Level Shifts in K-Mesonic Atoms*

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A recent experiment on K^- capture at rest on He^4 failed to observe any perturbing effect upon the Coulomb levels of the K-mesonic atom from the K^- nuclear interaction. This result was considered to be surprising because the S-wave $\bar{K}$-nucleon forces are known to be strong, at least in the isospin-zero channel. An attempt is made to explain the apparent weakness of the K^-He^3 forces by a Born-Oppenheimer estimate of the effective potential. The result contains the range of the $\bar{K}$-N force as an undetermined parameter; an upper bound is obtained from the result of the K^--He^4 experiment. Predictions (upper bounds) are then made for other very light nuclei.

I. INTRODUCTION

RECENTLY, Burleson *et al.*[1] have reported their observations of x rays emitted during K^- capture on He^4. They find lines with energies of 6.7 ± 0.2 and 34.7 ± 0.3 keV in good agreement with the Coulomb values (corrected for nuclear size and vacuum polarization) of 6.5 and 34.9 keV, respectively, for the L_α and K_α transitions in the K^--He^4 atom. It is also of interest that the width of the 34.7-keV line is not greater than the experimental resolution which is about[2] 5 keV.

In the approximation where the magnitude of the complex scattering length A is small compared with the atomic Bohr radius ($\sim$29 F) the complex energy shift for an s state is[3]

$$\Delta\epsilon_n \equiv \Delta E_n + i\Gamma/2 = -(4A/nB)E_n. \quad (1)$$

Here n is the principal quantum number, B the Bohr radius of K^--He^4, and E_n the energy of the state. The results of Burleson *et al.*[1] imply that the K^--He^4 s-wave scattering length satisfies ($A\equiv a+ib$)

$$a \leqslant 0.15\text{ F}, \quad (2a)$$

$$b \leqslant 0.4\text{ F}. \quad (2b)$$

These numbers may be compared with the nuclear radius[4] of He^4, which is about 1.7 F, and with the "best" known values of the $\bar{K}$-N scattering lengths which we can take to be (in fermis)[5]

$$A_0 \approx -1.6 + i0.51, \quad (3a)$$

$$A_1 \approx -0.2 + i0.44, \quad (3b)$$

in the isospin states 0 and 1, respectively. By either measure it would appear that $|A|$ is very small. In fact if one adopts the usual procedure of summing the scattering lengths for K^- on the constituent nucleons (a procedure that seems to work for the π^- level shifts[6]) then the predicted value of A would be $-2.2+i1.8$ F.

As Bethe and de Hoffmann have stressed[6] (within the framework of a special model), however, there is no justification for assuming that it is the scattering lengths that should be added rather than the potentials between the K^- and the constituent nucleons. The fact that addition of the scattering lengths does seem to work for π^- atoms can probably be accounted for by observing that low-energy π-N s-wave interactions are quite weak. Because of this the first Born approximation for the scattering from the effective potential should be applicable so that the scattering lengths and potentials are linearly related. One can conclude, then, that for π^- atoms the two ways of calculating are equivalent.

As a matter of fact it is not very difficult to argue that it is the potentials that are additive rather than the scattering lengths.[7] In order to see this consider the Schrödinger equation for a system of four nucleons (in the case of He^4) plus the K^- meson. If the nucleons are tightly bound together, and if the meson is "spatially distant" from them, then it should be a good approximation to factorize the wave function into a relative part (involving only the K^--He^4 relative coordinate) and the wave function of the nucleus itself. The resultant Hamiltonian for the relative wave function then has for its interaction part just the sum of the four K^--nucleon potentials. These will all have about the same value of the relative coordinate as their argument but will be "smeared" over the nuclear volume.[8]

By making use of this very simple minded argument it should be possible to gain at least a rough under-

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[1] G. R. Burleson, D. Cohen, R. C. Lamb, D. N. Michael, R. A. Schluter, and T. O. White, Phys. Rev. Letters **15**, 547 (1965). I am indebted to all the above authors for several illuminating discussions of their work. The upper bounds quoted correspond to 10% confidence limits.

