# Bound States and Bootstraps in Field Theory

MICHAEL M. BROIDO*†
*Courant Institute of Mathematical Sciences, New York*
*and*
*Palmer Physical Laboratory, Princeton University, Princeton, New Jersey*

AND

JOHN G. TAYLOR*
*Physics Department, Rutgers, The State University, New Brunswick, New Jersey*
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We present a general, model-independent theory of composite particles. Starting from a general local Lagrangian containing an elementary particle, we show how this particle becomes composite (in a precisely defined sense) as its wave-function renormalization constant tends to zero. As this limit is taken, the usual dynamical type of field equation changes its form. We show that the Green's functions of the theory still possess the usual physical interpretation in the limit; in particular, for renormalizable theories, if a certain coupling constant does not tend to zero, the "composite" propagator possesses a pole at the composite mass and with nonzero residue. We show that the operator form of the Green's-function equations describing the composite particle can be manipulated by natural approximations to explain a wide variety of known model-dependent results on composite particles, and in particular to obtain all the usual types of bootstrap equations. We present a preliminary classification of the operator equations, showing that many of them cannot possess physically meaningful solutions. Our results agree qualitatively with earlier, heuristic discussions. Finally we make some remarks about the possibility of "bootstrapped symmetry schemes."

## 1. INTRODUCTION

QUANTUM field theory has been very successful in numerous areas of elementary-particle physics, especially in quantum electrodynamics. It also serves as a useful heuristic framework in the discussion of both strong- and weak-interaction symmetries. It may now be possible to prove that the field equations have solutions for a certain class of interactions.[1] There are, however, two main difficulties in the use of field theory in elementary-particle physics. The first of these difficulties is to set up a general method for obtaining numerical solutions to the field equations for strong or weak interactions; the second is to know which strong or weak interactions to use. There are many possible interactions, all consistent with known conservation laws. Because we cannot yet obtain numerical solutions for any of these interactions we cannot appeal to nature to single out one or another of them. In particular, we cannot say at present which particles are to be taken as elementary and which as composite.

A partial solution to the computation problem has been given by the analytic $S$-matrix approach.[2] This emphasizes analyticity and unitarity of $S$-matrix elements, and enables equations for these quantities to be set up which contain on-mass-shell quantities only. In contrast, the equations of field theory (Green's-function equations) involve off-the-mass-shell quantities.[3] It has recently been shown[4] that field theory and $S$-matrix theory both have the same maximal analyticity (on the mass shell), so that we may regard them as different approaches to the same problem emphasizing different properties of $S$-matrix elements.

It has been possible to discuss the problem of which particles are elementary, which composite, in $S$-matrix theory,[2] and even to consider the case of a system composed only of composite particles.[5] Such a system of "bootstrapped" particles may have certain natural symmetries,[6] similar to those found recently in the strong interactions.[7]

From the equivalence of $S$-matrix theory and field theory, it should be possible to discuss bound states and bootstraps in field theory. In fact, we should be able to discuss these things without making any such approxi-

* Partly supported by the National Science Foundation at Rutgers University.

† Present address: Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, New York.

[1] J. G. Taylor, in Proceedings of the Third Eastern U. S. Theoretical Physics Conference, Maryland, 1964 (unpublished). See also the articles by A. Jaffe, E. Nelson, K. Symanzik, J. G. Taylor, and A. Wightman, in *The Proceedings of the Cambridge Conference on Mathematical Problems in the Theory of Elementary Particles, 1965* (M.I.T. Press, Cambridge, Massachusetts, to be published).

[2] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin and Company, New York, 1961); M. Jacob and G. F. Chew, *Strong Interaction Physics* (W. A. Benjamin and Company, New York, 1964).

[3] J. G. Taylor, Nuovo Cimento Suppl. **1**, 857 (1963).

[4] J. G. Taylor, Phys. Rev. **136**, B1134 (1964).

[5] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 349 (1961).

[6] These have been discussed in varying degrees of approximation, in the following papers: E. Abers, F. Zachariasen, and C. Zemach, Phys. Rev. **132**, 183 (1963); R. H. Capps, *ibid.* **132**, 2749 (1963); R. H. Capps, Phys. Rev. Letters **10**, 312 (1963); C. de Celles and J. E. Paton, *ibid.* **11**, 251 (1963); L. Cook, C. J. Goebel, and B. Sakita, Argonne report (unpublished); L. Cook and J. E. Paton, Princeton University report (unpublished); R. Cutkosky, Phys. Rev. **131**, 1888 (1963); J. S. Dowker and J. E. Paton Nuovo Cimento **30**, 450 (1963); J. Fulco and D. Wong, University of California report (unpublished); R. Hwa and H. Patil, Institute for Advanced Study report (unpublished). There have also been attempts to understand symmetry breaking by similar methods in R. Cutkosky and P. Tarjanne, Phys. Rev. **132**, 1355 (1963); R. Dashen and S. Frautschi, *ibid.* **137**, B1331 (1965).

[7] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962). Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).

mations (e.g., keeping only two-particle intermediate states) as are necessary in the present formulation of $S$-matrix theory. This is what we will do in this paper.

The problem of describing composite particles in field theory has been discussed already in numerous publications.[8] The general method of introducing quasi-local and local fields for composites was discussed by Haag and Zimmerman.[9] We use that method here. The more detailed problem of distinguishing between elementary and composite particles has been discussed for many model Lagrangian field theories.[8] The main result of these papers is that in suitable model theories an elementary particle becomes composite as its wave-function renormalization constant tends to zero. We wish to extend this result so that it is independent of any particular model.

A self-consistent set of equations in field theory, having no elementary particles, has been suggested by Salam.[10] The equations are derived under the condition that all vertex and wave-function renormalization constants vanish. We consider that system of equations here, and conclude that it is doubtful whether a nontrivial solution of these equations exists.

An alternative self-consistent set of equations may be set up under the weaker condition that only the wave-function renormalization constants for all particles vanish. We regard these equations as the complete form of the bootstrap equations which have been used in the two-particle approximation to generate symmetries.[6] These complete bootstrap equations are of a different type from that discussed in field theory so far.

The plan of the paper is to give, in the next section, a discussion of elementary and composite particles in field theory. This will enable us to define our terms precisely. In Sec. 3 we discuss the relation between these elementary and composite particles, and try to display both types of particle in field theory so that they appear in as similar a manner as possible. This section contains a good deal of discussion connected with the detailed behavior of the theory in the limit of the vanishing of the wave-function renormalization constant for the composite particle. In the fourth section we consider whether the vertex-function renormalization constant for a composite particle may be set equal to zero. The fifth section reviews consequences of these possibilities for the classification of particles as elementary or composite. In Sec. 6 we discuss how bootstrap equations arise in our general operator formalism, and how they reduce (by taking appropriate approximations) to the conventional bootstraps.

In Sec. 7 we give a preliminary classification of our operator bootstraps, and finally we make some remarks about the way "bootstrapped symmetries" arise in our theory. Three appendices are devoted to details of the mathematical discussion in Sec. 7.

## 2. ELEMENTARY AND COMPOSITE PARTICLES

In this section we will give our definitions of the terms "particle," "elementary particle," and "composite particle." We wish to do physics rather than metaphysics, so we must give these definitions in some reasonably precise physical framework. We take this framework to be local Lagrangian quantum field theory. By this we mean that there is a basic set of fields $\psi_1(x)\cdots\psi_N(x)$ for finite $N$, which satisfy a set of partial differential equations derived from a local Lagrangian in the fields $\psi_1\cdots\psi_N$. We assume further that the solutions to these equations satisfy the axioms of Wightman[11]; basically, the energy is positive and the $\psi_r(x)$ are local operator-valued distributions defined on suitable domains of some separable Hilbert space $\mathcal{H}$. We make no assumptions about the discrete spectrum of $P_\mu{}^2$ since that is given by the Lagrangian. Finally we assume that the Green's functions derivable from the fields exist. It is possible that there are no theories satisfying the above requirements, though it is likely that there are.[1]

In this framework we may consider the singularities of the Green's functions in their momentum variables. We say that there is a particle of mass $m$, composed of the fields $\psi_{\alpha_1}\cdots\psi_{\alpha_m}$ if all the Green's functions

$$G(p,q)\equiv\int\prod dx_i dy_j e^{-i\Sigma(p_ix_i-q_jy_j)} \times\langle 0|T\prod_{i=1}^{m}\psi_{\alpha_i}(x_i)\prod_{j=1}^{n}\psi_{\beta_j}(y_j)|0\rangle$$

have a simple pole at $(\sum p_i)^2=m^2$ for any choice of $\beta$'s and $q$'s. We further require that the residue at this pole is factorizable into the product of a function depending on the $\alpha$'s and $p$'s only, and a function depending on the $\beta$'s and $q$'s only. This definition is for a stable particle only. The number of types of truly stable particle in the world is very small; we expect at most six: the proton, electron, the two neutrinos, the photon, and the graviton. There are many other objects also called particles which decay in an observable lifetime. We may accommodate these, as unstable particles, if we extend our above definition of a stable particle to take account of complex singularities in the

[8] See, for example, D. Lurié and A. Macfarlane, Phys. Rev. **136**, B816 (1964), and references quoted there. See also S. Weinberg, *Brandeis 1964 Summer Institute Lectures* (Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1965), Vol. 2, and references quoted there.

[9] In particular, see R. Haag, Phys. Rev. **112**, 669 (1958); K. Hepp, Acta Phys. Austriatica **17**, 85 (1963); K. Nishijima, Phys. Rev. **111**, 995 (1958); D. Ruelle, Helv. Phys. Acta **35**, 147 (1962); W. Zimmermann, Nuovo Cimento **10**, 597 (1958).

[10] A. Salam, Nuovo Cimento **25**, 224 (1962). See also S. Weinberg, in *Proceedings of the 1962 High Energy Physics Conference*, (CERN, Geneva, 1962).

[11] A. Wightman, Phys. Rev. **101**, 860 (1956). See also R. Jost, *General Theory of Quantized Fields* (American Mathematical Society, Providence, Rhode Island, 1965), and A. S. Wightman and R. F. Streater, *PCT, Spin, Statistics and All That* (W. A. Benjamin, Inc., New York, 1964).

variable $(\sum p_i)^2$ on its unphysical sheets.[12] This requires a knowledge of the analytic structure of the Green's functions. We do not wish to go into this here, but assume that enough structure can be found[13] so that this definition of an unstable particle makes sense.

Now that we have particles we wish to introduce fields for them. We do this for stable particles. We start by obtaining the single-particle states. These are defined from the residues of the Green's functions at the poles. We denote the residue of the function $G(p,q)$ defined above to be equal to the product $F(p_i,\alpha_i)F(q_j,\beta_j)$ ($i=1$ to $m$, $j=1$ to $n$). The functions $F(p_i,\alpha_i)$ may be extended to be linear in the fields $\psi_{\alpha_i}$, so they may be regarded as the inner products of a certain state $|1\rangle$ with a subset $P(m)$ of these states obtained by applying polynomials in the fields $\psi_1\cdots\psi_N$ to the vacuum. The subset $P(m)$ is taken from those tates with total energy and momentum $p_\mu$ with $p_\mu{}^2=m^2$. Cyclicity of the vacuum ensures that the set $P(m)$ is dense in the set of all states with energy and momentum $p_\mu$ with $p_\mu{}^2=m^2$, so that $|1\rangle$ is well defined. We define $|1\rangle$ as the one-particle state of mass $m$.

It is evident that $m^2$ is a point in the spectrum of $P_\mu{}^2$ where $P_\mu$ is the total energy-momentum operator for our field theory. For us to call $|1\rangle$ a one-particle state it is necessary that $m^2$ be an isolated point of the spectrum $P_\mu{}^2$. We would expect $m^2$ to be in the continuum of $P_\mu{}^2$ if the Green's functions had a branch-cut type of singularity in $(\sum p_i)^2$ at $m^2$. It is the pole nature of the singularity which we require in order for $m^2$ to be in the discrete spectrum of $P_\mu{}^2$. It follows[14] that if $m^2$ were in the discrete spectrum of $P_\mu{}^2$ then pole singularities would arise in the manner we have assumed We are considering the converse of this, since we are given initially the Green's functions for a given set of fields, and not the spectrum of $P_\mu{}^2$. We cannot prove this converse here, though it seems very likely, so we assume it to be correct. Now that we know the discrete spectrum of $P_\mu{}^2$, we may use the Haag-Ruelle-Hepp[9,14] theory to define fields $\psi_a(x)$ for each one-particle state $|a\rangle$, as quasilocal functions of the basic fields $\psi_i(x)$ ($i=1$ to $N$).

We require each field operator describing a particle of definite mass and spin to transform according to the corresponding irreducible representation under Lorentz transformations. The only quasilocal functions are then local functions of the fields $\psi_i$. Local functions of local fields are not well defined, as the ultraviolet divergences in quantum electrodynamics show. We assume that a suitable prescription may be given to define these local functions.[15] The prescription is such as to enable an algebraic structure to be given to the set of fields $\psi_i$ at a given point $x$, with an associative law of multiplication.

It is possible that the laws of physics are most simply expressed in terms of fields for unstable particles as well as for stable ones. This appears to be the case in quantum electrodynamics, since, for example, the muon is described in terms of a field coupled via minimal coupling to other charged particles. But the muon is unstable under weak interactions, so that the discussion we have given above for stable particles no longer applies. We may avoid this difficulty by introducing fields and Lagrangians only for stable particles, the interactions being so chosen that they generate the known unstable particles and their effective interactions. This meets the problem we mentioned earlier: It is difficult to solve field equations. It may also require starting from nonlocal interactions.

To avoid these difficulties we will consider here only those theories for which we may also introduce a local function $\psi$ of the basic fields $\psi_i\cdots\psi_N$ to describe each unstable particle. To make sense of this requirement we have to specify what it means for a field $\psi$ to describe an unstable particle. We take this to correspond to the condition that the momentum-space Green's functions constructed from any number of $\psi_i$'s, and one $\psi$ has a simple pole singularity in the invariant squared momentum corresponding to the $\psi$, at a given complex point (independent of the number of $\psi_i$'s) on a suitable sheet of the invariant variable. It may be possible to show that such a field can always be introduced in the same manner as for a stable particle, by extending the space of states to include unstable particle states. This extension will give rise to an extension of the operator $P_\mu{}^2$ which will have complex eigenvalues and so will no longer be self-adjoint; however, there is no reason why $P_\mu{}^2$ should not still commute with its adjoint, so that it is *normal*, in which case it generates a commutative algebra closed under involution and we still have a relatively familiar type of spectral representation.[16] It may also still be possible to give a self-adjoint extension of the original field operator, though again the contrary case should not cause any trouble.

We thus have a field operator $\psi_\alpha$ for each particle $\alpha$. We term a field theory a particle theory if we can eliminate the basic fields $\psi_i$ from the field equations and replace them by the particle fields $\psi_\alpha$. If the resulting equations are local, involving the $\psi_\alpha$ at the same space-time point, we term the theory a local-particle theory.

