V. CONCLUSIONS

In summary, we have found the following:

(1) Using the two-body decay modes of the 2^+ octet and singlet one cannot determine whether they belong in a **189** or **405** representation of $SU(6)$.

(2) Ordering states by their n_λ eliminates the **189** because it gives the wrong order for the K^{**} and A_2 and has a combination of octet and singlet which disagrees with experiment.

(3) Using n_λ and invoking an interaction which moves the **27** representation high, allows the 2^+ mesons to be accommodated in the **405**. The ratio of octet to singlet for the f^* and f^0 is in good agreement with experiment.

(4) The Schwinger formula should apply to any $q\bar{q}$ system with a relative orbital angular momentum[24] and also to the **189** representation model. It should not apply to the **405** representation model, but Eq. (8) should be used instead. The present experimental data on the masses of the 2^+ mesons seem to favor Eq. (8) rather than the Schwinger formula.

(5) From the systematics of the meson and baryon masses it seems that several interesting features of the quark-quark and quark-antiquark interactions emerge. Their nature may be further tested in the classification of higher baryon and meson resonances.

ACKNOWLEDGMENTS

We wish to thank Professor J. Leitner and Professor N. Samios for useful discussions and for informing us of their results prior to publication. We also wish to thank M. Elitzur and Professor H. J. Lipkin for useful discussions.

PHYSICAL REVIEW VOLUME 147, NUMBER 4 29 JULY 1966

Exact Solution of the One-Photon-Exchange N/D Equations*

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(Received 2 March 1966)

The exact solution of the single-Yukawa-meson-exchange N/D equations is obtained in the zero-meson-mass (Coulomb) limit. The solution sums an infinite series of infrared-divergent ladder-graph fragments into finite, unitary partial-wave amplitudes. As with the Schrödinger equation for the hydrogen atom, one finds an infinite number of bound states (zeros of the D function) with an accumulation point at threshold. These bound-state poles of the scattering amplitude arise from the long-range force in much the same way as dynamical bound states arise generally in dispersion theory, thus allowing a discussion of the long-range force rather naturally in the usual dispersion-theoretic terms. The bound-state poles are neither so deeply bound nor so dense as those of the hydrogen atom, thus providing some understanding of the role of the one-photon-exchange force relative to the (long-range) multiple-photon-exchange forces. The possibilities for extending the technique to the relativistic one-photon case and the question of electromagnetic corrections to the strong interactions are briefly discussed. Finally, some possible approaches to including higher order photon exchanges are considered.

I. INTRODUCTION

ACCORDING to Chew,[1] there are several fundamental reasons why one should not expect to be able to formulate a self-consistent S-matrix theory for electrodynamics in which, for example, it would be possible to "bootstrap" the photon. On the other hand, there is the lesser goal of calculating electrodynamics and electrodynamic corrections to the strong interactions by using dispersion techniques that do not employ self-consistency as a calculational device, e.g., the N/D equations. Since the simplest electrodynamic problem is nonrelativistic Coulomb scattering, we shall concentrate our attention on that.

Several dispersive investigations of Coulomb scattering in the presence of the strong interactions have been carried out.[2–4] However, these studies have all treated the Coulomb part, philosophically, as "known". To date, the pure Coulomb problem has been investi-

* Work supported in part by the U. S. Atomic Energy Commission and in part by the Air Force Office of Scientific Research, Grant No. AF-AFOSR-232-63.

[1] G. F. Chew, University of California Radiation Laboratory Report No. UCRL-10845, 1963 (unpublished).

[2] R. F. Dashen and S. C. Frautschi, Phys. Rev. **135**, B1190 (1964).

[3] H. Cornille and A. Martin, Nuovo Cimento **26**, 298 (1962).

[4] J. Rix, thesis, Harvard University, 1965 (unpublished).

gated dispersively only in the context of its Regge behavior.[5,6]

One overriding difficulty in the study of Coulomb scattering is the occurrence of long-range divergences. Coulomb forces are sufficiently long range that one never has free-wave initial or final states, in contradiction to the basic assumptions of conventional scattering theory. If one considers electrodynamics as the zero-mass limit of a massive-vector-boson theory, one finds that, as the boson mass goes to zero, the phase shift and scattering amplitude go as[7]

$$\delta_l(s)\sim-\infty+\sigma_l(s), \tag{a}$$

$$A(s,\theta)\sim e^{-2i\infty}A_c(s,\theta), \quad \theta\neq 0, \tag{b}$$

where $\sigma_l(s)$ and $A_c(s,\theta)$ are the Coulomb phase shift and scattering amplitude, respectively, as calculated from the Schrödinger equation. We say that there appears an infinite phase common to every partial wave which fails to contribute to the differential cross section. This fact suggests two approaches to the problem of calculating Coulomb scattering by analyticity and unitarity, both of which will be discussed in this article.

(1) Do a unitary calculation using massive vector mesons (N/D etc.), then take the massless limit of the solution, being careful to separate off the infinite phase correctly. This will be the type of technique employed in Sec. II where we will develop an exact solution for the one-vector-meson-exchange N/D equations in the nonrelativistic and zero-meson-mass limit. The relativistic N/D solution will be discussed at the end of Sec. IV.

(2) Study the analyticity and unitarity properties of σ_l and A_c and ask whether they can be calculated directly from these principles, independently of any photon-mass limiting process. This approach will be discussed in Sec. VI. In that relations similar to (a) and (b) hold for quantum electrodynamics[8] (including bremstrahlung), these two approaches can be thought of as probably germane to any future dispersion theory of electrodynamics.