[2] Private communication from D. N. Michael.

[3] Equation (1) was first obtained by S. Deser, M. L. Goldberger, K. Baumann, and W. Thirring, Phys. Rev. **96**, 774 (1954). For more detailed discussions see N. Beyers, Phys. Rev. **107**, 843 (1957) and especially T. L. Trueman, Nucl. Phys. **26**, 57 (1961).

[4] George R. Burleson and H. W. Kendall, Nucl. Phys. **19**, 68 (1960).

[5] M. Sakitt, T. B. Day, R. G. Glasser, N. Seeman, J. Friedman, W. E. Humphrey, and R. R. Ross, Phys. Rev. **139**, 719 (1965). Note that Sakitt solution 2, which is used in the present work, actually predicts the $Y_0^*(1405)$ binding energy to be $\sim 25+i17.5$ MeV. This may be obtained from my Eq. (6) with $r=0$ which provides a better estimate than the one given by Sakitt *et al.*

[6] See the discussion in H. A. Bethe and Frederic de Hoffmann, *Mesons and Fields* (Harper and Row, New York, 1955), Vol. II, p. 103 ff.

[7] Or the K matrices as used by Beyers, Ref. 2.

[8] These arguments are only intended to provide an intuitive justification for using the "lowest order" optical-model potential. A more complete formal structure is provided, e.g., by H. Feshbach, Ann. Phys. (N. Y.) **5**, 357 (1958).

standing of the relationship between the result of the He^4 level shift measurement and the known $\bar{K}$-N scattering lengths. In order to do this it is necessary to construct the effective complex potentials responsible for the $\bar{K}$-N scattering. These may then be used to obtain the effective $\bar{K}$-He^4 nuclear interaction in terms of a single undetermined parameter, namely, the ratio of the range of the $\bar{K}$-N interaction (the "well" size) to the radius of the He^4 nucleus. A K^--He^4 level-shift measurement then provides a determination of this parameter and permits a prediction of the level shifts (or, equivalently, scattering lengths) in other light nuclei.

The discussion will be based upon the simplest possible model where all the interactions are described by complex square wells with the same radius for the real and imaginary parts. Further, identical radii will be assumed for the $\bar{K}$-N interaction in both of the possible isospin states. Although this is a relatively harmless assumption with regard to the elementary interactions, since only the "shape independent" quantity $2MVr^2$ is important (see next section for notation), it will be an embarrassment in the deduction of the K^--He^4 interaction. This is because the "smearing" of the potentials over the nuclear volume will affect them very differently if their radii are greatly different. Nevertheless, the amount of information presently available hardly justifies a more sophisticated approach.

II. $\bar{K}$-NUCLEON POTENTIALS

Let V be the potential defined with the convention that positive V corresponds to attraction. If κ is the wave number inside the well, r the radius, and M the reduced mass of the $\bar{K}^-$-p system, then

$$\kappa r \equiv \sigma + i\delta = [2MV]^{1/2} r. \quad (4)$$

The complex scattering length A then satisfies

$$A/r \equiv (a+ib)/r = (\kappa r)^{-1} \tan(\kappa r) - 1. \quad (5)$$

Unitarity requires that the imaginary part of V must be positive, and it is easy to see that $\sigma > \delta > 0$ if the real part is attractive whereas $\delta > \sigma > 0$ if it is repulsive.