We now wish to divide our particles into two classes, which we call elementary and composite. We want the composite particles to be "composed" of the elementary particles, and not vice versa. Thus, we must understand what is meant by a particle's being "composed" of other particles. By this we mean that the

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[12] R. Peierls, in Proceedings of the Fourth High Energy Physics Conference, Rochester, 1954 (unpublished).

[13] See, for example, J. Gunson and J. G. Taylor, Phys. Rev. **119**, 1121 (1960). This paper contains the first published statement of maximal analyticity; see also J. Gunson, J. Math. Phys. **6**, 827, 845, 852 (1965).

[14] J. Gunson and J. G. Taylor, Nuovo Cimento **15**, 806 (1960). K. Hepp, Institute for Advanced Study report, 1964 (unpublished).

[15] M. M. Broido (to be published); J. G. Valatin, Proc. Roy. Soc. (London) **A226**, 254 (1954) and earlier references quoted there; K. Wilson, Cornell University report (unpublished).

[16] M. M. Broido, Proc. Cambridge Phil. Soc. **62**, 209 (1966); Courant Institute report (unpublished).

description of the basic local interactions between particles does not require the use of the composite particles, but only of the elementary particles. So if we start from a basic local Lagrangian involving all the particle fields $\psi_\alpha$, the composite-particle fields are those which may be eliminated in a trivial manner by means of the equations of motion, giving rise to a local Lagrangian for the elementary-particle fields only. To preserve locality, this elimination must be in terms of local functions. The elimination may be in terms of self-consistent equations, in which the composite-particle field is a local function of both the elementary- and composite-particle fields. Such a possibility has to be included if we want to discuss a system composed of composite particles only. Conversely, if we start from a local Lagrangian in terms of the elementary-particle fields only, we are able by the Haag-Ruelle-Hepp theory to compose the composites in terms of suitable quasilocal functions of the elementary-particle fields. In order to have correct Lorentz-transformation properties for the fields, these quasilocal functions must be actually local. We extend this class of functions to include self-consistent functions, depending also on the composite fields. This extension may not produce any new theories, but we should consider it here to see whether it does or not.

To sum up, then, an elementary particle is one whose field enters in some local Lagrangian; a composite particle has a field which does not enter into the local Lagrangian, but is an implicit local function of the elementary fields.

This distinction can only be made if we have a local-particle theory. If we cannot eliminate all of the basic fields $\psi_i$ from the local Lagrangian, because, for example, there are too few particles, then it does not seem possible to talk sensibly about elementary or composite particles, but only about elementary or composite fields. We define one of the fields $\psi_i$ to be elementary if it cannot be represented from the equations of motion as a local function of some of the other fields $\psi_j$ and so be eliminated from the initial Lagrangian; a composite field is any local function of the fields $\psi_i$ [other than the trivial function $f(x)=x$].

We will only consider from now on a local-particle theory; our discussion can be extended to a local-field theory if we replace the work "particle" by "field" in our results.

In finishing this discussion we note that an elementary particle is that object which has a "core." This core cannot be reduced by removing particles from it; any further reduction by removal of particles beyond a certain number is impossible, and no change is achieved by this further reduction. In fact, it is possible to define our elementary particle as being associated with a certain external line, say $p_i$, in the Green's functions $G_n(p_1\cdots p_n)$, by requiring that the reduced functions $G_{n,r}(p_1\cdots p_n)$ obtained by removing all internal states containing up to and including $r$ particles between $p_1$ and $(p_2\cdots p_n)$, be the same given functions of the $p_i$ for $r$ beyond some given value $N$. It is then possible to show that the coupled Green's-function equations arise from a Lagrangian, which will in general be nonlocal.[17] If we require our theory to be evidently local we arrive at our previous definition of an elementary particle.

## 3. THE RELATION BETWEEN ELEMENTARY AND COMPOSITE PARTICLES

We now wish to discuss the similarities and differences between elementary and composite particles. We want to include a composite particle so that it looks as similar to an elementary particle as possible. We will do this in our Lagrangian formalism. We will not consider here what observational difference can arise, though Levinson's theorem and its extension to relativistic scattering indicate the sort of difference to expect.[18]

We suppose we have a local-particle field theory, with a local Lagrangian density $L(\psi_\alpha)$ at a space-time point $x$, in terms of the set of local elementary particle fields $\psi_\alpha$ at $x$. $L$ will be the sum of two terms, the first representing the free Lagrangian density at $x$ for the separate fields $\psi_\alpha$, the second the interaction between these fields, being a polynomial in the fields and their derivatives evaluated at $x$. The field equations for the fields $\psi_\alpha$ will be the Euler-Lagrange variational equations

$$\frac{\partial L}{\partial\psi_\alpha}-\partial_\mu\frac{\partial L}{\partial(\partial_\mu\psi_\alpha)}+\partial_\mu\partial_\nu\frac{\partial L}{\partial(\partial_\mu\partial_\nu\psi_\alpha)}\cdots=0\,, \tag{1}$$

where $\partial_\mu=\partial/\partial x^\mu$ and the summation convention is used.

We want to extend these Lagrangian equations to include the composite-field equations. From our previous section these equations for a given composite particle with field $\psi_c(x)$ will be

$$\psi_c(x)=\psi_c(\psi_\alpha,\psi_{c'})\,, \tag{2}$$

depending only on the values of the set of elementary- and composite-particle fields $\psi_\alpha$, $\psi_{c'}$, and their derivatives, evaluated at $x$. The function $\psi_c(.)$ on the right-hand side of (2) must transform under the Lorentz group as a tensor or spinor of the correct spin to describe the composite particle, and satisfying the corresponding subsidiary condition to contain only one spin value. Let us consider first the case that $\psi_c(.)$ does not depend on $\psi_c$ for any $c$. We construct the new Lagrangian

$$L'(\psi_\alpha,\psi_c)=L(\psi_\alpha)+a_c[\psi_c-\psi_c(.)]^2. \tag{3}$$

The quadratic term in (3) is written formally; in detail it will be the Hermitian invariant, formed by taking the inner product of $\psi_c-\psi_c(.)$ with its Hermitian conjugate

[17] J. G. Taylor, in *Proceedings of the Siena Conference on Elementary Particles and High Energy Physics*, edited by G. Bernandini and G. P. Puppi (Societa Italiana di Fisica, Bologna, 1963).

[18] This is discussed, for example, in Ref. 3, paper VI.

in the tensor space. The quantity $a_c$ is an arbitrary real constant. Then the field equation for $\psi_c$, obtained by varying (3), will be identical with (2), while the field equations for the fields $\psi_\alpha$ will be unchanged. We may extend $L'$ to include all the other composites if they are all given by Eqs. (2) without appearing on the right-hand side. We can do this by adding to (3) the term $a_{c'}[\psi_{c'}-\psi_{c'}(.)]^2$ where now $\psi_{c'}(.)$ is chosen to be independent of $\psi_c$ by using (2). We add similar quadratic terms, in turn, each term using the equation similar to (2) so that it depends only on the composite fields coming later in turn in the function $\psi_{c'}$.

In this process of extending $L$ we expect that at some state Eq. (2) will involve the same field on each side, so we will now take account of such a situation. In the case of $\psi_c(.)$ in (3) depending on $\psi_c$ (not on derivatives of $\psi_c$), we then obtain from (3)

$$[\psi_c-\psi_c(.)][1-\partial\psi_c(.)/\partial\psi_c]=0. \qquad (4)$$

Now $\partial\psi_c(.)/\partial\psi_c\neq 1$, since we are assuming (2) is not the trivial case, and again we obtain (2). In the more general case, when derivatives of $\psi_c$ appear on the right-hand side of (2) it seems difficult to set up a Lagrangian which will give rise to Eq. (2) alone. However, we have some freedom in the choice of the function $\psi_c(.)$ on the right-hand side of (2).[9] We assume that our choice is great enough to allow us to choose it so that it contains no higher than first derivatives of any fields, elementary or composite, and to be linear in any of these derivatives. We further require that no derivatives of $\psi_c$ itself appear on the right-hand side of (2). A set of composite particles which are such that their set of functions (2) all satisfy these conditions, and are derivable from a single Hermitian Lagrangian, is said to satisfy the unitarity condition. We use the word unitarity because we may then quantize the set of fields so as to correspond to a unitary $S$ matrix, even if there are no elementary particles present. We will not discuss the details of how we do that for elementary fields but refer to the discussion of this problem in the first paper of Ref. 3: no derivatives higher than the first must appear on the right-hand side of (2) for the elementary fields, basically so that the characteristics of the resulting differential equations for these fields are independent of the fields themselves. (A further general principle emerges which we discuss in Sec. 4 of this paper.) We extend this requirement to the composite fields by considering them as the limit, as $Z_c\to 0$, of elementary fields. These elementary fields have then to satisfy the unitarity requirements. We now turn to discuss this introduction of the wave-function renormalization constant $Z_c$, and its vanishing.

The extended Lagrangian

$$L(\psi_\alpha,\psi_c)=L(\psi_\alpha)+\sum a_c[\psi_c-\psi_c(.)]^2, \qquad (5)$$

where the quadratic functions of $\psi_c$ do not depend on the fields earlier in the summation in (5), will generate the original field equations and the composite-field equations (2) if the unitarity condition is satisfied for each term in (5). We can interpret the extra terms added to $L(\psi_\alpha)$ in (5) by adding further kinetic terms to (5) of form

$$[1-(1-Z_c)]L_0(\psi_c)$$

for each composite field $\psi_c$, where $Z_c$ is a constant equal to zero, and $L_0(\psi_c)$ is the free Lagrangian for a particle $\psi_c$ of arbitrary mass. The extra term

$$a_c[\psi_c^2-2\psi_c\psi_c(.)+\psi_c^2(.)] \qquad (6)$$

is to be interpreted as $a_c\psi_c^2=$ mass renormalization term, $a_c=\frac{1}{2}\delta\mu^2 Z_c$, where $Z_c$ is the wave-function renormalization constant of the renormalized field $\psi_c$; $-2a_c=gZ_c^{1/2}$ where $g$ is the unrenormalized coupling constant between $\psi_c$ and the fields which enter $\psi_c(.)$; and $a_c\psi_c^2(.)$ is a self-coupling term between these fields. Then the total Lagrangian (5), with $Z_c\neq 0$, is that for elementary particles $\psi_\alpha$ and $\psi_c$, and as $Z_c\to 0$ becomes the Lagrangian for the elementary particles $\psi_\alpha$ and the composite particles $\psi_c$. It may not be possible to write (5) as the correct Lagrangian, since successive elimination of the terms in the series in (5) may violate the unitarity condition. This will actually occur in a self-consistent model we consider later. In general, we require our theory to be unitary if we may form an extended Lagrangian

$$L(\psi_\alpha)+L(\psi_\alpha,\psi_c)$$

for all the composites (with certain restrictions on the way that derivatives enter, as discussed in paper I of Ref. 3), whose variation in each of the composites $\psi_c$ should result in (2). If we add to this the terms

$$\sum_c [1-(1-Z_c)]L_0(\psi_c)$$

and interpret the terms in $L(\psi_\alpha,\psi_c)$ as interaction terms or self-mass terms, then again we may interpret

$$L(\psi_\alpha)+L(\psi_\alpha,\psi_c)+\sum_c [1-(1-Z_c)]L_0(\psi_c) \qquad (7)$$

as a Lagrangian for elementary particles $\psi_\alpha$, $\psi_c$ which become composite as $Z_c\to 0$.

We may also reverse this step by starting from a Lagrangian like (7) for elementary $\psi_\alpha$ and $\psi_c$, and let $Z_c\to 0$. The resulting equation for $\psi_c$ will be of the form (2), the left-hand side of (2) arising from the self-mass term and the right-hand side from the interaction terms. If we require the field equation for any elementary field to have characteristics independent of the solution, and be of second order, then as $Z_c\to 0$ the right-hand side of (2) can depend at most linearly on derivatives of the given field itself. Thus formally we have proved the following result:


FIG. 1. An amputated bubble without propagator.

### Result

A composite particle in a unitary theory is an elementary patricle whose wave-function renormalization constant is equal to zero.

In this framework we may understand the results on the relation between a four-fermion theory with a boson bound state and a Yukawa theory, as discussed by Lurié and Macfarlane (Ref. 8). We take one elementary particle described by a spinor operator $\psi$, and

$$L(\psi)=L_0(\psi)+\mathcal{G}(\bar{\psi}\psi)^2.$$

Then with $\phi=\lambda\bar{\psi}\psi$ as the boson field,

$$L(\psi)+a(\phi-\lambda\bar{\psi}\psi)^2=L_0(\psi)+(\mathcal{G}+a\lambda^2)(\bar{\psi}\psi)^2 +a\phi^2-2a\lambda\phi\bar{\psi}\psi\,, \quad (8)$$

and the four-fermion interaction is removed if $\mathcal{G}=-a\lambda^2$. Further the boson self-mass is

$$\delta\mu^2=2a/Z\,,$$

where $Z$ is the boson wave-function renormalization,

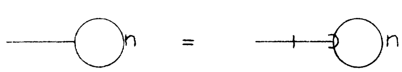



FIG. 2. Definition of an amputated bubble.

and the effective Yukawa coupling constant is

$$Z^{1/2}g_0=-2a\lambda.$$

Thus, the Lagrangian $L(\psi)$ is identical to that for a Yukawa coupling to a boson with $Z=0$ if

$$\mathcal{G}=-a\lambda^2,\quad \delta\mu^2=2a/Z,\quad g_0Z^{1/2}=-2a\lambda\,,$$

or eliminating $a$ and $\lambda$, if

$$\delta\mu^2=-g_0^2/2\mathcal{G}. \quad (9)$$

This is exactly the condition arrived at earlier.[8] However, this extra condition is very closely related to the particular type of interaction, while as we have seen above the $Z\to 0$ condition is the compositeness condition for any elementary particle interacting in a very general way with a set of other elementary particles, provided the theory is unitary.

So far the discussion has been purely formal in the following sense: We have pointed out how certain formal manipulations with the Lagrangian can cause a theory with an elementary particle to become formally identical in the limit $Z_3\to 0$ with a theory containing a composite particle. Now although at each stage of the limit process we have a well-defined theory, it is not yet clear in what sense the theory still exists in the limit. If the various renormalization processes were independent of one another, we could argue that since physically observable quantities are independent of the values of the renormalization constants, they must also have the correct values in the limit. However, these processes are not independent, as is clear from the model we have just discussed, and so we have to make sure that the theory still exists, and in particular, that there is still a composite-particle pole in the correct place with nonzero residue. This connection has indeed been discussed in many models,[8] and also in quantum electrodynamics.[19]

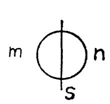



FIG. 3. A bubble without certain cuts.

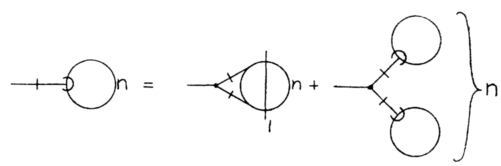



FIG. 4. Cubic self-interaction of a scalar field in Green's-function form, without charge renormalization.