In detail then, in Sec. II we will write down the N/D equations for one-vector-meson exchange in the nonrelativistic limit, the long-range aspects of this approximation being the same as in the relativistic case. In the (Coulomb) limit of zero meson mass, a finite solution to these non-Fredholm integral equations can be found explicitly even though every iteration of the equations is long-range divergent. The solution sums an infinite series of long-range-divergent contributions into finite, unitary partial-wave amplitudes. Since there is no perturbation expansion of the solution in powers of the fine-structure constant α, it is not surprising that the resulting amplitude (whose properties are discussed in Sec. IV) is not regular at zero coupling in the complex coupling plane. As with the Schrödinger equation for the hydrogen atom, we find an infinite number of bound states with an accumulation point at threshold. This fact allows insight into the dispersive connection between the long-range force and the hydrogen-like energy levels. One sees that the poles arise as zeros of the D function in an attempt to make $\alpha(s)D(s)$, the product of D with the (singular) input discontinuity on the left, integrable at threshold. Alternatively, we can say that the poles arise as an attempt by the N/D equations to subtract away some of the pathology of the photon exchange input near threshold, i.e., just the ordinary dispersive mechanism for the generation of bound-state poles.[9] The bound states are not so deeply bound as those of the Schrödinger equation, the one-photon exchange accounting for only 1% of the total binding. The remaining binding comes from the equally long-range multiple-photon exchanges. In addition, the density of poles near threshold is lower than in the hydrogen atom so that multiple-photon exchange is responsible for the binding of further poles as well. In particular, it is responsible for lifting the degeneracy of levels associated with different orbital angular momenta. In Sec. V and Sec. VI, several possible approaches to including the higher order photon exchanges are briefly discussed. In Sec. VI, rather peculiar analyticity and unitarity relations for the Coulomb case are obtained. It is noted that analyticity and unitarity do not allow calculation of the Coulomb scattering amplitude in any simple way.

II. THE NONRELATIVISTIC D-FUNCTION EQUATION

The nonrelativistic partial-wave N and D equations are[10]

$$N_l(s)=\int_{-\infty}^{0}\frac{ds'}{\pi}\frac{\beta_l(s')D_l(s')}{s'-s}, \tag{1}$$

$$D_l(s)=1-\int_{0}^{\infty}\frac{ds'}{\pi}\frac{(\sqrt{s'})N_l(s')}{s'-s}. \tag{2}$$

The function β_l is the imaginary part of the Born series expansion of the scattering amplitude, $A_l(s)=N_l(s)/D_l(s)$, for $s<0$. In the approximation where β_l is the imaginary part of the Born term for an attractive Yukawa potential of range λ^{-1}, one obtains

$$\beta_l{}^{(1)}(s)=\frac{\alpha m\pi}{2s}P_l\left(1+\frac{\lambda^2}{2s}\right)\theta(-s-\tfrac{1}{4}\lambda^2). \tag{3}$$

[5] Yu. M. Malyuta, Zh. Eksperim. i Teor. Fiz. **45**, 1167 (1963) [English transl.: Soviet Phys.—JETP **18**, 804 (1964)].
[6] V. Singh, Phys. Rev. **127**, 632 (1962).
[7] R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).
[8] D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. **13**, 379 (1961).
[9] S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1963).
[10] We use natural units, $\hbar=c=1$, and define s in terms of the nonrelativistic energy; $s\equiv 2mE$. Also $\alpha\cong 1/137$.

By substituting (1) into (2) and doing an internal integration, the following integral equation for $D_l(s)$ is obtained:

$$D_l(s)=1+\tfrac{1}{2}\alpha m\int_{-\infty}^{-\lambda^2/4}\frac{ds'}{s'}\frac{D_l(s')}{\sqrt{(-s)}+\sqrt{(-s')}}\times P_l\left(1+\frac{\lambda^2}{2s'}\right). \quad (4)$$

When $\lambda\to 0$, the Yukawa goes over to a Coulomb potential and Eq. (4) contains no explicit divergences:

$$D_l(s)=1+\tfrac{1}{2}\alpha m\int_{-\infty}^{0}\frac{ds'}{s'}\frac{D_l(s')}{\sqrt{(-s)}+\sqrt{(-s')}}. \quad (5)$$

This, then, is the one-photon-exchange D equation. It is, of course, an assumption that the $\lambda\to 0$ limit in the integral equation (as above) will yield the same result as first solving (4) then taking the limit of the solution.

If a solution to (5) exists, what properties do we expect it to have? In the first place, we note that (5) predicts the same D function for all values of l. Therefore the phase shift derived from (5) will be l-independent. That is, the one-photon exchange is not adequate to lift the degeneracy associated with different orbital quantum numbers. To split this degeneracy must be the task of the multiple-photon exchanges. Since our solutions will be the same in every partial wave, the sum over partial waves cannot sum to the Coulomb amplitude A_c derived from the Schrödinger equation. Next, we note that the iterative solution of (5) in powers of α diverges term by term. Evidently, the solution cannot be a function of α regular at $\alpha=0$. Furthermore, the D function cannot behave like the inhomogeneous term (one) near $s=0$, otherwise the integral in (5) would fail to converge at the upper limit of integration. We will discover, in fact, that the D function oscillates very rapidly near $s=0$ so as to average out the singular effect of the kernel. The resulting zeros of the oscillating D function correspond to an infinite number of bound states (the positronium poles).

We note that a formal attempt to derive an equation for the N function produces an integral equation with a kernel that diverges in the $\lambda\to 0$ limit. It can be shown, however, that the necessary interchange of the order of integration is invalid because of a nonuniformity of convergence caused by the rapid oscillations of the solutions. On the other hand, N can be obtained from D by using (1).