Let us first consider the isoscalar interaction. We shall assume that there is an s-wave bound state (the Y_0^* with mass 1405 MeV, and full width[9] about 35 MeV) with about 27-MeV binding energy. We know that the binding energy must be small compared with the well depth[10] so that σ in Eq. (5) must be close to $\pi/2$. Further, we can get an estimate of the well size from the relation

$$r = -A_0 - (2M)^{-1/2}(B + i\Gamma/2)^{-1/2} \approx 0.3 \text{ F}, \quad (6)$$

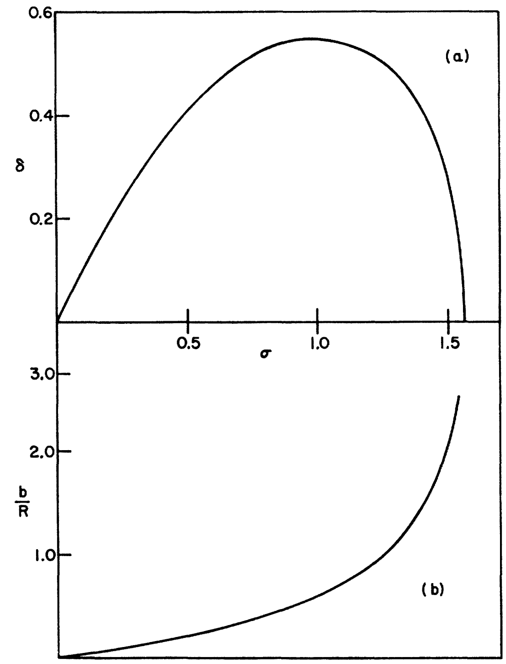

FIG. 1. (a) Relationship between real and imaginary part of the wave number in a complex square-well potential at zero energy when the real part of the scattering length vanishes. See text for notation. (b) Ratio of the imaginary part of the scattering length to radius of the square well for the situation as in (a).

where B and Γ are, respectively, the binding energy and width of the bound state. If the model is applicable, the imaginary parts of the two contributions to Eq. (6) should cancel identically. We shall ignore the fact that they do not quite do so.[11]

It is now possible to obtain an approximate solution to Eq. (5) by writing

$$\sigma = \pi/2 + \epsilon,$$

assuming σ and e small, and making use of the measured scattering length. The solutions are (inserting isospin indices)

$$\epsilon_0 \approx -(2/\pi)(r/a_0)[1+(r/a_0)]^{-1} \approx 0.15, \quad (7a)$$

$$\delta_0 \approx (\pi/2)(b_0/r)\epsilon_0^2 \approx 0.06, \quad (7b)$$

and those are small, as assumed.

Consider next the isotriplet interaction. The significant fact here is that the scattering length has a small, *negative* real part and a substantial imaginary part, thereby signifying that the real part of the potential is both attractive and strong provided that r is less than about 1 F (as one would expect). The last remark is best understood by inspection of Fig. 1(a) which shows the relationship between σ and δ when the real part of the scattering length *vanishes*. In this case either σ and δ are about equal (purely imaginary potential) as they are on the left-hand part of the curve, or $\sigma \gg \delta$ (attractive

[9] As tabulated by A. H. Rosenfeld, A. Barbaro-Galtieri, W. H. Barkas, P. L. Bastien, J. Kirz, and M. Ross, Rev. Mod. Phys. 37, 633 (1965).

[10] It is shown in Ref. 5 that the zero-range approximation gives a good fit to the scattering data up to about 30-MeV barycentric kinetic energy. It follows directly from this that 30 MeV must be small compared with the well depth.

[11] By about 0.04 F.

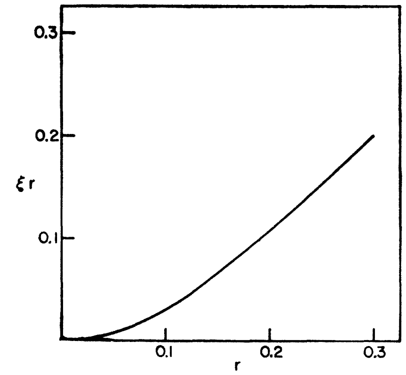

FIG. 2. Relativistic correction to the square-well parameters. See Eq. (12) of text.

potential as on the right. But Fig. 1(b) shows that b/r can only be appreciable in the latter situation.