What we must do, then, is to exhibit the mass and charge renormalizations explicitly and show that the quantities involved possess reasonable limits as $Z_3\to 0$. Now the wave-function renormalization can be carried out in the original operator form of the theory; this cannot yet be done for the mass and charge renormalizations. For this reason we will have to pass over to the Green's-function formalism[3] in which these other renormalizations can be carried out explicitly.[3] We refer the reader to those papers for detailed explanations of this formalism (see also Ref. 20), but for clarity we insert here a brief explanation which we hope will be sufficient for our further purposes and those of the reader. It should be noted that all our arguments apply outside perturbation theory.

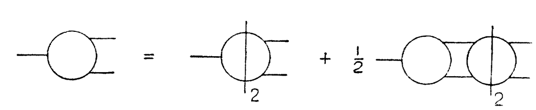



FIG. 5. Definition of a 2-cut-less amplitude.

The general $n$-particle "bubble" $\bigcirc n$ represents the formal sum of all unrenormalized perturbation graphs with the corresponding number of legs and consistent with the interaction under discussion. The line —— represents the bare propagator and has the value

$$\text{——}=i(p^2-m^2)^{-1}.$$

[19] I. Bialynicki-Birula, Phys. Rev. 130, 465 (1963).

[20] K. Symanzik, Hercegovni Summer School Proceedings, Jugoslavia, 1961 (unpublished).

The mass renormalization is carried out by removing the self-energy parts and absorbing them into the bubble to give a clothed propagator —+— and an amputated bubble as in Fig. 1; the exact definition is given in Fig. 2. More geneıally we consider bubbles of the form given in Fig. 3, which represent the sum of those graphs not possessing $s$-cuts in the indicated channel. The case $s=1$ represents the mass renormalization, so that for instance the field equation for the cubic self-interaction of a

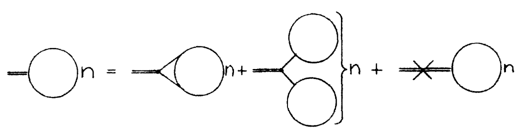


FIG. 6. Green's-function equation for two elementary particles [cf. Eq. (11)].

scalar-meson field is given in Fig. 4 [Eq. (17) of Ref. 3, paper I]:

The charge renormalization corresponds to the case $s=2$; it is given by definition, for instance as in Fig. 5 [Eq. (20) of Ref. 3, paper I]. Figure 5 says simply that certain contributions have been absorbed into the vertex part. The charge renormalization is then carried out by replacing the bare by the physical vertex everywhere, and removing the appropriate terms from the bubble. For more details, see p. 881 of Ref. 3.

We apologize to the reader for whom this extremely cursory introduction has been insufficient, and hope that careful referencing will help him through the following discussion.

Consider for concreteness a theory of two scalar particles described by fields $\phi_a$, $\phi_c$ with interaction

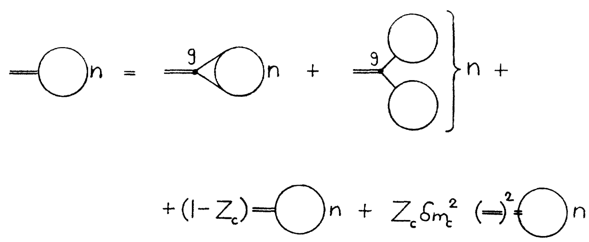


FIG. 7. Effect of $Z_c$ dependence in Fig. 6.

Lagrangian

$$\mathcal{L}_{\mathrm{int}}=g\phi_c\phi_a^2\,, \tag{10}$$

expressed in terms of unrenormalized quantities. Denote by $m_c$ and $\delta m_c$ the physical mass and the mass shift for the $c$ particle, and by $Z_c$ its wave-function renormalization constant. Let $g_r$ be the coupling constant in (10), renormalized with respect to the $a$ particle. In Green's-function equations we will denote the $a$-particle propagator by a single line —— and the $c$-particle propagator by a double line ═══. The clothed propagators —+— and ═╪═ are defined as in Fig. 2.

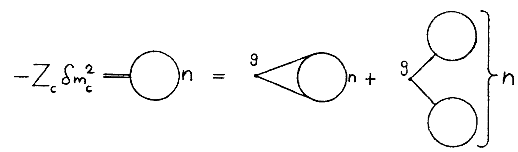


FIG. 8. The $Z_c\to 0$ limit process in Fig. 7.

For $Z_c\neq 0$, then, we use the Lagrangian-with-counterterms

$$\mathcal{L}=\sum_{i=a,c}\mathcal{L}_0(\phi_i)[1-(1-Z_i)]+\tfrac{1}{2}\sum_{i=a,c}Z_i\delta m_i^2\phi_i^2+\mathcal{L}_{\mathrm{int}} \tag{11}$$

and so in the limit $Z_c\to 0$, the equation for the $c$ particle becomes

$$\phi_c=\lambda\phi_a^2\,, \tag{12}$$

where

$$-\lambda=\lim_{Z_c\to 0}\frac{g}{Z_c\delta m_c^2}$$

In terms of $g_r$, we must thus discuss (13)

$$-\lambda=\lim_{Z_c\to 0}\frac{g_rZ_v}{Z_c\delta m_c^2}$$

(where $Z_v$ is the vertex-function renormalization con-

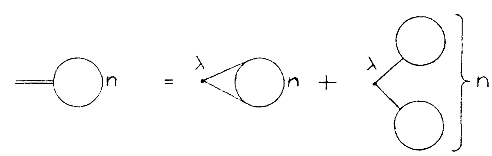


FIG. 9. Green's-function equation for a composite particle arising from Fig. 6 [cf. Eq. (12)].

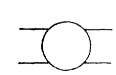
stant of the $c$-particle) so that if $Z_c\delta m_c^2\to 0$, as has been suggested for the Lee model, then also $Z_v\to 0$. We shall discuss the condition $Z_v\to 0$ below in more detail (where $Z_v$ is the vertex-function renormalization constant of the $c$-particle).

Now let us pass over to the Green's-function equations for the $c$ particle arising from (11). They are, by the general methods of Ref. 3, of the form given in Fig. 6. In Fig. 6 [compare Eq. (26) of paper I of Ref. 3] where

$$\mathbf{X}=(1-Z_c)(═══)^{-1}+Z_c\delta m_c^2 \tag{14}$$

and . denotes the coupling constant $g$, so that when we write out Fig. 6 explicitly, we get Fig. 7. Then as $Z_c\to 0$, the first and fourth terms cancel and we have (removing the factor ═══) the relation in Fig. 8, i.e., in terms of (13), provided the limit exists, we obtain Fig. 9. Figure 9 is simply a restatement of (12). This

FIG. 10. Four-point function.



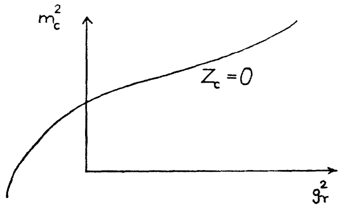



FIG. 11. Schematic dependence on the independent variables.

accomplishes the first part of our program, since it shows that the equations obtained by letting $Z_c \to 0$ in the Green's-function equations arising from (11) are the same as those obtained by putting $Z_c \to 0$ in (11) and then varying to give (12).

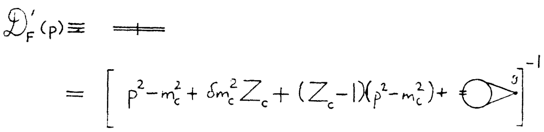


FIG. 12. Propagator equation for the composite particle.

We repeat that (12) *formally* describes $c$ as a bound state of $a$. We still have to show that this correspondence is more than merely formal. We now do just this.

In particular we must show that Fig. 10 has a pole at $m_c^2$, and that the residue is nonzero even in the limit $Z_c \to 0$. This is neither obvious nor trivial.

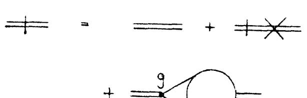


FIG. 13. Mass-renormalization of Fig. 12.

To clear the ideas, we will state unambiguously what are our independent variables. They will be the renormalized mass $m_c$ and the coupling constant $g_r$, which has been renormalized with respect to the wave-function renormalization of the elementary particle $a$ (since this will not cause any problems). These two quantities are the ones nearest to having some definite

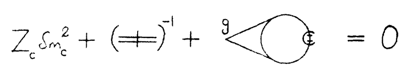


FIG. 14. Effect of $Z_c \to 0$ in Fig. 13.

physical significance which prevents us from regarding them as functions of other parameters in the theory. We take, then,

$$Z_c = Z_c(m_c^2, g_r^2),$$
$$\delta m_c^2 = \delta m_c^2(m_c^2, g_r^2),$$

and so forth. Now we consider $Z_c = 0$ as a curve in the $(m_c^2, g_r^2)$ plane (see Fig. 11), and we imagine ourselves

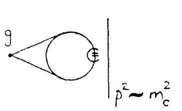


FIG. 15. Diagram for the bound-state coupling constant $\lambda$.

fixing $g_r^2$ and approaching the curve $Z_c = 0$ in such a way that the limit (13) remains finite and nonzero. If this is not possible for any $g_r^2$, then there will be no composite particle, regardless of how attractive the theory may be in other respects. If it is possible for many values of

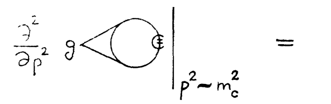


FIG. 16. Diagram form of the condition $Z_c = 0$.

$m_c^2$, we may have some choice in the selection of bound states (further restricted, perhaps by Levinson's theorem or by bootstrap conditions; we will deal with these later).

One sees how restrictive or otherwise this is also by considering the mass renormalization of the $c$ particle in

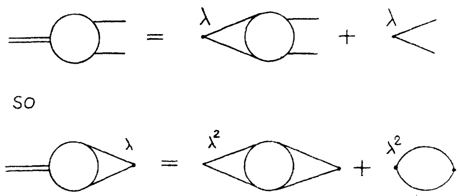


FIG. 17. Vertex-function and propagator in the composite-particle description.

the two prescriptions. In the $Z_c \neq 0$ prescription, we have the propagator equation for the $c$ particle given in Fig. 12 [Eq. (93) of paper I of Ref. 3]. In terms of previously introduced notation, this gives Fig. 13. As $Z_c \to 0$, we

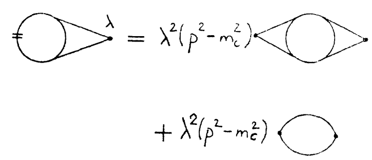


FIG. 18. Subtraction methods of Fig. 17.

obtain Fig. 14, so we require

$$\lambda^{-1} = \lim_{Z_c \to 0} \frac{-Z_c \delta m_c^2}{g}$$

to be given by the diagram in Fig. 15.

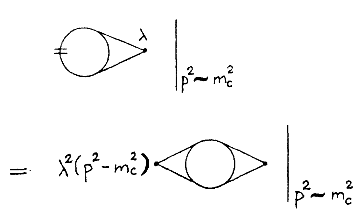


FIG. 19. Form of Fig. 18 at $p^2 \sim m_c^2$.

We also require the relationship expressed in Fig. 16, which is a restatement, in terms of the self-energy, of the requirement $Z_c = 0$. In fact, $Z_c$ is just the difference between the two sides of Fig. 16, and so we have the functional dependence of $Z_c$ on $m_c^2$ explicitly and on $g_r^2$ implicitly.

Now we wish to do the same in the composite-particle description. Consider Eq. (14). Figure 9 now gives us

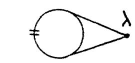



FIG. 20. The composite propagator.

Fig. 17 (for $n=2$) where the self-energy term consists of an arbitrary constant and a finite function of $p^2$ which does not have a pole at $p^2=m_c^2$, by the usual subtraction methods. This gives Fig. 18. The last term drops out at $p^2\sim m_c^2$, giving Fig. 19. This is precisely the behavior in Fig. 16. Now the diagram in Fig. 20 is, by definition,

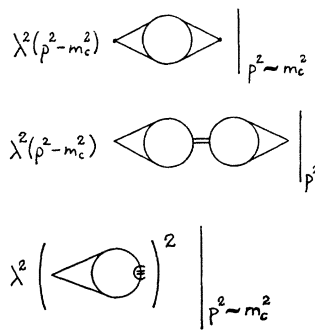


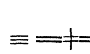

FIG. 21. Behavior of the composite propagator about the pole.

equal to

$$\int \langle 0|T(\phi_c(x)\phi_c(y))|0\rangle e^{ip(x-y)}d^4(x-y)$$

$$=\int \langle 0|T(\phi_a^2(x)\phi_c(y))|0\rangle e^{ip(x-y)}d^4(x-y)$$

[by Eq. (12)]



$$\sim\frac{1}{p^2-m_c^2} \quad \text{at} \quad p^2\sim m_c^2.$$

In terms of the pole in the composite propagator, this

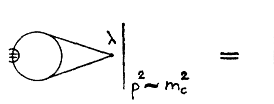



FIG. 22. Consistency condition arising in the composite formulation.

gives Fig. 21, i.e., in both cases (checking the consistency), we have Fig. 22.

This is precisely the same as the equation in Fig. 15 arising from the elementary-particle description as $Z_c\to 0$, and shows the consistency of the two descriptions with respect to the values of the coupling constants.

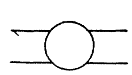


FIG. 23. Scattering amplitude.

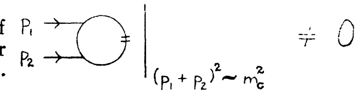



FIG. 24. Behavior of the vertex-function near the composite mass-shell.

Now at last we are in a position to show that the scattering amplitude in Fig. 23 still has a pole with nonzero residue *in the limit* $Z_c\to 0$ of the elementary-particle description of the $c$ particle. More precisely, the finiteness of $\lambda$ as given by Fig. 15 is enough to ensure this for Fig. 15 then gives Fig. 24. From Fig. 24

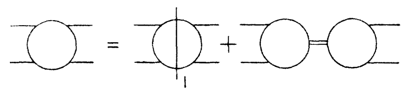



FIG. 25. Cut-structure of the scattering amplitude with respect to the composite propagator.

we obtain by definition the relationship given in Fig. 25, where the second term is finite as $p^2\to m_c^2$, whereas the third has the pole with nonzero residue.

This completes the proof of the assertion that, as $Z_c\to 0$, the $c$ particle becomes a composite of the $a$ particle, provided the expression in Fig. 15 has a finite limit.

What we have shown is not exactly that the two descriptions are equivalent, but that the elementary

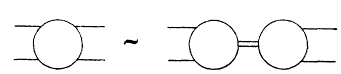



FIG. 26. Behavior of a scattering amplitude with a bound state of mass $m_c$ in the $s$-channel around the value $s=m_c^2$.

description gives the composite one in the limit. One will not expect to be able to show the converse, because there will be no means of recovering the terms that drop out in the limit $Z_c\to 0$. Nevertheless, one can establish a partial converse.

Consider a particle described by a field $\phi_a$, and a Lagrangian $\mathcal{L}(\phi_a)$. This time we suppose that there is a

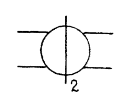



FIG. 27. 2-cut reduced amplitude (or "potential").

bound state of mass $m_c^2$ as in Fig. 26. We also suppose that the reduced amplitude (Fig. 27) has no pole at $s=m_c^2$, so that the equation in Fig. 28 [which is actually the definition of the reduced amplitude (Fig. 27)] can be viewed as a Bethe-Salpeter equation which generates the bound state ═══ by iteration of the "potential"

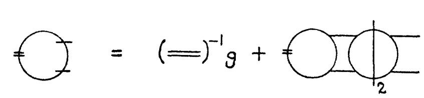



FIG. 28. Definition of the reduced amplitude.