Our method of solution employs techniques developed by Halpern[11] for the study of "nonrenormalizible" field theory problems where

$$\beta_l(s)\underset{s\to-\infty}{\cong}\Gamma s^m\theta(-s),\quad m>0.$$

These techniques are also valid for $m<0$, Eq. (5) corresponding to $m=-1$. For greater detail in what follows, the reader is referred to Ref. 11.

III. SOLUTION OF THE D EQUATION

We introduce into (5) the following substitutions.

$$(-s)^{-1/2}\equiv e^{\xi};\quad (-s')^{-1/2}\equiv e^{\eta};\quad \phi(\xi)\equiv D_l(s). \quad (6)$$

Then (5) becomes

$$\phi(\xi)=1-\alpha m\int_{-\infty}^{\infty}d\eta\frac{e^{\eta}\phi(\eta)}{1+e^{-(\xi-\eta)}}. \quad (7)$$

The factor e^{η} prevents (7) from becoming algebraic in Fourier transform space; instead one obtains a finite-difference equation:

$$\phi(\omega)=\delta(\omega)+\alpha m\pi i\,\mathrm{csch}\pi(\omega-i\epsilon)\phi(\omega+i), \quad (8)$$

where ϵ is an infinitesimal positive number and

$$\phi(\omega)=\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}e^{-i\xi\omega}\phi(\xi). \quad (9)$$

In order that the solution of (8), when transformed, satisfy (7), it is necessary that $\phi(\omega)$ be regular in the "period strip" $0<\mathrm{Im}\omega<1$. The method of solution will guarantee this condition.

First, let us consider the homogeneous counterpart of (8):

$$\phi_H(\omega)=\alpha m\pi i\,\mathrm{csch}\pi(\omega-i\epsilon)\phi_H(\omega+i). \quad (10)$$

We can guess a solution of the form

$$\phi_H(\omega)=\exp\{F(\omega)\}, \quad (11)$$

where $F(\omega)$ satisfies

$$F(\omega+i)-F(\omega)=-\ln[\alpha\pi m i\,\mathrm{csch}\pi(\omega-i\epsilon)]. \quad (12)$$

Equation (12) can be solved immediately by Fourier transform.

$$F(\omega)=\int_{-\infty}^{\infty}\frac{dx}{2\pi}\frac{e^{-ix\omega}}{1-e^{x}}\int_{-\infty}^{\infty}d\omega' e^{ix\omega'}\times\ln[\alpha m\pi i\,\mathrm{csch}\pi(\omega'-i\epsilon)]. \quad (13)$$

The factor $(1-e^{x})^{-1}$ guarantees that ϕ_H will be regular in the period strip. The integrations of (13) can be done exactly. For real ω,

$$F(\omega)=-\tfrac{1}{2}i\pi\epsilon(\omega)(\omega^2+\tfrac{1}{6})+(i/2\pi)\epsilon(\omega)\mathrm{Li}_2[\exp(-2\pi|\omega|)]-i\omega\ln[1-\exp(-2\pi|\omega|)]+i\omega\ln 2\pi\alpha m+\pi\omega, \quad (14)$$

where

$$\epsilon(\omega)\equiv+1\quad\omega>0$$
$$\equiv-1\quad\omega<0$$

$$\mathrm{Li}_2(z)\equiv-\int_0^z\frac{dx}{x}\ln(1-x)\quad\text{(dilogarithm)}.$$

[11] M. B. Halpern, Phys. Rev. **140**, B1570 (1965); also, J. Math. Phys. **7**, 1226 (1966).

Since $\mathrm{Li}_2(1)=\frac{1}{6}\pi^2$, then $F(0)=0$ and $\phi_H(0)=1$. Now, returning to the inhomogeneous equation, we assume a solution of the form

$$\phi(\omega)=G(\omega)\phi_H(\omega). \tag{15}$$

The "Green's function," $G(\omega)$, satisfies

$$G(\omega)-G(\omega+i)=\delta(\omega). \tag{16}$$

The solutions to (16) can be obtained exactly and typified by their behavior for large and small ω.

$$G^{(\pm)}(\omega)=\tfrac{1}{2}i\,\mathrm{csch}\pi(\omega+i\epsilon)e^{\pm\pi\omega};$$

$$G_p(\omega)=\tfrac{1}{2}i\coth\pi(\omega+i\epsilon). \tag{17}$$

Since the D function must be real for $s<0$, its transform $\phi(\omega)$ must obey the following condition:

$$\phi(\omega)=\phi^*(-\omega^*). \tag{18}$$

Therefore we must use $G^{(-)}$ in (15) to guarantee condition (18). The final exact expression for D is then

$$D_l(s)=\tfrac{1}{2}i\int_{-\infty}^{\infty}d\omega\phi_H(\omega)\,\mathrm{csch}\pi(\omega+i\epsilon)\times\exp[-i\omega\ln(-s)^{1/2}-\pi\omega]. \tag{19}$$

Equation (19) is not the only solution to (5). Instead of using ϕ_H as given in (11) and (14) we could have multiplied it by any function, $a(\omega)$, regular in the period strip and on its boundaries with the properties

(1) $a(\omega)=a(\omega+i)$,

(2) $a(\omega)=a^*(-\omega^*)$,

(3) $a(0)\neq 0$.

Then the resulting product $a\phi_H$ would also satisfy (10) (and be analytic in the period strip). Examples of $a(\omega)$ which satisfy the conditions are

$$a_n(\omega)=\cosh(2\pi n\omega)\quad(n=\text{integer}).$$

However, it can be shown that the solution (19), obtained by exponentiation and direct Fourier transform, is distinguished by being the $\lambda\to 0$ limit of the sum of the Neumann-Liouville series solution of (4). Therefore we conclude that it is the desired solution of (5) and we will now study several of its properties.

