma(\mathbf{r})=\int d^3r\, U(\mathbf{r})\rho(\mathbf{r}), \quad \text{(A1)}$$

where $U(\mathbf{r})$ is an externally applied potential. The Fourier transform of the density autocorrelation function

$$\tilde{S}(x,x')=-i\{\langle T(\rho(x)\rho(x'))\rangle-\langle\rho(x)\rangle\langle\rho(x')\rangle\} \quad \text{(A2)}$$

may be written in terms of the Fourier transform of the irreducible polarization part,

$$\tilde{P}(x,x')=2i\int d^4x_1d^4x_2\tilde{G}(x,x_1)\frac{\delta G^{-1}(x_1,x_2)}{\delta A(x')}\tilde{G}(x_2,x^+), \quad \text{(A3)}$$

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[12] Using a self-energy of the form $\Sigma(\mathbf{p},p_0)=-p_0\varphi(\epsilon(p))$, and taking the limits indicated in (25) we find

$$f(s)=\frac{1}{\pi^2}\int_{\frac{-\mu}{1+\varphi(-\mu)}<E<0} p^2dp\frac{1}{1+\varphi(\epsilon)}$$
$$\times\left\{\left[\frac{1}{m}\frac{\partial E}{\partial\epsilon}+s^2\left(\frac{\partial^2E}{\partial\epsilon^2}\right)\Big/\left(\frac{\partial E}{\partial\epsilon}\right)^2\right]\left(1\Big/s^2-\left(\frac{p}{m}\frac{\partial E}{\partial\epsilon}\right)^2\right)\right.$$
$$+\left(\frac{\partial^2E}{\partial\epsilon^2}\Big/\left(\frac{\partial E}{\partial\epsilon}\right)^2\right)+\frac{sm}{p}\left(\frac{\partial^2E}{\partial\epsilon^2}\Big/\left(\frac{\partial E}{\partial\epsilon}\right)^3\right)$$
$$\left.\times\ln\left|\left(s-\frac{p}{m}\frac{\partial E}{\partial\epsilon}\right)\Big/\left(s+\frac{p}{m}\frac{\partial E}{\partial\epsilon}\right)\right|\right\}$$

with $\epsilon=(p^2/2m)-\mu$, $E=\epsilon/[1+\varphi(\epsilon)]$. Since $s$ and the chemical potential $\mu$ are independent parameters, we can see that $f(s)$ is not identically zero. This may be shown, for example, by looking at the power-series expansion of the integrand for large $s$.

The proposal of Balian and Fredkin corresponds to setting

$$\varphi(\epsilon)=+[2\alpha(1+v/c)\ln|B/\epsilon|]^{1/2}.$$

[13] There is a discussion of the deformation-potential theorem in solids given by P. W. Anderson, in *Concepts in Solids* (W. A. Benjamin, Inc., New York, 1963), pp. 116–119.

[14] It is perhaps of interest to mention that some time ago one of the authors (S. E.) noted that an electron coupling to phonons which went to a constant for long wavelengths resulted in the singular behavior given in (5). At that time it was noted that this singularity could lead to a fit of the observed tunneling anomalies in metals and semiconductors (Ref. 15) and also lead to specific-heat anomalies. However, as in the case of He$^3$, the impossibility of such coupling in the materials used ruled out this as a possible explanation. One reason such coupling is impossible in metals and heavily doped semiconductors was pointed out by Dr. C. Herring (private communication). Such coupling implies that an almost uniform ionic displacement (creation of a phonon) causes an infinite-range electric field. The presence of amost free electrons will screen this field out. In fact this is precisely the qualitative argument given above. Of course such behavior can be found in some insulating crystals, which are then called piezoelectric. We have received a report prior to publication of work by G. Mahan and C. Duke which discusses the above idea in relation to tunneling in semiconductors.

[15] A. Wyatt, Phys. Rev. Letters **13**, 401 (1964); R. Logan and J. Rowell, *ibid.* **13**, 404 (1964).

[16] J. Schwinger, Proc. Natl. Acad. Sci. (U.S.) **37**, 452 (1951).