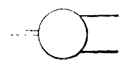


FIG. 29. Bound-state vertex-function.

Fig. 27. (Note that the argument is crossing symmetric and relativistic; it is the analogy that is not.) In particular, on the mass shell of the composite particle, Fig. 28 reduces to a homogeneous Bethe-Salpeter equation for the bound-state vertex function, Fig. 29.

We will attempt to describe the bound state by the field $\lambda\phi_a^2$ *à la* Zimmermann.[9] We immediately obtain Fig. 9, which can be written as in Fig. 30, with $\mu=(p^2-m_c^2)\lambda$. This is similar to Fig. 6; in fact, the only difference is that we have lost the mass counterterm given in Fig. 31 (which anyway underwent a great transformation as $Z_c\rightarrow 0$), and that we have a momentum-dependent coupling constant $\mu$. The theory will be internally self-consistent provided $\lambda$ is determined by the usual residue condition, Fig. 22.

All these arguments can be extended without trouble to particles of different spin types and having different interaction Lagrangians. This is clear from the general graph structure developed in Ref. 3, which is the essential foundation of all the above discussion.

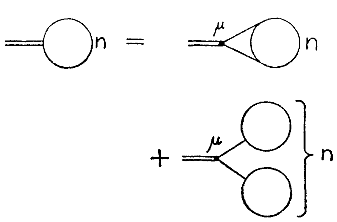



FIG. 30. A bound state looking similar to an elementary particle.

We have attempted by similar methods to discover whether the self-consistency of the theory at this stage requires some definite behavior of the vertex-function renormalization constant $Z_v$ of the system. The manipulations involved are complicated; we will not go into them here. We have concluded that the theory does not yet impose a condition on $Z_v$. As we will see later, this conclusion does not necessarily hold when one considers the more demanding conditions associated with bootstrap mechanisms. Finally, we remark that if we attempt to identify a coupling constant in the standard way from Fig. 9, this "constant" would be $\lambda(p^2-m_c^2)$ and so would vanish on the mass shell. One might interpret this as a requirement that $Z_v=0$, though such an interpretation is evidently inconsistent in field theory, since $Z_v$ would be dependent on $p^2$. In the next section we turn to a more complete discussion of whether or not $Z_v$ need vanish.

*Note added in proof.* It seems that the condition $Z_3=0$ may not be the only one for compositeness in the quite special case where one starts with two elementary particles having the *same quantum numbers* and tries to make them composite [R. L. Zimmermann and D. Alexanian (private communication); J. C. Houard and J. C. le Guillou, Collège de France (unpublished)].

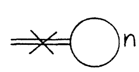



FIG. 31. Mass counter-term in the elementary description (Fig. 6) not arising in the composite description (Fig. 30).

One may wonder whether this has any physical significance. The problem seems to arise only in the case of $\phi$-$\omega$ mixing. We may try to deal with this problem by the method of bootstrapped symmetries (see Sec. 7D below). If the symmetry is $SU_3$, since $\phi$ and $\omega$ belong to different representations, the situation of identical quantum numbers does not really arise. In the case of $SU_6$, we have to deal with a badly broken symmetry. A natural way of introducing symmetry-breaking terms into our bootstraps is by kinematic terms, which will tend to give increasing violations at higher energies, as is needed experimentally. But then the particles are, after all, still elementary; the problems encountered in the above discussions do not arise. Altogether we feel that the case of two different particles with all the same quantum numbers is unlikely to be physically very important.

However there remains the formal problem of justifying the method of the present paper in this special case by showing that the results of the above authors can be obtained by our methods. We have done this. Since certain special considerations arise, we will give the details elsewhere.

## 4. FURTHER COMPOSITENESS CONDITIONS

Our discussion of the previous section results in just one essential difference between an elementary and a composite particle, depending on the value of $Z$, the wave-function renormalization constant for the particle. $Z$ is defined in any local Lagrangian theory, since it is always possible to expose the self-energy in the propagator, and to define $Z$ as the residue at the resulting pole. Thus, we may define $Z$ even for a theory which is nonrenormalizable in the usual sense.[21] Earlier discussions of composite particles have used the further condition of the vanishing of the corresponding vertex-function renormalization constant, not just at a single value of $p^2$,[22] but for all values of $p^2$ (where $p$ is the momentum of the composite particle entering the vertex function). Such a condition has also been used as a residue condition[23]; this aspect has been discussed in Sec. 3. We wish to consider here whether such an additional condition is required or even consistent in order to make composite particles look like elementary ones.

We first remark that our discussion in the preceding section shows that this additional condition is not in general required; *any* composite particle can be considered as the limit as $Z_c\rightarrow 0$ of an elementary-particle

[21] This question of mass renormalization is discussed further in Ref. 3, particularly paper I of that reference.
[22] See Ref. 3, paper VI.
[23] P. Kaus and F. Zachariasen, Phys. Rev. **138**, B1304 (1965).

$C$ interacting in a local fashion with the original elementary particles through a possible local interaction $\mathcal{L}_{\text{int}}^{(c)}$. Provided a local interaction $\mathcal{L}_{\text{int}}^{(c)}$ may be chosen which is renormalizable, then we expect the limit $Z_c \to 0$ to exist for all the Green's functions of the theory. Thus, we have only to consider whether one can impose the additional condition consistently.

In order to discuss this consistency, we may proceed in either of two possible ways. The first is to look at models which are soluble, and discuss the consistency for them; the difficulty is that our results may not extend to more realistic theories which cannot be solved completely. The alternative is to discuss realistic field theories in some approximation scheme.

We can consider as models illustrating the first alternative potential-scattering theory,[24] the Lee model,[25] the Zachariasen model,[26] the Chew-Low model,[27] and two-particle unitarity models.[28] In potential scattering the additional condition arises that the vertex-function renormalization constant vanishes. But if this condition is imposed we find that the corresponding vertex function $\Gamma(E)$ vanishes except at the bound-state energy $B$, when it takes a finite value. Evidently such a function cannot be treated in a consistent fashion without vanishing effectively everywhere. To avoid this paradox, we should interpret the nonrelativistic results not in terms of a local field theory, but rather in terms of a nonlocal field theory with an energy-dependent coupling constant vanishing at the bound-state energy. However, the condition does arise in a special way in the Lee model.[25]

When we turn to discussions associated with more realistic field theories[29] we see that they are completely nonrigorous. The models discussed are of nonrenormalizable theories, so they are even more "beyond the pale" of rigor than is usual in such discussions. We may consider the problem for a slightly less pathological theory, say a theory in two space dimensions. Here it is still not possible to say anything exact about the existence of solutions to the equations when the vertex-function renormalization constant is set equal to zero. However, there are indications that no such solution exists. This is the case for the two-particle-exchange approximation equations for pion scattering,[30] and also in a suitable region in function space for any approximate equation obtained by neglecting the Green's functions with more than a certain number of variables for any local interaction.[31] Such results lead one to believe that no solution to the complete equations, except the trivial free-field equations, will exist if the vertex-function renormalization constant is set equal to zero. We will thus not consider this condition further, but go back to the condition we discussed in the previous section—that the wave-function renormalization constant alone vanishes.

We saw that this condition allows us to regard a composite particle as an elementary one, and to introduce a suitable interaction between it and the original elementary particles. In so doing we have not done anything new. Of course, it may be a very helpful reshuffling of perturbation expansions to introduce the "elementary" composite, as has been much advocated by Weinberg,[8] with his quasiparticles. Thus, it may help us towards the problem of computing, which we discussed in the Introduction. It may also help us to decide which interactions we should be using. However, it will not help us in any simple or direct fashion in this latter investigation, at least not at present. We now wish to see if we can understand better the possible interactions which may occur.

Many different interactions may give rise to the same physical predictions. In the case of a system of elementary particles interacting with each other our problem is then to divide possible local interactions into equivalence classes, each class containing all these interactions with the same on-the-mass-shell predictions. This is a very interesting task, but is at present impossible, since we have not yet got the tools to deal effectively with a single interaction.

When we turn to the problem of classifying interactions when "elementary" composite particles are included, we see that there is one case in which there is only one class—that is, when there are only composite particles. For we know that any quasilocal field function of the other fields may be used to describe the composite, provided the vacuum-to-one-composite-particle matrix element is nonzero.[9] Hence, for a world in which there are only a certain number of composite particles almost any interactions may be used, and will give the same physical results. This system—the bootstrap system—is evidently of interest because of

[24] N. Bertocchi, Nuovo Cimento **31**, 1352 (1964); R. Rockmore, Phys. Rev. **132**, 878 (1963) and Brookhaven National Laboratory report 8388 (unpublished); S. Weinberg, Phys. Rev. **132**, 776 (1963).

[25] I. S. Gerstein, University of Pennsylvania report (unpublished); J.-C. Houard, Ann. Inst. H. Poincaré **2**, 105 (1965); J.-C. Houard and B. Jouvet, Nuovo Cimento **18**, 446 (1960); M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).

[26] R. Acharya, Nuovo Cimento **24**, 870 (1962); J. S. Dowker, *ibid.* **25**, 1135 (1962); **29**, 551 (1963); C. R. Hagen, Ann. Phys. (N. Y.) **31**, 185 (1965); M. L. Whippman and I. S. Gerstein, Phys. Rev. **134**, B1123 (1964). (In the paper of C. R. Hagen there is an independent derivation of the degenerate equations arising from the $Z_3=0$ condition; however we disagree with his further discussion of these equations.)

[27] K. Huang and F. E. Low, J. Math. Phys. **6**, 795 (1965); K. Huang and H. Mueller, Phys. Rev. Letters **14**, 396 (1965).

[28] W. Guettinger, Nuovo Cimento **36**, 968 (1965); B. Lee, K. Mahantappa, I. S. Gerstein, and M. Whippman, Ann. Phys. (N. Y.) **28**, 466 (1964); M. Ida, Progr. Theoret. Phys. (Kyoto) **34**, 92 (1965).

[29] B. Jouvet, Nuovo Cimento **5**, 1 (1957); (this appears to be the first published discussion of the $Z_3=0$ condition in relation to composite particles.) E. G. P. Rowe, Nucl. Phys. **45**, 593 (1963). See also the papers of Refs. 8 and 28 and references quoted therein.

[30] M. M. Broido, J. Math. Phys. **6**, 1702 (1965).

[31] J. G. Taylor, J. Math. Phys. **6**, 1148 (1965); J. G. Taylor, *ibid.* (to be published).

this lack of arbitrariness in physical predictions. We will have either that such a system of composites cannot "glue each other together," or if they can then the type of glue is unimportant. This result is very appealing; we would like to find further that there is only one such possible bootstrapped system, which is our world. However, the results turn out to be rather different from what one would expect, namely, the field-theoretic equations which (as we see in the next section) reduce in the single-particle-exchange approximation to the usual bootstrap equations, turn out in many cases to have no nontrivial solution. Now the principle we have just enunciated allows us to claim that if one given form of interaction leads via our bootstrap mechanism to particle-like solutions and a (perhaps trivial) $S$ matrix, then any other interaction with the same mass spectrum satisfying the given conditions will give the same $S$ matrix. It does not, strictly speaking, allow us to infer that if a given interaction *fails* to give particle-like solutions, then so must any other; yet it is to this latter, heuristic extension of the "glue is unimportant" principle that we will have to appeal if we are to avoid considering separately each possible interaction. This is an important general problem of quantum field theory to which no attention has yet been paid in the literature, except in general relativity. In order to give some foundation for this extension of the principle, we will consider in Sec. 6 examples where different Lagrangians do in fact all fail to give particle-like solutions. Thus, what we now wish to do is to consider in detail bootstrapped systems and the possibility of their existence.

*Note added in proof.* Since this paper was originally written, innumerable articles have appeared proposing further compositeness conditions related to $Z_1=0$. This work is almost all dependent on two-particle unitarity. We mention in particular the ingenious classification proposed by Ida,[28] based on the asymptotic behavior of the Lehmann spectral function, and applied by him in a series of papers [e.g. M. Ida, Progr. Theoret. Phys. **34**, 990 (1965); K. Hayashi *et al., ibid.* **34**, 636 (1965)], and also new work based on the Zachariasen model [see N. G. Deshpande and S. A. Bludman, Phys. Rev. **143**, 1239 (1966) and references quoted there]. The various conditions which arise ($Z_1/Z_3=0$: Kaus and Zachariasen[23]; $Z_1^2/Z_3=0$: Deshpande and Bludman) are not in any way unique, as has been emphasized by Ida. It seems likely to us that such conditions arise only in two-particle unitarity, and are not at all general. In the absence of reasonably realistic soluble models with inelasticity, we therefore still feel that $Z_3=0$ should be the only *general* condition, as suggested by our Green's-function analysis.

## 5. CLASSIFICATION OF PARTICLES

There have been a number of discussions in the literature concerning the classification of particles into elementary and composite. In this section we will review these attempts in the light of the work of the last two sections.

The deuteron has been discussed by Weinberg.[32] Using the Low equation, he has shown that, in the limit of vanishing binding energy, the success of the effective-range approximation can be explained by putting $Z_3<0.01$. His theory exists in the limit $Z_3\to 0$. The $Z_3=0$ results can also be explained in the Zachariasen model (Dowker, Ref. 26).

Weinberg's results have been used by Amado *et al.*[33] to discuss the triton, taking the value $Z_3=0.048$ for the deuteron and putting the deuteron potential into the Fadeev equations.

These results appear to show that $Z_3=0$ is a physically viable condition in nonrelativistic situations. Their importance for us is based mainly on the fact that they are model-independent.

Passing to the most relativistic problem of all, we consider the possibility that the photon is a bound state. Now if we consider the usual wave equation for a massive vector boson with interaction:

$$(\Box^2+m^2)A_\mu=\lambda J_\mu \\ +(1-Z_3)(\Box^2+m^2)A_\mu+Z_3\delta m^2 A_\mu \quad (15)$$

and put the physical mass $m=0$ from experiment (see below) and the bare mass zero by gauge invariance, we get simply $J_\mu=0$. [Strictly speaking, we have $Z_1^{-1}J_\mu=0$; but there seems to be no reason why $Z_1$ should vanish either in general (Sec. 4) or, in particular, in quantum electrodynamics.] We may consider the possibility that $\lambda=Z_3/Z_1$ approaches a finite limit [(cf. Eq. (13)]; but the completely renormalized wave equation for the photon will be

$$Z_3\Box^2 A_\mu^{(r)}=e_r Z_1 J_\mu^{(r)},$$

where we suppose that as $Z_3\to 0$, we have $Z_1\sim Z_3^{\alpha}$ say, for some real $\alpha$. There are now three cases:

$$\begin{array}{ll} \alpha>1\Rightarrow \Box^2 A_\mu^{(r)}=0: & \text{free field,} \\ \alpha=1\Rightarrow \Box^2 A_\mu^{(r)}=e_r J_\mu^{r}: & \text{elementary photon,} \\ \alpha<1\Rightarrow J_\mu^{(r)}=0: & \text{no photon at all.} \end{array}$$

This argument is independent of power behavior of $Z_1$ in $Z_3$ since if $Z_1$ is more singular than any power of $Z_3$, evidently we still get the third case. The conclusion that the photon is elementary thus seems inescapable, as long as the physical mass really vanishes.