IV. PROPERTIES OF THE SOLUTION

First we note that D is finite but, according to (14) and (19), it is evidently a function of $\ln[m\alpha/\sqrt{(-s)}]$. Therefore (19) cannot have a convergent series solution in powers of α about the point $\alpha=0$. One concludes then that (4) has a branch point in the complex α plane whose position depends on λ and which moves to $\alpha=0$ when $\lambda\to 0$. The particular form of the branch point at zero coupling will be exhibited below. [See Eq. (37)].

Behavior Immediately below Threshold

For very small negative s ($0>s\gg-\alpha^2m^2$), the integrand of (19) develops saddle points at

$$\omega_\pm\cong\pm\frac{1}{\pi}\ln\left(\frac{2m\pi\alpha}{\sqrt{(-s)}}\right)+i.$$

The application of the saddlepoint method to (19) gives

$$D_l(s)\underset{s\to 0-}{\sim}-\frac{\sqrt{(-2s)}}{m\pi\alpha}\sin\left[\frac{1}{2\pi}\ln^2\left(\frac{2m\pi\alpha}{\sqrt{(-s)}}\right)-\frac{5\pi}{6}\right]. \tag{20}$$

The zeros of (20) correspond to poles of the scattering amplitude. They are infinite in number and, near threshold, appear at

$$s_n\cong-(2\pi m\alpha)^2\exp[-2\pi\sqrt{(2n)}]\quad(n\text{ integer}\gg 1). \tag{21}$$

This result is to be compared with the location of the poles in the exact Coulomb amplitude.

$$s_n=-(m\alpha/n)^2\quad(n\text{ integer}).$$

Although Eq. (5) predicts an infinite number of poles in the scattering amplitude, the number of these poles in a given interval of s is much lower than for the exact Coulomb case. The ratio of these densities near threshold is

$$\frac{\text{No. of one photon exchange poles}}{\text{No. of Coulomb poles}}\underset{s\to 0-}{\sim}\frac{\sqrt{(-s)}}{\pi^2\alpha m}\times\ln\left[-\frac{(2\pi\alpha m)^2}{s}\right].$$

Thus we learn that the higher order photon exchanges must create more poles. A numerical calculation of the D function (19) reveals the position of the ground-state pole at about $s_0=-10^{-2}m^2\alpha^2$. On comparison with the exact Coulomb ground-state position ($s_0=-m^2\alpha^2$), we learn that the one-photon exchange provides only about 1% of the binding. The balance of course comes from the higher order exchanges. Furthermore, the bound-state spectrum predicted by (19) is completely degenerate with respect to the angular momentum l, so that the observed l dependence and binding energy of the Coulomb bound-state spectrum must be accounted for by the inclusion of higher order terms in β_l of Eqs. (1) and (3).

The N Function and the Residues at the Poles

Instead of calculating the N function directly by using (1) and (19), one can work in Fourier transform space with (2). Then (2) becomes

$$\phi(\omega)=\delta(\omega)+i\,\mathrm{csch}\tfrac{1}{2}\pi(\omega-i\epsilon)\,\chi(\omega-i), \tag{22}$$

where

$$N_l(s)=\int_{-\infty}^{\infty} d\omega\, \chi(\omega)e^{-i\omega\ln\sqrt{s}}. \tag{23}$$

One can show that a translation of the contour of integration in (23) to $\mathrm{Im}\omega=-1$ is legitimate. Then, the desired relation is

$$N_l(s)=-\frac{i}{\sqrt{s}}\int_{-\infty}^{\infty} d\omega\, \sinh(\tfrac{1}{2}\pi\omega)\phi(\omega)e^{-i\omega\ln\sqrt{s}}. \tag{24}$$

Below threshold $\sqrt{s}\to+i\sqrt{-s}$ and (24) becomes

$$N_l(s)=-\frac{1}{\sqrt{(-s)}}\int_{-\infty}^{\infty} d\omega \times\sinh(\tfrac{1}{2}\pi\omega)\phi(\omega)e^{-i\omega\ln\sqrt{(-s)}+\frac{1}{2}\pi\omega}. \tag{25}$$

When $s\to 0-$, there is a dominant point of stationary phase at

$$\omega_+=(1/\pi)\ln(2m\pi\alpha/\sqrt{(-s)})\gg 1.$$

The application of the method of stationary phase to (25) gives

$$N_l(s)\underset{s\to 0-}{\sim}\frac{1}{\sqrt{(-2s)}} \times\exp\left\{i\left[\frac{1}{2\pi}\ln^2\left(\frac{2m\pi\alpha}{\sqrt{(-s)}}\right)-\frac{5\pi}{6}\right]\right\}. \tag{26}$$

Notice that, near threshold, Eqs. (20) and (26) demonstrate explicitly that

$$\mathrm{Im}N_l(s)=(m\pi\alpha/2s)D_l(s)=\beta_l(s)D_l(s)(s<0). \tag{27}$$

In fact, Eq. (27) can be demonstrated for all $s<0$ by using (19) and (25).