[17] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959), (see p. 1371).

where $x^+$ denotes $(\mathbf{x}, t+0^+)$. The effective field $A(x)$ generated by the external potential $U(x)$ is given by

$$A(x)=U(x)+\int d^3x''\, V(|\mathbf{x}-\mathbf{x}''|)\langle\rho(x'')\rangle. \quad \text{(A4)}$$

The reason for calling $P$ the irreducible polarization part can be seen diagrammatically or from the equation for the change in the potential $A$ generated by a change in the external potential,

$$\begin{aligned}\tilde{K}(x,x')&\\ &=\frac{\delta A(x)}{\delta U(x')}=\delta(x-x')+\int d^3x_1\, V(|x-x_1|)\tilde{S}(x_1,x')\\ &=\delta(x-x')+\int d^3x_1 d^4x_2\, V(|x-x_1|)\tilde{P}(x_1,x_2)\tilde{K}(x_2,x').\end{aligned}$$

Thus the quantity usually denoted as the inverse dielectric function is given by

$$K(k)=[1-V_kP(k)]^{-1}. \quad \text{(A5)}$$

The equation for the single-particle Green's function may be written as

$$\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}-A(x)\right)G(x,x') - \int d^4x_1\, G(x,x_1)\Sigma(x_1,x')=\delta(x-x'), \quad \text{(A6)}$$

where the self-energy is

$$\begin{aligned}\Sigma(p)&=\int d^4(x-x')\, e^{-ip\cdot(x-x')}\tilde{\Sigma}(x,x')\\ &=+i\int \frac{d^4k}{(2\pi)^4}V_kG(p+k)\Gamma(p,k)K(k). \quad \text{(A7)}\end{aligned}$$

The vertex function is defined by the equation

$$\begin{aligned}\Gamma(p,k)=&\int d^4(x-x')d^4(x-z)\\ &\times e^{-ip\cdot(x-x')-ik\cdot(x-z)}\frac{\delta\tilde{G}^{-1}(x,x')}{\delta A(z)}. \quad \text{(A8)}\end{aligned}$$

We note from (A3) that the irreducible polarization part may be written as

$$P(k)=2i\int \frac{d^4p}{(2\pi)^4}G(p+k)G(p)\Gamma(p,k). \quad \text{(A9)}$$

The equation for $\tilde{S}(x,x')$ is obtained by calculating

$$\begin{aligned}\tilde{S}(x,x')&=\frac{\delta}{\delta U(x')}\sum_\sigma\langle\psi_\sigma^\dagger(x)\psi_\sigma(x)\rangle=-2i\frac{\delta}{\delta U(x')}\tilde{G}(x,x^+)\\ &=2i\int d^4x_1d^4x_2d^4x_3\,\tilde{G}(x,x_1)\frac{\delta\tilde{G}^{-1}(x_1,x_2)}{\delta A(x_3)}\\ &\qquad\times\tilde{G}(x_2,x^+)\frac{\delta A(x_3)}{\delta U(x')}, \quad \text{(A10)}\end{aligned}$$

where the factor of 2 arises from the sum over spins. Using (A5), (A8), and (A9), we may write the density autocorrelation function as

$$S(k)=P(k)/[1-V_kP(k)]. \quad \text{(A11)}$$

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PHYSICAL REVIEW VOLUME 148, NUMBER 1 5 AUGUST 1966

# Theory of Liquid Helium Three

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(Received 21 February 1966)

The present work contains a detailed study of two types of heat-capacity anomalies present in liquid He$^3$. One of these is the plateau formation at intermediate pressures, $p \lesssim 10$ atm. The other anomaly consists in the appearance of a flat constant-pressure heat-capacity maximum, followed by a shallow minimum, between the temperatures of 0.15 and 0.30°K, and the pressures of 10 and 30 atm, approximately. At somewhat higher pressures, over a small pressure range, only the heat-capacity maximum becomes observable. These heat-capacity anomalies originate with the competition of the thermal excitations of the spin and nonspin degrees of freedom, within the theoretical approach advanced here. The observation of the predicted low-amplitude heat-capacity extrema in compressed liquid He$^3$ may require heat-capacity measurements of increased accuracy, preferably at higher pressures. The problem of the heat-capacity behavior at low and very low temperatures has also been explored under a restrictive hypothesis. It will be assumed that the nature of the thermal excitations which appear in incontrovertible and independent heat-capacity data, available only at saturation and at temperatures above 0.2°K, persists on the spin and nonspin degrees of freedom down to the absolute zero. With the recently established universal character of the nuclear-paramagnetic-susceptibility ratio law of liquid He$^3$, down to 0.05°K, the stated assumption leads one to predict that the ratio of the total heat capacity to the temperature approaches parabolically in temperature its finite limit at the absolute zero. In the present theory, deviations from this behavior require a new paramagnetic-susceptibility law at the lowest temperatures.

## 1. INTRODUCTION

IN our initial approach[1] toward a theoretical formulation of the equilibrium thermal properties of liquid He$^3$, it was possible to prove that the observed peculiar shape of the saturated-liquid heat capacity as a function of the temperature had to arise with the competitive behavior of two component sets of degrees of freedom. One of these sets referred to the nuclear-spin

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[1] L. Goldstein, Phys. Rev. **96**, 1455 (1954).