Now the upper limit on the physical mass of the photon is given (to order of magnitude) by the inverse of the radius of the universe; we have $1/R_H\sim 10^{-55}$ g. As is so often the case with arguments of this type based

[32] S. Weinberg, Phys. Rev. **137**, B672 (1965). The work of Weinberg has recently been extended by Ida [M. Ida, Progr. Theoret. Phys. (Kyoto) **35**, 104 (1966)]. For our purposes this new work may be regarded as further confirmation of the conclusion in the text.

[33] R. D. Amado, R. Aaron, and Y. Y. Yam, Phys. Review Letters **13**, 574 (1965).

on experiment, a hostile critic can always assert that the mass could be still lower and yet nonzero.

Ignoring this possibility, we can thus see why the ingenious arguments of Białynicki-Birula[19] fail. This author showed that quantum electrodynamics can be obtained from the four-fermion theory discussed in Sec. 3 by an appropriate interpretation of the renormalization constants in the two theories, i.e., the photon appears as a bound state of positron and electron. It has been claimed[8] that he did not take the condition $Z_3 \rightarrow 0$ into account, and that "identification of $\mu_0'$ with $\mu_0$ is not justified at all since Birula [*sic*] has used $\mu_0'=0$ whereas $\mu_0$ is strictly undetermined$\cdots$" (see Ref. 8, footnote 47). The first of these remarks is factually incorrect,[34] as one sees from Eq. (36) of Ref. 19. The second seems to us to be equally erroneous; $\mu_0'$ is an arbitrary parameter in Ref. 8, and any appropriate value may be chosen. There is no reason why one should not take the same value for $\mu_0$ in Ref. 19.

Summing up, the photon is not a bound state because as $Z_3 \rightarrow 0$, either the current vanishes, or the dynamical term *fails* to vanish, depending on the behavior of $Z_1$. We have gone into this problem in some detail because of the importance of quantum electrodynamics as a physical theory and because of the above-mentioned confusions in the literature.

The same arguments will also be valid, of course, for the neutrino.

With regard to other types of particle, we do not know exactly what is the correct interaction by which to describe them. However, we can write down the appropriate composite-particle equations corresponding to certain types of interaction, and can discuss the possibility of *those particular interactions* giving rise to a composite particle in the limit $Z_3 \rightarrow 0$.

Consider for example the possibility of describing the neutron as a bound state of neutral pions. Starting from a Yukawa-type interaction

$$\mathcal{L}_I = gN\pi\pi \tag{16}$$

we obtain, in the limit as the wave-function renormalization constant of the neutron tends to zero,

$$N = \lambda\pi N\,,$$

where $N$ is a spinor field and $\pi$ a scalar. Now clearly this equation holds for each component $N_\alpha$ say of $N$. If we can cancel the factors $N$, we would have an equation $\pi=$const. This would appear to be inconsistent with a $\pi$ having nontrivial dynamics. The following questions then arise:

(a) When can we cancel factors?

(b) Does an equation $\pi=$const imply an essentially trivial $\pi$?

It turns out that for a bootstrapped system we can discuss (a) in many cases of interest; this is done in Sec. 7 of this paper. Again, question (b) can be answered in the bootstrap context. (The answer is "yes" because the equation $\pi=$const then *determines the dynamics* of the $\pi$, which it does not now do.)

We will now consider briefly a few nonbootstrap possibilities, assuming that we can give a positive answer to both questions posed above.

*Scalar $\pi$ with derivative coupling:* The composite condition is $N=(\partial_\mu\pi)\gamma_\mu N$ and we can say nothing because the $\gamma_\mu$ cannot be simultaneously diagonalized.

*Pseudoscalar $\pi$ with Yukawa coupling:* We get

$$N=\pi\gamma_5 N$$

and if our assumptions are correct, $\pi$ is again trivial.

*Pseudoscalar $\pi$ with derivative coupling:*

$$N=\partial_\mu\phi(\gamma_\mu\gamma_5 N)$$

and we can say nothing.

Our conclusions are as follows: If cancellations are allowed and if our "triviality condition" is correct, we must have derivative coupling in order to describe the neutron as a composite of $\pi$ and $N$. This agrees with a recent result of Huang and Low.[27]

In particular, if the particles are completely bootstrapped, we can give a much fuller discussion of our two conditions; see Sec. 6.

## 6. BOOTSTRAPPED SYSTEMS

We define a bootstrapped system of particles to be one composed of a system of particles interacting through a local Lagrangian, with each particle having its wave-function renormalization constant tending to zero. That is, we first quantize with all $Z_c$'s nonzero, then we let them tend to zero. In this case the resulting system of field equations reduces to a system of constraint equations. We have already discussed in Sec. 3 the manner in which we may interpret the field equations for one particle as its wave-function renormalization constant tends to zero; this interpretation was in terms of the Green's functions of the theory. We now extend that discussion to the case of a bootstrapped system. Our previous discussion showed that the composite particles we were describing were identical to those considered by others. Now we wish to relate our bootstrap equations to those discussed elsewhere.

As a preliminary problem, we could discuss the partially bootstrapped system of Sec. 3, in which a composite particle bootstraps itself out of itself and an elementary particle. Thus Eq. (2) becomes, in terms of the composite-particle field $\phi_c$ and the elementary-particle field $\phi_a$:

$$\phi_c(x)=\lambda\phi_c(x)\phi_a(x)\,. \tag{17}$$

The corresponding Lagrangian for this will be, for constant $\mu$,

$$\mathcal{L}'=\mathcal{L}(\phi_a)+\mu\phi_c(\tfrac{1}{2}\phi_c+\tfrac{1}{3}\lambda\phi_c\phi_a)$$

[34] I. Bialynicki-Birula (private communication).

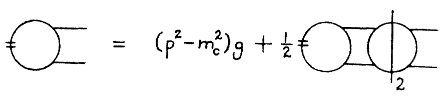



FIG. 32. Charge-renormalized vertex-function equation.

so can be derived, in the limit $Z_c \to 0$, from the complete Lagrangian

$$\mathcal{L} = \mathcal{L}(\phi_a) + [1-(1-Z_c)]\mathcal{L}_0(\psi_c) + \tfrac{1}{2}Z_c\delta m_c^2\phi_c^2 + g\phi_c^2\phi_a\,, \quad (18)$$

where $Z_c\delta m_c^2 = \mu$, $g = -\frac{1}{3}\lambda\mu$ and $\mathcal{L}(\psi_c)$ is the free Lagrangian for the $c$ particle with physical mass $m_c$. Thus, we may calculate with (30), treating $c$ as elementary, and add the condition $Z_c \to 0$. Alternatively we may attempt to solve Fig. 2 directly by operator methods.[29] We can make the whole system bootstrapped if we take $Z_a \to 0$ in $\mathcal{L}(\phi_a)$. With

$$\mathcal{L}(\phi_a) = [1-(1-Z_a)\mathcal{L}_0(\phi_a) + \tfrac{1}{2}Z_a\delta m_a^2\phi_a^2 + \tau\phi_a\mathcal{L}(\phi_a)\,,$$

the resulting equation is

$$\phi_a = \nu\mathcal{L}'(\phi_a)\,, \quad (19)$$

where $\nu = \lim_{Z_a\to 0}(-2\tau/Z_a\delta m_a^2)$. Thus, our complete bootstrapped system is composed of the two coupled equations (17) and (19). This is the type of equation we wish to discuss.

As we saw in Sec. 3 we may write down such a system of equations in terms of Green's-function equations; the

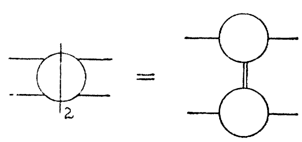



FIG. 33. Approximation to the relativistic "potential."

extension of the discussion in Sec. 3 to Fig. 9 or to the more general equations with all wave-function renormalization constants equal to zero is straightforward and need not be given here.

We would like to relate our bootstraps to those of the $N/D$ and vertex-function type. This cannot be done directly, because of the approximations in the latter equations. We proceed from the composite equations of Sec. 3. The charge-renormalized vertex-function equation is given in Fig. 32, where the composite particle is not present in the potential of Fig. 27 but is generated by its iteration (cf. Fig. 28). We now approximate the relativistic potential as indicated in Fig. 33. Thus Fig. 32 becomes Fig. 34.

When we approximate the vertex function (Fig. 35) to be a constant $g_0$ in Fig. 34, we see that on the mass shell of the composite particle, Fig. 34 is now just the vertex equation of Cutkosky (see his paper in Ref. 6). We may also reduce Figs. 34 and 16 to the more usual set of bootstrap equations given by Rockmore.[24] We do this in the approximation of constant vertex functions and by using the potential approximation of Fig. 33 in dispersion theory. The equation of Fig. 34, evaluated by dispersion methods keeping only two-particle intermediate states, is the condition that the $D$ function $D(s)$ vanish at $s=M_c^2$ in the $N/D$ bootstrap method (Rockmore, Ref. 24). This follows because we are dealing with single-particle exchange, and so the left-hand discontinuity for the $N$ function is given. Further, Fig. 16, evaluated by dispersion methods, is the residue condition

$$g^{-2} = -[(dD/ds)/8\pi N(s)]|_{s=m_c^2}$$

in the $N/D$ bootstrap method (Rockmore, Ref. 24). We see that this type of bootstrap is only an approximation

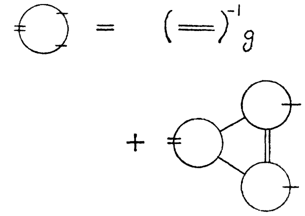



FIG. 34. Approximate vertex-function equation.

to our composite equations; it is not a complete bootstrap. However, if the $a$ particle is treated in a similar fashion, we arrive at a complete system of $N/D$ or vertex-function bootstraps. These will be an approximation to the complete set of Green's-function equations with all the wave-function renormalization constants now put equal to zero.

We may extend this argument to the two-particle unitarity approximation, following the arguments of Lee *et al.* and Ida.[28] For our theory with an elementary particle $A$ in it will satisfy unitarity and have the usual analyticity properties.[4] We now keep only two-particle intermediate states in the dispersion analysis of the $A$-particle propagator and vertex-function and the two-particle scattering amplitude. The resulting equations, in the limit $Z_A \to 0$, will be identical with the $N/D$ equations arising from a model in which the $A$ particle is composite, as shown by the above authors.

We can also relate our equations to the Reggeized bootstraps discussed by Kaus and Zachariasen[23] and by Pran Nath.[35] This discussion is carried out in the two-particle unitarity approximation by means of the identification of composite particles as Regge poles in

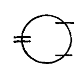


FIG. 35. Form of the vertex approximated by a constant in certain bootstraps.

[35] Pran Nath, University of Pittsburgh report (unpublished). Since the present paper went to press, considerable further work has appeared on Reggeized bootstraps: See W. J. Abbe, P. Kaus, P. Nath, and Y. N. Srivastava, Phys. Rev. **141**, 1513 (1966) and references quoted there. The detailed connection of such Regge-pole discussions of bootstraps with field theory is not yet well understood, at least by the present authors. We wish to thank Dr. Pran Nath for a correspondence about this connection.

the Bethe-Salpeter equation,[36] The first-named authors find that the vertex-function renormalization constant has to be set equal to zero. We have already mentioned our general objections to this, although it may arise in certain approximations.

Summing up, then, our bootstrap systems are a complete operator form of all the currently used approximate bootstrap systems.

## 7. ARE THERE SOLUTIONS TO THE BOOTSTRAP EQUATIONS?

We have seen how the usual Lagrangian variational equations such as (1), which would "normally" be expected to have a typical form (pseudoscalar-meson theory[3])

$$(\Box^2+m^2)\phi=G\phi^3 \quad \text{(or other interaction term)}, \tag{20}$$

may lead instead, through the use of elementary-composite particles and the associated conditions on the renormalization constants, to an equation of a completely different type, namely

$$\phi=g\phi^3 \quad \text{where} \quad g=G(\delta m^2)^{-1}. \tag{21}$$

There is a vast literature, reviewed, for instance, in Ref. 3, on field equations similar to (20); but for (21) there is virtually nothing. In this section we will set up an apparatus by which such equations can be handled.

Obviously the first question about such an equation as (21) or, say, (17), is: Can one cancel the factor common to both sides? If one can, there result the essentially trivial equations $\phi^2=$constant (or $\phi_a$ constant). If not, even the apparently simple equations (17) or (21) may perhaps have physically nontrivial solutions (compare Hagen[26]). This can partly be reduced to an algebraic question, and we have summarized briefly some of the relevant algebraic concepts in the Appendices 1 and 2.

Related to this is the following physically interesting problem: Under what circumstances does the structure of the Eq. (17) or (21) impose a restriction on, or even fix, the values of the coupling constants $g$ or $\lambda$? We will see that it is possible in certain cases to give a definite answer to this question.

Just as do the usual field equations, our equation includes interaction terms containing products of fields at the same point. We (and others) have discussed this question elsewhere.[15,37] In any case, it will not matter *how* we define the product, provided it is associative and commutative.

There are evidently serious difficulties to be faced in making these assumptions. The basic one is that in general we expect the fields to be operator-valued distributions, and so not defined (as perhaps unbounded operators) at each point of space-time. The necessity of considering such distributions has been proved in local field theory under very natural assumptions by Wightman.[38] Since we do not wish to put ourselves "beyond the pale" of general field theory by violating one or other of these assumptions, it is necessary for us to say a few words on this point here.

The problem we face in defining a theory of products for operator-valued distributions is already present when one tries to set up a theory of products for scalar-valued distributions.[37] There is the well-known paradox of Schwartz[39] which shows that an associative product of distributions cannot in general be defined. However, one can set up an associative and commutative product for the distributions arising in perturbation theory; these products are local.[37] The extension of this discussion to field operators has not yet been done, but there is no reason why it should not be possible. By contrast with many people working in field theory, we regard this problem as one of the central ones of the subject, because of its intimate connection with the renormalization. In all the usual cases the corresponding Green's functions become well-defined through renormalization[3] and so implicitly define a product. The experimental successes of quantum electrodynamics may be regarded as a further justification of such a way of thinking. Thus, our assumption that the fields generate at a point an associative and commutative algebra of polynomials is fully supported by the known renormalization procedure in the case of renormalizable interactions and is not at all inconsistent with the basic properties of fields which we have been using in earlier arguments.

On the basis we will deal first of all with a single scalar particle bootstrapping itself; this case can be treated in detail and all the essential ideas carried through explicitly. Then in succeeding subsections we treat more briefly several different scalar particles, particles of higher spin and derivative couplings, and finally the possibility of building symmetries ($SU_3$, etc.) into the crossing matrix.