The *residue* of the nth pole is obtained from (20) and (26):

$$R_n\cong -m\pi\alpha/2\sqrt{(2n)}\ (n\ \text{integer}\gg 1). \tag{28}$$

The sign of R_n is correct for *physical bound states.*

Repulsive Input

For a repulsive Coulomb input, Eq. (5) is changed by having $\alpha\to-\alpha$.[12] Nevertheless, the procedure outlined above is applicable and one finds the counterpart of (14) to be

$$F_R(\omega)=-\tfrac{1}{2}i\pi\epsilon(\omega)(\omega^2+\tfrac{1}{6})+(i/2\pi)\epsilon(\omega)\mathrm{Li}_2(e^{-2\pi|\omega|}) -i\omega\ln(1-e^{-2\pi|\omega|})+i\omega\ln(2\pi|\alpha|m). \tag{29}$$

Now one must use G_p of (17) to obtain the solution for D corresponding to condition (18). A steepest descent calculation of D near threshold yields an infinite set of zeros as in the attractive case. However, the residues of the corresponding poles of the repulsive scattering amplitude are now *positive* and, therefore, correspond to *ghost states*:

$$R_n^{(\mathrm{rep})}\cong m\pi\alpha/\sqrt{(2n)}.$$

The mechanism by which the bound states and ghosts are produced in this dispersive model is the usual one. That is, they arise so as to subtract off the most singular part of the attraction or repulsion.[9] (The δ-function contributions of the bound states to the absorptive part on the left are of the opposite sign relative to the input. The fact that they accumulate toward threshold is demanded by the need to subtract off more and more singularity from the input itself as $s\to 0-$.)

It should also be emphasized that, for the attractive case, the higher order contributions to β_l are all attractive and l-dependent, in which case we expect that a better approximation will bind the states more deeply as well as remove the total degeneracy in l. For the repulsive situation the higher order contributions are alternately attractive and repulsive. Therefore, there is reason to hope that the ghost-state spectrum will vanish as one goes to increasingly better approximations.

High-Energy Behavior

Extracting the behavior of (19) when $|s|\gg m^2\alpha^2$ is quite difficult since there is no saddle point. One must proceed in a less direct manner. By changing variables, Eq. (5) can be rewritten

$$D_l(x)=1-\alpha m x\int_0^{\infty} dx'\frac{D_l(x')}{x+x'}, \tag{30}$$

where $x\equiv(-s)^{-1/2}$. We are free to define a new function $H(x)$:

$$\begin{aligned} D(x)&\equiv 1+H(x) \quad x<(\alpha m)^{-1}\\ &\equiv D(x) \quad x>(\alpha m)^{-1}, \end{aligned} \tag{31}$$

where

$$H(x)=-\alpha m x\ln(1+1/\alpha m x)-\alpha m x A(x); \tag{32}$$

$$A(x)\equiv\int_0^{(\alpha m)^{-1}}\frac{dx'}{x+x'}H(x')+\int_{(\alpha m)^{-1}}^{\infty}\frac{dx'}{x+x'}D(x'). \tag{33}$$

Our problem, now, is to find $A(x)$ near $x=0(s\to-\infty)$. In fact, $A(0)$ is just a finite constant:

$$A(0)=\int_0^{(\alpha m)^{-1}}\frac{dx'}{x'}H(x')+\int_{(\alpha m)^{-1}}^{\infty}\frac{dx'}{x'}D(x'). \tag{34}$$

By using (31) and the explicit solution (19), we can write

$$\begin{Bmatrix}D(x')\\H(x')\end{Bmatrix}=\tfrac{1}{2}i\int_{-\infty}^{\infty} d\omega\,\mathrm{csch}\pi(\omega\pm i\epsilon) \times\exp\{i\omega\ln x'+F(\omega)-\pi\omega\}. \tag{35}$$

Substituting (35) into (34) and interchanging the order

[12] Recall that (19) for D has a branch point at $\alpha=0$, in which case one cannot obtain the repulsive D function simply by replacing $\alpha\to-\alpha$ in (19).

of integration gives

$$A(0)=+i\frac{dF}{d\omega}\bigg|_{\omega=0}-i\pi-\ln\alpha m=1-\ln 2\pi. \quad (36)$$

Therefore, the leading behavior of D as $s\to-\infty$ is given by

$$D_l(s)\underset{s\to-\infty}{\sim}1-\frac{\alpha m}{\sqrt{(-s)}}\left\{\ln\left(\frac{\sqrt{(-s)}}{\alpha m}\right)+1-\ln 2\pi\right\}. \quad (37)$$

In that the D function is a function only of $(m\alpha/\sqrt{(-s)})$, it is clear that large s corresponds also to small α. This then is the promised form of the branch point at zero fine-structure constant.

The D function for large positive s can be obtained from a similar calculation. It turns out to be just the analytic continuation

$$D_l(s)\underset{s\to+\infty}{\sim}1-\frac{\alpha mi}{\sqrt{s}}\left\{\ln\left(\frac{-i\sqrt{s}}{\alpha m}\right)+1-\ln 2\pi\right\}. \quad (38)$$

The leading terms of the phase shift can be obtained from (38):

$$\delta_l(s)\underset{s\to+\infty}{\sim}\frac{\alpha m}{\sqrt{s}}\left\{\ln\left(\frac{\sqrt{s}}{\alpha m}\right)+1-\ln 2\pi\right\}. \quad (39)$$

This should be compared with the leading term of the (attractive) Coulomb phase shift, calculated from the Schrödinger equation

$$\delta_l^{\text{Coul}}(s)\underset{s\to+\infty}{\sim}-(\alpha m/\sqrt{s})\Psi(l+1). \quad (40)$$

Where $\Psi(z)$ is the logarithmic derivative of the Γ function. Our phase does not go to zero as fast as the Coulomb phase (40), having developed a logarithmic creep related to the fact that we have not had to factor off any divergent or unobservable part of the phase shift.