In the present case the fact that the real part of the scattering length is negative suggests that the potential is nearly strong enough for a bound state to occur (the condition is that $a/r+1<0$) so that is again makes sense to write

$$\sigma_1\approx\tfrac{1}{2}\pi+\epsilon_1$$

and assume that ϵ_1 is small. From Eq. (5), the scattering length of Eq. (36), and the assumption that $r\approx 0.3$ F we find after a little numerical work that

$$\epsilon_1\approx\delta(1-4/\pi^2)(r/b_1)+(a_1/b_1)\approx 0, \qquad (8a)$$

$$\delta_1\approx\left(\frac{2}{\pi}\right)r/b_1\approx 0.45. \qquad (8b)$$

III. THE NUCLEAR POTENTIAL

We now write the $\bar{K}$-*nucleus* potential in two-component form

$$U=\tfrac{1}{2}\{Z[(U_1+U_0)+(U_1-U_0)\tau_1]+N(1-\tau_3)U_1\}, \quad (9)$$

where the Pauli matrices are in the representation

$$\tau_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \tau_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

and the two-component wave function is $\begin{pmatrix}\bar{K}^0\\K^-\end{pmatrix}$. The nuclear proton and neutron numbers are denoted by Z and N. The U_i represent the smeared potentials which we now take to be (neglecting ϵ_1 and ϵ_0 as small corrections)

$$U_0\approx(2M)^{-1}(r/R_N)^3[(\pi/2)^2r^{-2}+2ib_0a_0^{-2}r^{-1}], \quad (10a)$$

$$U_1\approx(2M)^{-1}(r/R_N)^3[(\pi/2)^2r^{-2}+2ib_1^{-1}r^{-1}]. \quad (10b)$$

The effect of the smearing over the nuclear volume is represented by the factor $(r/R_N)^3$ which is, of course, just the ratio of "volumes" of the nucleon and nucleus (of radius R_N). Equation (10) is something of a hoax as it stands because the $\bar{K}$-N potentials are so deep that the interacting particles move with relativistic velocities in the inner region. We shall later make a simple-minded correction for this effect by using relativistic kinematics "inside" the potential.

The near equality of the dominant real parts of the U_i provides a great simplification in the form of the potential Eq. (9), which is nearly diagonal in the $(\bar{K}^0,K^-)$ representation (except in the "degenerate" case $N=0$). Consequently, we may finally write the K^--nuclear interaction in the form of the dimensionless square-well parameter

$$(K_NR_N)^2=\xi(M_n/M)(r/R_n)\{\mathcal{A}(\tfrac{1}{2}\pi)^2 + 2i[(N+\tfrac{1}{2}Z)(r/b_1)+(\tfrac{1}{2}Z)(b_0/a_0)(r/a_0)]\}, \quad (11)$$

where a factor ξ has been inserted to allow for the "internal" relativistic corrections, M_n is the K^--nucleus reduced mass, $\mathcal{A}$ is the nucleon number, and K_N the wave number inside the nuclear potential. The correction factor ξ is obtained from the relationship

$$\xi=(2M)\left(\frac{2}{\pi}\right)^2r^2\left\{\left[M_p^2+\left(\frac{\pi}{2}\right)^2r^{-2}\right]^{1/2} + \left[M_K^2+\left(\frac{\pi}{2}\right)^2r^{-2}\right]^{1/2}-(M_p+M_K)\right\}. \quad (12)$$

Equation (12) is the consequence of the stated assumption that the potential can be defined by using relativistic kinematics in the "inner" region for the $\bar{K}$ nucleon system.