### A. A Single Scalar or Pseudoscalar Particle Bootstrapping Itself

We already know[3] how to make sense outside perturbation theory of a Green's-function treatment of a field equation with interaction term

$$\sum_{n=2}^{k} g_n\phi^n;$$

correspondingly we get the bootstrap system

$$\delta m^2\phi=\sum_{n=2}^{k} g_n\phi^{n-1} \tag{22}$$

[36] B. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962). J. G. Taylor, paper V of Ref. 3.
[37] J. G. Taylor, Nuovo Cimento **17**, 695 (1960), and further references quoted therein. See also H. G. Bremerman, Berkeley report (unpublished).
[38] A. S. Wightman, Ann. Inst. Henri Poincaré **1**, 403 (1964).
[39] L. Schwartz, Compt. Rend. **239**, 847 (1954).

or, more simply and generally,

$$p(\phi)=0, \tag{23}$$

where $p$ is a polynomial whose coefficients are to be interpreted as coupling constants.

As throughout the paper, we assume we possess an adequate theory of products. Then the field generates algebraically *at each space-time* point a commutative ring $\mathbf{\Phi}$.

Because no derivatives appear in the bootstrap equation, it can be regarded as a relation on the ring $\mathbf{\Phi}$, and so the ring $\mathbf{\Psi}$ of solutions will be a quotient ring of $\mathbf{\Phi}$, as discussed in reference.[40]

We now pass on to a discussion of the other physical conditions which will affect the structure of the ring. Because of our special construction in which each point can be considered separately, the Lorentz group will not enter in, nor will causality, except insofar as it is connected with the use of canonical commutation relations (CCR). The CCR's themselves will not appear in the discussion either, since the equation contains no derivatives; however, they must be fixed *a priori* in accordance with our earlier discussion in Sec. 3, since the bootstrap is a $Z_3\rightarrow 0$ limit of the quantized field equation. The asymptotic condition will alter the values of the constants in the equation in a way we will demonstrate in detail below, but will not have any other effect. For the moment, then, we can ignore all these features; it is not that they are irrelevant, but rather that we have chosen a method which isolates aspects of the dynamical equation unconnected with them.

Our arguments will thus have a satisfying generality, but will certainly be incomplete; on the other hand, we can derive from them enough information to give a preliminary classification, as we shall see. Thus, the structure of our ring is determined solely by the bootstrap itself.

Denote then the complex field by $C$, and let $\theta$ be an indeterminate. Considering for definiteness equation (21) the ring of solutions will be isomorphic to the polynomial domain $C[\theta]$ of all polynomials with coefficients in $C$, in which polynomials differing by differences of which $m^2\theta-g\theta^3$ is a factor are to be identified. In other words, $\mathbf{\Psi}$ is the quotient of $C[\theta]$ with respect to the *ideal generated by the bootstrap:*

$$\mathbf{\Psi}\approx\frac{C[\theta]}{(\theta-g\theta^3)}, \tag{24}$$

where as usual in ideal theory[41] (Appendix 1), $(\cdots)$ denotes the ideal generated by whatever appears between the brackets.

Now the polynomial $\theta-g\theta^3$ is, for nonzero $g$, without repeated factors. Hence the ideal $(\theta-g\theta^3)$ is the intersection (cf. Theorem A9) of prime (Def. A4)—and not merely primary, Def. A5—ideals.

Correspondingly, $\mathbf{\Psi}$ is isomorphic to the direct sum of the three fields

$$\frac{C[\theta]}{(\theta)},\quad \frac{C[\theta]}{(1\pm g^{1/2}\theta)}, \tag{25}$$

and in particular is semi-simple. This is just the "compound" situation described in Appendix 2, and is of no interest because within each of the three direct summands the quantized field reduces to a constant. We will have to use the word "field" ambiguously in the mathematical and physical senses. We adopt the convention of always associating the physical use of the word with a symbol or with such a word as "quantized" or "operator."

We conclude, then, that the bootstrap system (21) cannot lead to anything of physical interest. This argument can be extended to the more general polynomial bootstrap (22) or (23) and indeed to other situations capable of description by commutative rings. This is done in Appendix 3. In the ring structure we have been dealing with, the result is the following:

*Result:* Let $p(\psi)=0$ be a bootstrap in the quantized field variable $\psi$, where $p$ is a polynomial and the variable $\psi$ takes values in some as yet unspecified ring $\mathbf{\Psi}$. Then if $p$ has no repeated zeros, all solutions of the bootstrap $p(\psi)=0$ will come from the superimposition of trivial solutions $\psi=\text{const}$. However, if $p$ has repeated zeros, then $p(\psi)=0$ will have other solutions, the "peculiar" solutions (Appendix 2) in appropriate rings $\mathbf{\Psi}$.

More explicitly, the semi-simple case will be a direct sum of $n$ solutions of the form $\psi(x)=f(x)I$, where $f(x)$ is a distribution and $I$ is the identity operator; there can be no particle creation or annihilation.

We note that one of the main problems of this (and other) discussions of quantum field theory is that no suitable criteria are known for restricting the classes of rings (or other categories, in the sense of homological algebra) in which solutions should be sought. In fact we consider the category to be *one of the variables in the problem* since we are certainly not in a position to talk about correctness classes, etc. We return to this problem below.

The simplest case of a bootstrap of this type is that for scalar or pseudoscalar mesons bootstrapping themselves through a square interaction and is of the form $\psi=g\psi^2$. We now discuss the effect of the asymptotic condition. This causes one to modify the equation to

$$\psi_1=g_1\psi_1^2+b, \tag{26}$$

where the constant $b$ and the renormalized coupling constant $g_1$ are chosen so that there is no vacuum polarization:

$$\langle 0|\psi_1(x)|0\rangle=0 \tag{27}$$

[40] M. M. Broido, Cambridge University report, 1964 (unpublished).

[41] B. L. van der Waerden, *Modern Algebra* (Frederick Ungar Publishing Company, New York, 1953), especially Chaps. 12–13.

and so that there is a pole of the right type at the physical mass:

$$\int d^4x\, e^{ipx}(p^2-m^2)\langle 0|T(\psi_1(x)\psi_1(0))|0\rangle|_{p^2=m^2}=1\,,$$

where $T$ denotes the time-ordered product. Here the double root condition for (26) is

$$1-4g_1b=0\,,$$

while (27) gives us

$$g_1\|\psi_1|0\rangle\|^2+b=0$$

so that the double root occurs only if

$$1+4g_1^2\|\psi_1|0\rangle\|^2=0\,,$$

which is impossible. One again concludes that the equation cannot have "peculiar" solutions of physical interest.

One can apply similar arguments to the case of a single higher interaction term, with the same result. Tentatively, then, we draw the following conclusion: a scalar or pseudoscalar meson bootstrapping itself cannot be described by a field theory of the usual type with a single nonderivative interaction. Though such a system may perhaps be described approximately by, say, an $N/D$ calculation, one now knows that such a calculation cannot be an approximation to a full field-theoretic description.

In spite of the physically negative result, it may be worth mentioning the problems associated with the calculation and interpretation of the "peculiar" solutions. In more usual situations one can most simply consider function algebras (obtained for instance from analogs of the Gelfand-Neumark theorem[16]). Since these are always semi-simple, they are useless here. One would have to use non-normal operators, which of course generate algebras without involution. Any detailed work will then have to be done without the aid of representation theorems. Since there is no physical continuity because of special values of the coupling constants, calculations would have to be done by algebraic and not transcendental methods. This last difficulty is likely to arise in any theory which aims to determine the constants internally rather than feeding them in from the outside, and so should not deter one. It should be observed that the situation is not similar to that of a linear eigenvalue problem, since there is no "eigenfunction expansion" available.

On the credit side we see that we may have here a field-theoretic way of obtaining bootstrap solutions for special values of coupling constants. This has been constantly advocated by Chew as the way the world may work; we may have here a definite way of achieving this possibility. We hope to return to this elsewhere.

### B. Several Scalar Particles

Again we use the "glue is unimportant" idea; any interaction Lagrangian with a genuine interaction must give the same result, so we try the simplest possible. An $\mathcal{L}_i=\sum_i \phi_i^2$ is trivial (mass terms only), so we consider a Yukawa-type 3-field interaction. We mention briefly various cases:

(1) All particles have the same quantum numbers. Then, according to the previous section, one can in principle use just one field and the arguments of the previous subsection apply.

(2) Consider a neutral $\pi$ and a neutral $K$ interacting through $\mathcal{L}_I=G\pi K^2$: we get the bootstraps

$$\pi=\lambda_1K^2\,,$$
$$K=\lambda_2\pi K\,,$$

and immediately $K=\lambda_1\lambda_2K^3$, which is clearly uninteresting by the arguments of the previous subsection.

Now suppose $K$ is charged, $K\neq\bar{K}$, and that we have the interaction

$$\mathcal{L}_I=G\pi K\bar{K}\,,$$

there will also be the usual self-mass term $K\bar{K}$, so we get

$$\begin{aligned}\pi&=\lambda_1K\bar{K}\,,\\ K&=\lambda_2\pi K\,,\\ \bar{K}&=\lambda_2\pi\bar{K}\,.\end{aligned}\tag{28}$$

Even if we regard the variations in $K$ and $\bar{K}$ as independent, the last two equations are not independent. We then have to consider the ring obtained by eliminating one of the variables, say $\pi$:

$$\frac{C[\theta_K,\theta_{\bar{K}}]}{(K)(1-\lambda_1\lambda_2\bar{K}^2)}\,,\tag{29}$$

which is obviously semisimple.

If $K$ and $\bar{K}$ are varied together, it is convenient to go over to the usual linear combinations

$$K_{1,2}=K\pm\bar{K}$$

and the Lagrangian

$$\mathcal{L}_I=G'\pi K_1K_2\,,$$

giving

$$\begin{aligned}\pi&=\lambda_1'K_1K_2\,,\\ K_1&=\lambda_2'\pi K_1\,,\\ K_2&=\lambda_2'\pi K_2\,.\end{aligned}\tag{30}$$

We are now dealing with three quantities which are independent over the real field, so we can discuss the ring

$$\frac{R[\theta_\pi,\theta_{K_1},\theta_{K_2}]}{(K_1)(K_2)(\pi-\lambda_1'K_1K_2)(1-\lambda_2'\pi)}\,,\tag{31}$$

where $R$ is the real field.

That this is the correct ring, and that it is semi-simple, are nontrivial statements; they are justified in Appendix 3.

### C. Mesons and Fermions

As usual, we will consider Yukawa-type Lagrangians.

For a scalar meson, we have $\mathcal{L}_I = g\psi\bar{\psi}\phi$ with the bootstraps $\psi = \lambda\phi\psi$, $\phi = \lambda'\bar{\psi}\psi$ giving for the spinor components

$$\psi_\alpha = \Big(\sum_{\beta=1}^{4} (\psi_\beta(x)\bar{\psi}_\beta(x)\Big)\lambda\lambda'\psi_\alpha\,,$$

which is trivial in spin space (though not yet necessarily in the space of operators). However, it can now be treated by the arguments of subsection B, and again the result is negative.

The other natural Lagrangian is

$$\mathcal{L}_I = g\bar{\psi}\gamma_\mu\psi\partial_\mu\phi$$

but this is identically zero by Green's theorem, since the current $\bar{\psi}\gamma\hat{\mu}\psi$ is conserved.

If the meson is a pseudoscalar, we may consider $\mathcal{L}_I = g\bar{\psi}\gamma_5\psi\phi$, giving

$$\psi = \lambda\gamma_5\psi\phi\,, \tag{32}$$

$$\phi = \lambda'\bar{\psi}\gamma_5\psi\,, \tag{33}$$

so that

$$\psi = \lambda''\gamma_5\psi(\bar{\psi}\gamma_5\psi)\,,$$

which gives either $\psi = 0$ or $\phi = \text{const}$: no solution. We may also consider adding in the Matthews term $\mu\phi^4$, say, to the Lagrangian. This does not change the bootstrap (32), but the bootstrap (33) now becomes

$$\phi = \lambda'\bar{\psi}\gamma_5\psi + \mu\phi^3\,. \tag{34}$$

The coupled equations (32) and (34) can be handled as follows: Write

$$a = \bar{\psi}\gamma_5\psi\,,$$
$$b = \bar{\psi}\psi\,.$$

Then the system (32), (34) becomes

$$\begin{aligned} a &= \lambda b\phi\,,\\ b &= \lambda a\phi\,,\\ \phi &= \lambda' a + \mu\phi^3\,, \end{aligned} \tag{35}$$

so that

$$\phi[1 - (\lambda^2 + \mu)\phi^2 + \mu\lambda^2\phi^4] = 0\,.$$

This equation in $\phi$ will have double roots if $\lambda^2 = \mu$. Then we have

$$\phi(\phi^2\lambda^2 - 1)^2 = 0\,.$$

Passing to the quotient ring as usual, we see at once that the factor $\phi$ can be cancelled. (This is also an example of theorem A6.) We get

$$(\phi^2\lambda^2 - 1)^2 = 0\,. \tag{36}$$

Now we can ignore the part of the expressions $a$, $b$ which are not in the ideal of the total algebra generated by $C[\phi]$, since for those parts the Eqs. (35) vanish identically. From (35) we have the relations

$$\begin{aligned} b(1 - \lambda^2\phi^2) &= 0\,,\\ a(1 - \lambda^2\phi^2) &= 0\,. \end{aligned} \tag{37}$$

If we had enough relations of this form relating to different variables $a$, $b$, say, $c$, $d\ldots$ we could immediately infer that the common factor $(1 - \lambda^2\phi^2)$ must vanish, which would be enough to show that there were no peculiar solutions. Even the relations $0 \neq a \neq b \neq 0$ represent a strong plausibility argument for claiming that necessarily $(1 - \lambda^2\phi^2) = 0$ identically. In other words, if we add the Matthews term $\mu\phi^4$, there can be no solution unless $\lambda^2 = \mu$, and this peculiar solution is even more pathological than usual, if it exists at all. This is an excellent illustration of our extension of the "glue is unimportant" principle of Sec. 4. This and subsection 7B are the examples of the way this principle works in practice that we promised in that section.

So far we have failed to find a set of bootstraps which could be hoped to give a nontrivial result, even admitting very general rings. However, when we come to consider derivative couplings, the situation cannot be handled by our methods. Consider, say, the partially conserved axial-vector current (PCAC) derived from the Lagrangian

$$\mathcal{L}_I = g\bar{\psi}\gamma_5\gamma_\mu\psi\partial_\mu\phi\,.$$

The bootstraps are

$$\psi = \lambda\gamma_5\gamma_\mu\psi\partial_\mu\phi\,,$$
$$\phi = \lambda\partial_\mu(\bar{\psi}\gamma_5\gamma_\mu\psi)\,,$$

and these do seem to be nontrivial, even if we can somehow cancel factors.

This gives, we feel, excellent support to the result of Huang and Low, who suggest[27] that only the PCAC interaction can lead to anything. We will not here consider further how to solve the resulting equations, since they are outside the scope of the above discussion, but will return to this topic in a later article.