Comparison with the Dashen-Frautschi Technique

A few comments are in order here comparing our technique with that of Dashen and Frautschi,[13] which also purports to use the one-photon-exchange force as input into the N/D equations. Instead of our procedure as outlined above, they give the photon a small mass λ and use essentially only the first iteration of the N/D equations. As $\lambda\to 0$, they find that a logarithmically divergent phase

$$\exp\left\{\frac{-ie^2}{2\sqrt{s}}\ln\left(\frac{g(s)}{\lambda^2}\right)\right\},\quad g(s)=4s \quad (41)$$

factors out (we have learned that this divergence would not have appeared if Dashen and Frautschi had summed all the iterations of the N/D). They take the position that this factor should simply be set to one; that is, they take the peculiar energy-dependent photon mass

$$\lambda=\sqrt{g(s)}=2\sqrt{s}. \quad (42)$$

With this prescription they show that (at least in the nonrelativistic case) electromagnetic mass shifts come our correctly (i.e., they agree with the Schrödinger approach). Evidently, although they start with only the one-photon exchange (which we solve exactly), using the photon mass in this way includes most of the bound-state shift information from all photon exchanges. On the other hand, from our form of the phase shifts (39), it is evident that our equations have, roughly speaking, set

$$\lambda=2\alpha\quad(m=1). \quad (43)$$

The way in which this happened is quite clear. Actually (38) can be written as

$$D_l(s)\sim 1+\tfrac{1}{2}\alpha\int_{-\infty}^{-\alpha^2}\frac{ds'}{s'}\frac{1}{\sqrt{(-s)}+\sqrt{(-s')}},$$

which emphasizes that, roughly speaking again, the oscillations of the D function are so violent for $-\alpha^2<s'<0$ that (for large s) the input is simply oscillated to death in that region. This is of course equivalent to having the asymptotics dominated by the first and second iterations of a massive vector meson ($\lambda=2\alpha$). There is nothing extraordinary about the photon having an "effective mass" in this range; in fact, because we know Dashen and Frautschi's procedure is roughly correct in the nonrelativistic case, ours is certainly less accurate. Actually, if one formally attempts to calculate strong mass shifts with our technique they will, not unexpectedly, come out proportional to $\alpha\ln\alpha$ instead of the desired α. We shall have more to say about the subject of corrections to strong interactions in Sec. V.

The Relativistic One-Photon Exchange

In principle, we can extend our treatment of the nonrelativistic case to the relativistic one-photon exchange. Instead of (1) and (2), we start with the relativistic N/D equations for pseudoscalar-pseudoscalar scattering[9]

$$N_l(\nu)=-\frac{\nu}{\pi}\int_{\text{left}}\frac{d\nu'}{\nu'}\frac{\beta_l(\nu')D_l(\nu')}{\nu'-\nu},$$

$$D_l(\nu)=1-\frac{\nu}{\pi}\int_{\text{right}}\frac{d\nu'}{\nu'}\frac{\rho(\nu')N(\nu')}{\nu'-\nu}, \quad (44)$$

where we have set the S-wave subtraction to zero for simplicity. (The discussion goes through unchanged if we include it.) Just as above, we use a massive vector-meson-exchange input and *then* take the zero-mass limit in the resulting integral equations. The limiting

[13] R. Dashen and S. Frautschi, Phys. Rev. **135**, B1190 (1964).

(one-photon) discontinuity on the left is

$$\beta_l(\nu)=\pm\tfrac{1}{4}\alpha\pi((2\nu+\mu^2)/\nu), \tag{45}$$

where μ is the external mass and the upper sign is the attractive, the lower the repulsive case. The methods developed by Halpern allow an exact solution only in the case of a power-law input, but a method of iteration about the most singular part was outlined there for inputs like these. In this problem, one would solve first by approximating $\beta_l(\nu)$ and $\rho(\nu)$ near threshold. The form of the resulting equations are exactly those studied above ($\beta_l \propto \pm\nu^{-1}$ near threshold). In particular, the positronium poles appear in this first approximation. One then takes the resulting D function, substitutes it into the Hilbert transform (44) giving N (now with the full β_l) to get a better approximation for N and so on.

V. EXTENSIONS OF THE ONE-PHOTON EXCHANGE

In this section, we will consider two generalizations of the one-photon-exchange calculation discussed above. The first will be a conjecture about the role played by the higher order photon exchanges in this N/D framework. The second extension will be a discussion of the possibility of using our technique to calculate electromagnetic corrections to strong mass shifts.

Strictly speaking, the discontinuities of the multiple-photon exchange graphs (for $s<0$) all diverge as $\lambda\to 0$. However the divergence occurs in a manner which permits us to carry it along in our investigation. In particular, when β_l is the imaginary part of the first and second terms of the Born series, the following terms survive in the limit $\lambda\to 0$.

$$\beta_l\equiv\beta_l^{(1)}+\beta_l^{(2)},$$

$$\beta_l^{(2)}(s)\equiv\frac{\alpha^2m^2\pi}{s\sqrt{(-s)}}\left\{\ln\left(\frac{4|s|}{\lambda^2\gamma}\right)-\Psi(l+1)\right\}\theta(-s), \tag{46}$$

where $\ln\gamma=0.577\cdots$ is the Euler-Mascheroni constant and $\Psi(z)$ is the logarithmic derivative of the gamma function. Note that the divergent term (as $\lambda\to 0$) of (46) is independent of l. The new D equation is

$$D_l(s)=1+\int_{-\infty}^{0}\frac{ds'}{\pi}\frac{\beta_l^{(1)}(s')+\beta_l^{(2)}(s')}{\sqrt{(-s)}+\sqrt{(-s')}}D_l(s'). \tag{47}$$

Equation (47) allows latitude for an interesting conjecture. Since (46) represents an input which is singular at $s=0$, we anticipate that the solution to (47) must oscillate very rapidly as $s\to 0$— as was the case with (5). Therefore we expect the same type of damping mechanism for $0\leq s\lesssim -\alpha^2m^2$ to be operative as in the case of (5). If we assume then that we can simply cut this region out of the range of integration in (47), as was essentially the case in the asymptotic analysis of the D function (19), then we can carry out an analysis similar to that following (30) to calculate the large $|s|$ behavior of the solution to (47). In that case, one discovers that the second-order terms of (46) actually contribute *order* α effects in D. It seems then that higher order contributions to the input discontinuity can cause order α contributions in D. If this is true, then it is easy to see how the discrepancy between (39) and (40) can be resolved, dispersively, by the inclusion of higher order input information.