### D. Symmetries

We finally discuss whether it is possible to bootstrap symmetries by our operator methods; much work has been done on this in the $N/D$ approach,[6] and we may regard our discussion as an attempt to go beyond the two-particle unitarity and one-particle-exchange approximations made there. In the process of doing this we will, of course, have to make certain *a priori* unjustified assumptions concerning the nature of solutions to our bootstrap equations; we will spell out these assumptions in full whenever we make them, but will not justify them here. We feel that our results are of enough interest to warrant this. We hope to give such justifications elsewhere.

Let us consider first a set of scalar neutral fields $\phi_1,\ldots,\phi_N$, and let them interact with each other through a Lagrangian

$$\mathfrak{L}=\mathfrak{L}_0+A_{ijkl}\phi_i\phi_j\phi_k\phi_l+\sum_{i=1}^{N}\delta m_i^2\phi_i^2 Z_i, \qquad (38)$$

where the summation convention is being used, and $\mathfrak{L}_0$ is the kinetic energy part of $\mathfrak{L}$. Here $A_{ijkl}$ is a completely symmetric tensor of rank 4. In the limit $Z_i\to 0$ the equations of motion derived from (38) become

$$\phi_i=B_{ijkl}\phi_j\phi_k\phi_l, \qquad (39)$$

where $B_{ijkl}=\lim_{Z_i\to 0}A_{ijkl}/\delta m_i^2 Z_i$ is assumed to exist and be nonzero, as we discussed earlier, in Sec. 3, for a composite. The problem we are now faced with is: Under what conditions does (39) have a nontrivial solution? This problem is more difficult than the similar ones met earlier in this section since we cannot, in general, obtain a polynomial equation involving only one of the fields from (39); if we can, then we may deduce that there are only trivial solutions except when there are repeated roots, in which case a peculiar solution may occur. It has not been possible to obtain simple conditions on $B_{ijkl}$ corresponding to the requirement of obtaining a polynomial in a single field which has a multiple root. We may attempt to argue in a less precise fashion as follows. We suppose that the $N$ equations (39) will have only trivial solutions unless they are linearly dependent, so at least one of the equations is redundant. In this case we would require a linear relation between the $n$ equations, so that there exists a set of $n$ real numbers $\lambda_1,\cdots,\lambda_N$ so that

$$\lambda_i[\phi_i-B_{ijkl}\phi_j\phi_k\phi_l]=0 \qquad (40)$$

for any $\phi_1,\cdots,\phi_N$. This is evidently not possible, so we conclude that, barring peculiar solutions, it is unlikely that (38) allows a bootstrap. If we use the "glue is unimportant" argument we may conclude that it is unlikely that a system of neutral scalar particles can bootstrap itself, independent of any symmetry. We note that this further condition on (39), that it have less than $N$ independent equations in order that a bootstrap be possible, is a very reasonable one; it allows for an infinite number of values of the field at a point, and this is a prerequisite for obtaining a particle-like structure.

We may immediately extend our argument to scalar charged particles and particles with higher spin and find similarly that no bootstrap is possible, unless, of course, derivative couplings are admitted. If we do not allow these—we would like the world to be as "smooth" or as "renormalizable" as possible—then we have to turn to fermions to get a possible bootstrap. Let us consider, then, a set $(q_1,\cdots,q_N)$ of spin-$\frac{1}{2}$ fields. The Lagrangian for them will be similar to (38)

$$\mathfrak{L}=\mathfrak{L}_0+A_{ijkl}\bar{q}_i\bar{q}_j q_k q_l, \qquad (41)$$

where now $A_{ijkl}$ is antisymmetric in the pairs $(i,j)$ and $(k,l)$ and $A_{ijkl}^*=A_{klij}$. The bootstrap equations arising from (41) are

$$\begin{aligned} q_i&=\bar{q}_j A_{ijkl}q_k q_l, \\ \bar{q}_k&=\bar{q}_i\bar{q}_j A_{ijkl}q_l. \end{aligned} \qquad (42)$$

We may proceed as we did for (39), and attempt to reduce the $2N$ equations (42) by a condition of linear dependence between them. In this case the impossibility of the equation similar to (40) is not so immediate, since the fields $q_i$ and $\bar{q}_i$ are complex, and satisfy the conditions arising from anticommutation:

$$q_i^2=\bar{q}_i^2=0. \qquad (43)$$

It is condition (43) that prevents us from concluding that the linear dependence of the Eqs. (42) is impossible. We may still attempt to reduce the number of independent equations in (42) as follows; since we are most interested in symmetries and conjecture that the glue is unimportant we will proceed from the interaction term in the Lagrangian

$$\sum\nolimits_\alpha(\bar{q}\Gamma_\alpha q)(\bar{q}\Gamma_\alpha q), \qquad (44)$$

where $1\leqslant\alpha\leqslant N$ and $\{\Gamma_\alpha\}$ is a set of $N\times N$ matrices. The bootstrap equations are now

$$\begin{aligned} \lambda_i q_i&=[(\Gamma_\alpha q)_i,(\bar{q}\Gamma_\alpha q)]_+, \\ \lambda_i\bar{q}_i&=[(\bar{q}\Gamma_\alpha)_i,(\bar{q}\Gamma_\alpha q)]_+, \end{aligned} \qquad (45)$$

where $\lambda_i$ is a suitable nonzero real constant for each $i$. We wish to reduce the number of independent equations arising from (45). To see how this may be achieved we will derive some conditions from (45) and attempt to make these conditions become identities. For this we take any $N\times N$ Hermitian matrix $M$ and construct $(\bar{q}Mq)=\bar{q}_i M_{ij}q_j$. We now specialize further, and choose $\lambda_i=\lambda$, all $i$. This corresponds to taking the effective composite coupling constants to be equal. We then obtain from (45) that

$$\begin{aligned} \lambda(\bar{q}Mq)&=[(\bar{q}M\Gamma_\alpha q),(\bar{q}\Gamma_\alpha q)]_+-(\bar{q}\Gamma_\alpha M\Gamma_\alpha q) \\ &=[(\bar{q}\Gamma_\alpha Mq),(\bar{q}\Gamma_\alpha q)]_+-(\bar{q}\Gamma_\alpha M\Gamma_\alpha q), \end{aligned}$$

so

$$[(\bar{q}[M,\Gamma_\alpha]_-q),(\bar{q}\Gamma_\alpha q)]_+=0. \qquad (46)$$

We see that there are as many equations in (46) as there are independent $N\times N$ Hermitian matrices, i.e., $(N^2-1)$ (where we have taken out a common modulus, so chosen all to have determinant $+1$, say). We wish that a number of the equations (46) are identities in $q$ and $\bar{q}$. We have not yet obtained the general condition under which this is possible; but an evident possibility is when the set $\{\Gamma_\alpha\}$ are the generators of a simple Lie algebra, so

$$[\Gamma_\alpha,\Gamma_\beta]_-=C_{\alpha\beta\gamma}\Gamma_\gamma, \qquad (47)$$

where $C_{\alpha\beta\gamma}$ is totally antisymmetric in its indices. Suppose that we take $\{\Gamma_\alpha\}$ to be the generators of the Lie algebra $SU_n$; then if $M$ is in the algebra it is a linear combination of elements $\Gamma_\beta$, and for each of these (47) is true for all $q$, $\bar{q}$:

$$[(\bar{q}[\Gamma_\beta,\Gamma_\alpha]_-q),(\bar{q}\Gamma_\alpha q)]_+=-C_{\alpha\beta\gamma}[(\bar{q}\Gamma_\gamma q),(\bar{q}\Gamma_\alpha q)]_+\equiv 0$$

because of the antisymmetry of $C_{\alpha\beta\gamma}$ in $\alpha$ and $\gamma$. Since there are $(n^2-1)$ independent Hermitian generators of $SU_n$ then if $N\leqslant n$ the equations (46) are identities for any $M$; if $N>n$ then there are $(N^2-n^2)$ equations which are not necessarily identities following from the argument related to (47); while the remaining $(n^2-1)$ equations of (46) will be identities in $q$ and $\bar{q}$. The best chance of having a bootstrap will be when $N\leqslant n$; we do not expect a bootstrap at all if the number of nontrivial equations in (46) is larger or equal to $N$, i.e., if $N^2-n^2\geqslant N$. Thus, a bootstrap will have a chance of occurring if $N(N-1)<n^2$, and so only if $N=n$. Hence we can only bootstrap the fundamental representation of any $SU_n$ group (since we cannot have $N<n$).

It is still possible that particular values of $n$ may be rejected because all fields except one can be eliminated and the resulting polynomial in this field is trivial. As an example of this, let us look at $SU_2$ with $\Gamma_0=1$, $\Gamma_i=\sigma_i$. Then (45) becomes, after suitable algebraic reductions:

$$\begin{aligned} q_1\bar{q}_1&=\mu_1 q_2\bar{q}_2 q_1\bar{q}_1\,,\\ q_2\bar{q}_2&=\mu_2 q_1\bar{q}_1 q_2\bar{q}_2\,,\end{aligned}\tag{48}$$

where $\mu_1$, $\mu_2$ are nonzero constants. If we denote $q_1\bar{q}_1=a$, $q_2\bar{q}_2=b$, we have $a=\mu_1 ab=\mu_1 b/\mu_2$, since $a$ and $b$ commute. Thus $a=\mu_2 a^2$, and we see easily that $\mu_1=\mu_2=1$ and $a^2=a=b=b^2$. The dynamical information in Eq. (48) is completely contained in these equalities, and therefore (48) cannot give rise to bootstrapped particles. Thus, it does not seem to be possible to bootstrap $SU_2$ by our methods. This corresponds to reality, in that we should get $SU_3$. We hope to discuss $SU_3$ and other Lie algebras elsewhere.

*Note added in proof.* Since this paper went to press, there has been great interest in the subject of bootstrapped symmetries. We mention briefly the relation of some of the prominent currents in this region to our present work.

(a) Mass formulas and coupling constants. An extensive series of calculations has been carried out by Chan and others [e.g. Chan Hong-Mo and C. Wilkin, CERN report, 1965 (unpublished) and references quoted there] using the Zachariasen-Zemach bootstrap mechanism [F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962)], together with $SU_3$, dominance of one-particle exchange and various other assumptions. There is reasonable qualitative agreement with experiment. As these authors themselves admit, their work can be criticized for lack of generality (see the *Note added in proof* at the end of Sec. 4); it is difficult to see to what extent their conclusions are independent of their special assumptions.

(b) Weak and electromagnetic effects. Dashen and Frautschi [e.g. Phys. Rev. **143**, 1171 (1966)] have written down a kind of generalized set of coupled dispersion relations for amplitudes (in the various channels) of the type $\gamma+a\rightarrow b$; these will be linear if one works to first order. They discuss general properties of such sets of equations. This discussion can be regarded as an $S$-matrix-theory equivalent of part of our work in the present paper, inasmuch as it is substantially model-independent. However the work in Sec. 5 of the present paper suggests that the photon cannot (at least in field theory) be included in such a bootstrap discussion. The discussion of Dashen and Frautschi appears to contain a determination of the subtraction constant in the nucleon electromagnetic form factor, i.e. a bootstrap mechanism for the photon. These authors have, in all fairness, already considered a difficulty of this type arising in $S$-matrix theory [S. Mandelstam, Nuovo Cimento **30**, 1113 (1963); **30**, 1127 (1963)], and have advanced counter-arguments. However our objection is more serious. Until $S$-matrix theory is capable of accounting for quantum electrodynamics with a mechanism containing a composite photon (say a bootstrap) it will be difficult to understand $S$-matrix treatments of electromagnetic corrections to strong-interaction processes *based on bootstrap calculations*.

A number of authors have also considered certain weak decays of strongly-interacting particles by similar methods: we cannot go into details here.

(c) Dynamical information on strong interactions. The present authors have performed a preliminary analysis of the bootstrap equations (39) and (45) for the $SU_3$ case, in a quark model. Without further assumptions we have been able to show that *commutation* relations are actually inconsistent with $SU_3$; *anticommutation* relations are not only consistent with $SU_3$, but allow a partial dynamical explanation of octet and decouplet enhancement. Details will appear elsewhere.

## 8. CONCLUSION

In this paper we have shown how bound states and composite particles can be described in a Lagrangian field theory without reference to specific models. In particular, we have shown how field equations arise which describe systems of particles which are all composites of each other. We have called these bootstraps, and have justified the description by showing that, in suitable approximations they give rise to the usual types of bootstrap system. Until this point, the discussion is (we claim) conceptually complete and as rigorous as the present condition of Lagrangian field theory allows.

It follows unambiguously from these arguments that the photon is an elementary particle and cannot take part in bootstrapped symmetry schemes.

We have then gone on to classify the field-theoretic bootstraps which we have obtained into those *a priori* capable and those *a priori* incapable of determining the dynamics of a particle-like system. The mathematical treatment of individual equations is here quite rigorous and is complete modulo the solution of the product problem. However, in order to assert that there can be no bootstrap generating the types of particle we have discussed, we have been forced to appeal to an extension of the Haag-Zimmermann "the glue in unimportant" principle. This principle is well-established provided the equations have particle-like solutions; our heuristic extension is to the case when they lack such solutions. If this extension can be justified, we have the following statement:

(1) In our description of composite particles, there cannot be unapproximated bootstrap systems describing the following sets of particles: one or several scalar or pseudoscalar mesons, charged or neutral; neutral scalar mesons coupled to fermions; "bootstrapped symmetries" of a Lie algebra type for integer spin particles.

We cannot make this statement about pseudoscalar mesons coupled to fermions (PCAC model, necessarily with derivative coupling).

Under certain added assumptions, we have shown that it may be possible to bootstrap a set of $n$ spin-$\frac{1}{2}$ particles interacting through an $SU_n$-invariant Lagrangian, for $n>2$. It does not appear possible to obtain $SU_2$ or any $m$-dimensional representation of an $SU_n$ with $m>n$.

We cannot prove that ours is the *only possible* field-theoretic method of obtaining the usual bootstraps, but we regard it as likely. If we are correct in this, the following further statement will doubtless hold:

(2) In the cases where statement 1 holds, any physically meaningful results which apparently arise from standard bootstrap techniques in fact merely reflect the inadequacy of the approximations which are used.

Our work appears to bring the following problems into the limelight in this field:

(1) Devise methods of dealing with bootstraps involving derivative couplings.

(2) Continue the preceding discussion of symmetries.

(3) Discuss symmetry breaking of possible symmetric bootstraps due to small nonzero values of wave-function renormalization constants (especially for $SU_6$).

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## APPENDIX 1: IDEALS IN COMMUTATIVE RINGS

References are to van der Waerden's *Modern Algebra* (Frederick Ungar Publishing Company, New York, 1953).

*Definition A1.* Let $R$ be a ring, $I$ an ideal of $R$. $I$ is said to be *maximal* if it is not properly contained in any other ideal of $R$.

*Theorem A2.* If $R$ is a ring with identity and $I$ a maximal ideal of $R$, then the residue class ring $R/I$ is a field (Sec. 17).

For the remainder of this appendix, let $R$ denote a *commutative* ring. One can then go some way towards classifying the ideals according to how "near" the residue class rings are to being fields:

*Definition A3.* The ring $R$ is said to be a *domain of integrity* if it has no divisors of zero.