One might think that the finite solution of (5) would be useful for the calculation of finite, lowest order Coulomb corrections to the binding energy of strong interaction bound states, thereby circumventing the divergence difficulties associated with the Dashen-Frautschi technique. This is not so, as the following formal manipulations will indicate.

Consider the formal partial-wave Green's-function equation for two interfering central potentials, V and P.

$$G_l=G_l^{(0)}+G_l^{(0)}(V+P)G_l, \tag{48}$$

where $G_l=G_l(r,\bar{r};s)$ and $G_l^{(0)}$ is the outgoing free-wave Green's function. According to Fredholm theory, the solution of (48) can be written most generally as

$$G_l(r\bar{r};s)=N_l(r\bar{r};s)/D_l(s), \tag{49}$$

where[14]

$$D_l(s)=\det[1-G_l^0(V+P)]. \tag{50}$$

On the other hand, it can be shown that $D_l(s)$ is just the D function for the partial-wave scattering amplitude generated by the action of V and P jointly.[4] The "matrix" of (50) can be factored

$$1-G_l^0(V+P)=(1-G_l^0P)[1-(1-G_l^0P)^{-1}G_l^0V]. \tag{51}$$

Since the determinant of a product is the product of determinants we can write (50) as

$$D_l(s)=D_l^P(s)\det[1-(1-G_l^0P)^{-1}G_l^0V], \tag{52}$$

where $D_l^P(s)\equiv\det(1-G_l^0P)$ is the D function for the potential P alone. Thus, the D function factorizes and it is clear that the zeros of D corresponding to the bound states of V shifted by the action of P can only come from $\det[1-(1-G_l^0P)^{-1}G_l^0V]$. The function D_l^P contains no information about the perturbed bound states. In our case, P corresponds to the Coulomb limit of the Yukawa and D_l^P to the solution of (5) or (47), etc.

One can determine from a study of the structure of $\det[1-(1-G_l^{(0)}P)^{-1}G_l^{(0)}V]$ that, in effect, the information about the Coulomb shift of a strong-interaction bound state comes in only through those terms of β_l (the input discontinuity for the N/D equations) which contain powers of α *and at least one power of* g^2

[14] The "matrix" notation of (50) refers to the continuous variables r and $\bar{r}$ which are approximated by a discrete set of points r_i. The manipulations are carried out with respect to the discrete labels after which one passes back to the continuous variables.

(the strength of the strong potential).[4,15] Thus an input of one-photon exchange *plus* a strong force cannot be adequate for mass-shift calculations.

VI. THE COULOMB PROBLEM DISPERSIVELY

Until now, our emphasis has been on combining dispersive techniques (primarily the N/D method) with a $\lambda \to 0$ limiting process (although we have managed to get away without a great deal of the latter). As mentioned in the Introduction, one might also hope to find a dispersive scheme for the direct calculation of the Coulomb amplitudes themselves ($A_l(s)$, $A_c(s,t)$), independent of any mass-limiting process. It is to the analytic properties of these functions we now turn our attention.

Coulomb Partial-Wave S Matrix

The (attractive) Coulomb partial wave amplitudes are

$$A_l(s)=\frac{S_l^c(s)-1}{2i\sqrt{s}},\quad S_l^c(s)=\frac{\Gamma(l+1-i\eta)}{\Gamma(l+1+i\eta)}, \tag{53}$$

where $\eta=\alpha m/\sqrt{s}$. The amplitude is real analytic, satisfies the usual unitarity condition on the right, and has only the positronium poles on the left, with the accumulation at threshold. (All the cut structure of the many-vector-meson exchange is factored out with the infinite phase in the $\lambda \to 0$ limit.) It is possible to write an N/D decomposition[16]

$$A_l(s)=\frac{N_l(s)}{D_l(s)}=\frac{(1/2i\sqrt{s})\{[\Gamma(l+1+i\eta)]^{-1}-[\Gamma(l+1-i\eta)]^{-1}\}}{[\Gamma(l+1-i\eta)]^{-1}}, \tag{54}$$

in which both N and D are real analytic and $\mathrm{Im}D_l(s)=-(\sqrt{s})N_l(s)$. The decomposition is useless in any ordinary way on at least two counts. (a) There are no dynamical discontinuities to provide input information for the N/D dispersion relations. (b) In addition to the omnipresent nonisolated essential singularity at threshold due to the poles, both N and D have an isolated (exponentially increasing) essential singularity at threshold, which makes it impossible to write down dispersion relations for either. Hopefully one could factor the harmful part (the exponential increase) of the essential singularity out of N and D simultaneously as their ratio A_l is free of this difficulty. We have not yet been able to effect these two factorings simultaneously. Even if it can be done, there will still remain the problem of input forces, that is, there are none (beyond the poles which presumably one would have as zeros of D)—as there are no left cuts in A_l. There is the hope that one could feed an infinite number of poles into N as input, with undetermined masses and residues, then determine the parameters in a self-consistent way, by, e.g., requiring no new poles or something like that. The trouble with this is that there do not seem to be enough constraints on the problem to come anywhere near uniqueness. In conclusion, we feel that the analyticity of the Coulomb partial waves, although resembling that of the Yukawa potential in some respects, is sufficiently different to kill any simple attempt at an S-matrix-theoretic dynamic calculation.