*Definition A4.* The ideal $I$ of $R$ is *prime* if the residue class ring $R/I$ is a domain of integrity.

*Definition A5.* The ideal $I$ of $R$ is *primary* if in the residue class ring $R/I$, every divisor of zero is nilpotent.

*Theorem A6.* Every primary ideal $I$ is contained in some prime ideal $I'$, which can be taken as the ideal of those elements of $R$, each of which has some power in $I$ (Sec. 86, Prop. 1). $I$ is said to *belong to* $I'$.

*Definition A7.* The *divisor chain condition* is said to hold in $R$ if for every chain of ideals $I_1, I_2 \cdots$ for which $I_n$ is properly contained in $I_{n+1}$ for all $n$, the chain comes to an end after a finite number of terms.

*Theorem A8.* The divisor chain condition holds in every field, in the ring of integers, in every finite ring; if it is valid in $R$, it is valid in every residue class ring $R/I$; if it is valid in the ring $R$ with identity, it is valid in the polynomial domain $R[\theta]$ (Sec. 84).

*Theorem A9.* If the divisor chain condition holds in $R$, every ideal can be represented as the intersection of a finite number of primary ideals (Sec. 87).

*Definition A10.* An ideal which belongs to only one prime ideal (cf. Thm. A6) is said to be *single-primed.*

*Theorem A11.* If the divisor chain condition holds in $R$, every single-primed ideal is primary (Sec. 90).

## APPENDIX 2: THE USE OF RINGS IN FIELD THEORY

The purpose of this appendix is to give an informal account of the types of ring structure which are relevant to quantum field theory and in particular to the special type of problem discussed in this paper. Although one of us has done something of the sort elsewhere in connection with the use of Banach algebras[40] that discussion was not wide or detailed enough for the present context. We will by no means attempt a complete survey, will deal with algebraic matters only (no topology), will not be in any way rigorous, allowing oursevles some vagueness for the sake of more intuitive clarity. In short, we simply attempt to supply some general reasons why certain types of ring structure might be thought to lend themselves to use in certain problems of field theory.

### Semi-simple Rings

A *simple* ring is one without proper ideals. A *semi-simple* ring is essentially one which is the direct sum of simple rings. The usual rings of functions are semi-simple; so are the types of operator algebra normally used in field theory; see Ref. 42. This is why when in the paper we need to discuss more general situations, we find ourselves using rather unusual constructions.

The *radical*[43] of a ring is the intersection of all its maximal (say left) ideals. If $R$ is a ring and $I$ is its radical, which is always a two-sided ideal, then the residue class ring is semisimple. Thus if we have a local field theory described by a ring $R$ with radical $I$, then any concrete realization of it, i.e., any representation of it as an operator algebra (in Hilbert space), is always a representation of $R/I$ and is semisimple. We might just as well have used $R/I$ in the first place, and this is why $C^*$ algebras (which are always semisimple) are used in local field theory.

In the present paper the following problem frequently arises: Under what circumstances is it permissible to *cancel a factor* say from both sides of an equation on a ring

$$\phi\eta=\phi\mu$$

and to assert that $\eta=\mu$? The solution $\phi=0$ is usually of no interest.

Even if the ring is not semisimple we may have the following situation: it is the direct sum of two ideals $I_1$ and $I_2$ with $\phi$ in $I_1$ and $\eta-\mu$ in $I_2$, such that $I_1I_2=0$. In such case we cannot cancel a factor (each factor is a divisor of zero). This situation is normally of no interest, since the problem has been split up into two subproblems (about the ideals $I_1$ and $I_2$) *about which the equation gives no information.*

More generally, if the ring describing the solutions of the equation is a direct sum, we shall refer to the problem as *compound.* Such a problem always splits up into a collection of similar subproblems, one for each direct summand.

As far as quantum field theory is concerned, the point is clearest when (as always in this paper) there is a finite number of direct summands. This corresponds essentially to a finite number of degrees of freedom, whereas for particle-like solutions we require an infinite number of degrees of freedom. We require also that the physical quantities associated with these degrees of freedom be determined by the equation itself; in the mutually annihilating case we mentioned above, this does not happen, i.e., the equation contains insufficient physical information. We continue this discussion after some remarks about the other cases.

[42] Essentially because all irreducible operator algebras are semisimple: M. A. Neumark, *Normed Rings* (English translation: P. Noordhoff, Ltd., Groningen, The Netherlands, 1959), Sec. 7.5.
[43] M. A. Neumark, Ref. 42.

Consider now another situation: the ring is a *domain of integrity* (Defn. A3). In this case we may certainly cancel factors: $\phi(\eta-\mu)=0$ does imply either $\phi=0$ or $\eta=\mu$ and so if for instance we have an equation in one unknown:

$$\sum_{k=1}^{n} a_k\phi^k=0,$$

which (for example) factorizes in the form

$$\prod_{k=1}^{n}(\phi-b_k)=0,$$

we can confidently assert that $\phi=b_k$ for some $k$. This is also of no interest to us, at least when the constants $b_k$ are multiples of the identity, for the cancellation simplifies matters too much. This situation (direct decomposition possible into trivial factors) we shall refer to as *trivial.*

In Sec. 7 of the present paper we are essentially trying to classify the bootstrap equations which have arisen into two classes, those which are obviously trivial in this sense, and the others, which we shall call *peculiar.* We give more detailed discussion of the peculiar cases in the text; they are the ones which have to be considered most carefully, whose special features give rise to physically interesting possibilities such as restrictions on the coupling constants.

We may sum the possibilities up by a little table:

| Description of situation | Type of ring | Interpretation |
|---|---|---|
| Compound | Direct sum | Consider each direct summand separately as below |
| Trivial | Domain of integrity | We can cancel common factors |
| Peculiar | Not a domain of integrity, but also not a direct sum | Must consider for each case separately whether cancellation is possible. |

We can now discuss in more detail how we break down the situations which we described above as compound. Suppose the ring $R$ is the direct sum of mutually annihilating ideals $I_n$, where $n$ runs over some index set. As usual, we consider an equation of the form

$$\phi\eta=0$$

but we do not suppose that $\phi$ and $\eta$ are in mutually annihilating ideals.

We have to solve the problem separately for each $I_n$, and each $I_n$ (if it is not itself a direct sum) can be treated as trivial or peculiar as briefly mentioned above. In particular, if each $I_n$ is simple, $R$ is semisimple; if in addition $R$ is commutative, and so (as is the case in all the usual situations of analysis) the $I_n$ are fields with unique factorization, then the whole solution in $R$ is built up from the trivial solutions in each of the $I_n$ by a kind of orthogonal superposition. Consider for example a problem about a self-adjoint operator in Hilbert space satisfying an equation $p(A)=0$, where $p$ is a polynomial. Solutions will be those operators $A$ whose spectra consists only of points $\lambda$ satisfying $p(\lambda)=0$, and so may be regarded as superpositions of one-dimensional projections with those weights.

If now the ring $R$ is semi-simple but not commutative, the simple rings $I_n$ will in general be *complete matrix rings* over some field, i.e., essentially, rings of all linear operators on some space. The subproblems may then be very difficult to deal with (a problem of this kind is discussed in Ref. 30), but one of the main purposes of the way the bootstrap problems are formulated in this paper is to avoid having to consider this kind of noncommutative situation.

## APPENDIX 3: SOME IDEAL-THEORETIC ARGUMENTS FROM THE TEXT

We develop in detail the ideal-theoretic argument by which one can analyze a bootstrap of the form (22) or (23) taking values in a commutative ring. We rely heavily on the terminology and results of the ideal theory of commutative rings as given in Ref. 41 say, some of which we summarized in Appendix 1; and we also refer by implication to the general discussion of our use of rings given in Appendix 2.

If the polynomial $p$ in the domain $C[\theta]$ is without repeated roots, then the ring $C[\theta]/(p)$ is the direct sum of $n$ copies of $C$, where $n$ is the degree of $p$. The proof is trivial.

Equally important for us, in view of the classification remarks in Appendix 2, is the converse: if $p$ *has* a repeated root, then $C[\theta]/(p)$ is *never* semi-simple.

For consider the repeated factor $\theta^2$ in $p$. This situation is easily shown to have all features of the most general one, provided that the base field is algebraically closed or at least has a unique factorization theorem. It is possible[44] to consider other situations, but they are rather pathological and we will not go here into the complications of interpreting them.

Now the ideal $(\theta^2)$ of $C[\theta]$ is not itself prime, but in the description given by Theorem A6 and the remark after it, it is single-primed (Defn. A10). Now the divisor chain condition (Defn. A7) holds in $C[\theta]$ (Theorem A8), and so by Theorem A11 the single-primed ideal $(\theta^2)$ is also primary, so that every divisor of zero in $C[\theta]/(\theta^2)$ is nilpotent.

In order to show that $C[\theta]/(\theta^2)$ is not semi-simple, it then suffices to show that it possesses a divisor of zero, for the radical (Appendix 2) of a ring can be defined as the ideal generated by the nilpotent elements.[41] In the particular situation we are dealing with, this is trivial, for the image of in the natural homorphism

$$C[\theta]\rightarrow C[\theta]/(\theta^2)$$

is already nilpotent; in other situations this may not be so easy.

All this leads to the result written out explicitly in the text. We have carefully couched this argument in the most general possible terms in order to emphasize that it does not depend in detail on the choice of the base field nor on the particular nonprime single-primed primary ideal causing the trouble. Roughly speaking, in order to establish a result of this type in a general situation the following steps must be performed:

(1) Established a sufficient condition for semi-simplicity formulated in terms of the equation.

(2) Show that the condition is also necessary by demonstrating that if it is violated, nilpotent elements arise.

We show above how, in many cases of interest, this last can be split into 3 substages: establishment of a divisor chain condition; use of primary ideals, etc.; proof that divisors of zero arise in the quotient ring. The first of these is usually trivial, and the third will often follow directly from the equation (unique factorization), so one's ideal-theoretic arguments can be directed exclusively to the second substage.

In Sec. 6B we also use polynomial domains in several variables. We add the following remarks:

The prime ideals are not necessarily maximal, so that the quotient rings are not fields. However, when each irreducible factor is a first-degree polynomial in one variable [e.g., (29), we immediately have a direct sum

$$\frac{C[\theta_K]}{(K)}\oplus\frac{C[\theta_{\bar{K}}]}{(1-\lambda_1\lambda_2\bar{K}^2)}$$

and we can use the previous arguments. Again in (31) the irreducible factors are not all first degree, but each of them obviously does generate a prime ideal. Then each element of (31) is uniquely determined by the remainders when an arbitrary representative from $R[\theta_\pi,\theta_{K_1},\theta_{K_2}]$ is divided by each of the three factors in the denominator of (31), respectively. Since the converse is trivial, this establishes the assertion of the text that (31) is a direct sum of the three quotient domains.

We still have to show that (31) is the correct ring for the problem which it describes. That we *may* use the real field $R$ follows from the reality of the coupling constants $\lambda_1'$, $\lambda_2'$. That we *must* use it follows from the meaninglessness of claiming that, over the complex field, two indeterminates are independent 'except for being complex conjugates.' Finally, we consider the factor $(1-\lambda_2\pi)$ in (31); since it appears twice in the field

[44] M. M. Broido, Churchill dissertation, 1965 (unpublished), especially Chap. 2.

equations (30), why does it not appear twice in (31)—in which case (31) would not be semi-simple? The reason is that we require the ideal representing *all* the solutions of all Eqs. (30) simultaneously; this is just the denominator of (31). To include the factor $(1-\lambda_2'\pi)$ twice would exclude certain such solutions. [Notice that if *one* equation contained a repeated factor, we would have to include it in the analog of (31), unless we could find other arguments why it should be excluded.]

The general philosophy behind this method of dealing with these equations will be discussed more fully in a forthcoming publication.[45]

[45] M. M. Broido, Courant Institute report (to be published).

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# $N$-Body Relativistic Scattering Theory*

LEONARD ROSENBERG
*Department of Physics, New York University, University Heights, New York, New York*
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A set of coupled linear integral equations is proposed as a means of generating Lorentz-invariant multiparticle scattering amplitudes which satisfy truncated unitarity relations. Steady-state nonrelativistic scattering theory, in particular the version based on the generalized Faddeev equations, is used as a guide in the formulation and physical interpretation of the equations. The input to the integral equations is a set of scattering amplitudes for subsystems of particles. Creation and annihilation processes are described by the interchange of terminal-state scattering amplitudes with vertex functions, in complete formal analogy with the nonrelativistic treatment of break-up and capture events. A wave function is introduced, a conserved current is defined, and a perturbation theory for discrete states is set up. Formulas for transition rates, for reaction and decay processes, are derived which agree with the familiar results of time-dependent Hamiltonian theory. The possibility that self-consistency criteria might provide a basis for the determination of the input amplitudes is noted.

## 1. INTRODUCTION

A FORMULATION of steady-state scattering theory for nonrelativistic multiparticle systems in terms of integral equations of the Faddeev type[1] has been described recently,[2] and some of the practical advantages of such a formulation have been discussed. In particular, it was pointed out that since the input to the integral equations does not involve the potentials directly, but rather scattering amplitudes for subsystems of particles, equations of this type may be useful even when the potential picture breaks down. Furthermore, there is the possibility of determining the input amplitudes experimentally.

A relativistic extension of the $N$-body integral equations is proposed here.[3] We have no dynamical principle, equivalent to the Schrödinger equation, from which such an equation can be derived. However, the particular choice of the structure of the equations may be strongly restricted by requiring the equations to have certain reasonable properties. We first note that the structure of the nonrelativistic equations is such that the output of the integral equations will satisfy $N$-body unitarity provided that the input amplitudes satisfy the appropriate subsystem unitarity relations. The relativistic equations have been set up in a similar form so that, as shown in Sec. 3, this unitarity property has been preserved. Here the phrase "$N$-body unitarity" implies a unitarity relation in which intermediate states containing more than $N$ particles are ignored. Of course such a relation can not be correct in the relativistic case since at sufficiently high energies the number of particles in intermediate states may be arbitrarily large. (It is also recognized that crossing symmetry will be violated.) Nevertheless, it is possible to conceive of a successive approximation procedure in which $N$ is increased from one stage to the next. The assumption that such a procedure has reasonable convergence properties is implicit in the present approach. (In the following we shall be concerned only with $N$ finite, and the term "unitarity" is always to be interpreted in the restricted sense discussed above.)

We take the Green's function for the noninteracting system to be a product of Feynman propagators. The Lorentz invariance of the integral equations is then assured if the input amplitudes are chosen to be invariant. The choice of Green's function must be consistent with unitarity but a degree of arbitrariness still remains. Choices different from the one made here have been discussed previously[3] for $N=3$.

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[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. 39, 1459 (1960) [English transl.: Soviet Phys.—JETP 12, 1014 (1961)].
[2] L. Rosenberg, Phys. Rev. 140, B217 (1965).
[3] Different types of relativistic extensions of the three-body Faddeev equations have been reported by V. A. Alessandrini and R. L. Omnes, Phys. Rev. 139, B167 (1965), and by C. Lovelace, D. Z. Freedman, and J. M. Namyslowski (unpublished).