[15] Y. S. Kim, Phys. Rev. **142**, 1150 (1966).
[16] L. S. Brown, Phys. Rev. **135**, B314 (1964).

Coulomb Scattering Amplitude

The attractive Coulomb scattering amplitude, calculated from the Schrödinger equation, can be written down in terms of the usual scattering variables, $s\equiv p^2$ and $t\equiv-|\mathbf{p}'-\mathbf{p}|^2$.

$$A_c(st)=-\frac{2\alpha m}{t}\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)}\exp\left\{i\eta\ln\left(\frac{-t}{2s}\right)\right\}. \tag{55}$$

In the case of the Yukawa potential, the knowledge of the Born term, the (elastic) unitarity statement, and the Mandelstam representation is adequate to generate the scattering amplitude (the so-called Mandelstam iteration procedure). Here we address ourselves to the problem of dynamically calculating (55) from such of these relations as we can derive. Let us first obtain the very peculiar unitarity relation of the Coulomb amplitude.

One would expect that the usual partial-wave projections of (55) would be the $A_l(s)$, as given in Eq. (54). Surprisingly, this is not the case. What can be shown in fact is that

$$\frac{1}{2}\int_{-1}^{+1}P_l(z)A_c(s,t)dz=\frac{S_l^c(s)}{2i\sqrt{s}}. \tag{56}$$

That is, (55) is more like an S matrix than a scattering amplitude. At first glance, it seems that (56) must be incorrect because the left side of (56) seems to go to zero as $\alpha \to 0$, whereas the right side evidently goes to a constant $(2i\sqrt{s})^{-1}$. In fact, however, it is not true that the left side vanishes as $\alpha \to 0$. For example, consider the S wave projection in which the integration is simply

$$\int_{-1}^{+1}\frac{dz}{1-z}\exp\left\{\frac{i\alpha}{\sqrt{s}}\ln\left(\frac{1-z}{2}\right)\right\}=-\frac{i\sqrt{s}}{\alpha}. \tag{57}$$

We see that, without the Regge-like exponential factor, the integral would fail to converge, and, related to this, one cannot get the perturbation expansion of the partial-wave projection by perturbatively expanding A_c before the projection. The partial-wave projection in the Coulomb case is not a uniformly convergent integration.

Because of (56), the Coulomb amplitude cannot be expected to have the ordinary unitarity statement. However, using (56) it is easy to construct an amplitude unitary in the usual sense

$$A_u(s,t)=A_c(s,t)+(i/\sqrt{s})\delta(1-z). \tag{58}$$

Substituting (58) into the usual unitarity statement, we obtain the unitarity relation for the Coulomb amplitude ($s>0$)

$$4\pi\delta(t)=\int d\Omega' A_c^*(s,t_f)A_c(s,t_i). \tag{59}$$

It is clear that this unitarity statement is going to severely hinder attempts at iterative schemes of calculation.

Now let us examine the analyticity of (55). Singh[6] has noted that the Coulomb amplitude has a cut structure similar to the Yukawa case. In particular, one can calculate directly from (55) discontinuities across the positive s and t axes to obtain spectral functions $A^s(s,t)$, $A^t(s,t)$, $\rho(s,t)$. The latter two are given explicitly by Singh. The remaining question is whether or not Hilbert transforming the spectral functions regains the amplitude. One can easily prove by direct integration over the indicated spectral functions that

$$A_c(s,t)=\frac{1}{\pi}\int_0^\infty \frac{A^t(s,t')dt'}{t'-t},$$

$$A^s(s,t)=\frac{1}{\pi}\int_0^\infty \frac{\rho(s,t')dt'}{t'-t}. \tag{60}$$

That is, one does have the dispersion relations for constant s, but because of exponential increase near $s=0+$ ($t>0$) in the Coulomb amplitude and all its spectral functions, one cannot do integrals over s, and the dispersion relations for constant t are lost. In particular, there is no full Mandelstam representation. Notice how peculiar even these provable dispersion relations are. A glance at A^t in Singh shows that it is of order α^2 as $\alpha\to 0$, thus one might expect the Born term to appear in addition to the integral over A^t in the first relation of (60). We emphasize that this is not the case. Again because of the Regge-like exponential factor, the integral over A^t comes out of order α.

The relations (59) and (60) (plus, of course, the Born term) are all we can obtain then of the ordinary analyticity and unitarity scaffolding that usually goes into a dynamical calculation. We remind the reader that the first step in a Mandelstam iteration procedure is to substitute the Born term in on the right side of the unitarity statement, thus obtaining A^s to fourth order. In the case of (59), this iterative approach is not possible. In that the dispersion relations (60) are also "homogeneous" (contrary to the Yukawa case—in which the Born term appears in addition to the integrals over the weight functions), they are also useless in any simple iterative scheme. As far as we can see, the problem remains essentially a nonlinear one. (It is not even clear that these relations uniquely specify A_c.)

ACKNOWLEDGMENTS

The authors would like to thank C. Sheppey (CERN) for the numerical calculation of the lowest Coulomb bound state predicted by (19). One of the authors (M.B.H.) would like to thank Professor L. Van Hove for the hospitality extended to him at CERN during part of this work and to acknowledge the financial aid provided by a NATO postdoctoral fellowship during that period. The other (J.R.) wishes to acknowledge the many helpful discussions held with Professor W. Gilbert of Harvard University.