Up to this point we have guided the discussion by assuming the value of r deduced from the experimental isoscalar scattering length in Eq. (6). We shall now turn the game around and attempt to obtain an upper bound on r from the He^4 level shift experiment. Since the experimental scattering length is known to be small it is reasonable and proper to simplify matters by using the first-Born-approximation result

$$A\approx\tfrac{1}{3}K_N^2R_N^3\approx[4.4+i1.4(r/R_n)]\xi r. \quad (13)$$

A plot of ξr versus r is shown in Fig. 2. Using the upper bound on the scattering length from the level shift experiment we see that (taking the two-standard deviation limit)

$$r\lesssim 0.15\text{ F}. \quad (14)$$

IV. CONCLUSIONS

It may not have escaped the reader that the discussion leading to the derivation of the nuclear potentials has followed along the general lines of the Born-Oppenheimer approximation.[12] The conditions for validity are essentially that the K^- motion be "slow" enough for the nucleons to make many collisions among

[12] M. Born and J. R. Oppenheimer, Ann. Physik 84, 457 (1927).

themselves during the collision process and that the perturbing effect of the K^- upon the nucleus be negligible (in the present application). That the first condition be satisfied may be verified *a posteriori* from Eq. (1) and (13) from which one may estimate the capture time to be $\sim 10^{-18}$ sec, slow compared with any nuclear time. The second condition is presumably satisfied as a consequence of the strong binding of the α particle and the absence of low-lying excited states. Under these conditions the effective potential seen by the K^- is just

$$V(r)=\sum_i \int d^{3a}x|\Psi(\mathbf{x})|^2 V_i(|\mathbf{r}-\mathbf{x}_i|), \qquad (15)$$

where V_i is the potential between the K^- and the ith nucleon and $\Psi(\mathbf{x})$ the wave function of the nucleus. From Eq. (15) may be seen one major deficiency in the smearing procedure used to obtain Eq. (10) which essentially treated $\Psi(\mathbf{x})$ as a product of one-body wave functions. Thus, the first improvement one might seek in the estimation of $V(r)$ is the inclusion of correlation effects among the nucleons. These may be expected to reduce the effective interaction because of the inhibiting effect of the Pauli principle.

Under the circumstances it would not appear that one should view with alarm the discrepancy between the value of r deduced from Eq. (6) and the upper bound, Eq. (14), indicated by the level shift experiment. The as yet somewhat unsettled state of K^--p scattering analyses, the admittedly approximate nature of the present discussion, and the fact that Eq. (14) represents some sort of average over the isosinglet and isotriplet radii probably can account for the difference in the two estimates.

Nevertheless, both estimates of the radius are uncomfortably small if one wants to interpret it as a measure of the exchanged mass giving rise to the $\bar{K}$-N interaction. One can do this by relating the square-well to the Yukawa potential that has the same volume integral when both potentials have a zero-energy bound state.[13] The result is

$$M_{\rm ex}\approx 3r^{-1}\approx 4.2 \text{ GeV} \qquad (16)$$

for $r\approx 0.15$ F, a mass that is far too large to correspond to vector-meson exchange.[14] Despite this it seems reasonable to take seriously the two experimental indications that the effective interaction radius is small and thereby give somewhat less credence to the theoretical interpretation in terms of vector meson exchange. In doing so one can bear in mind the oft-repeated observation that s-wave interactions are sensitive to very short-range forces which are "screened" by the centrifugal barrier in higher partial waves.

TABLE I. Predicted scattering lengths for K^- on very light nuclei (fermis).

Nucleus	$A=a+ib$
He^4	$0.29+i0.06$
He^3	$0.21+i0.04$
H^3	$0.21+i0.05$

Having taken the foregoing viewpoint it then makes sense to predict upper bounds on the K^- scattering lengths for the light nuclides of nucleon numbers three and four. These are readily obtainable from Eq. (11) and the Born approximation [first equality in Eq. (13)]. The results are given in Table I. No prediction is made for the deuteron because its large size and weak binding would probably make the present approach inapplicable.

It is interesting to observe that the K^- scattering lengths discussed here may only be measurable in level-shift experiments. The reason may be seen by reviewing the criteria for validity of the Born-Oppenheimer potentials which imply that the K^- momenta must be small compared with ~ 50 MeV/c (taking nucleon velocities in a nucleus to be about $\beta\approx 0.1$), and scattering experiments at such low energies are probably impractical.

ACKNOWLEDGMENT

Some of the conclusions contained in the present work are the result of an interesting and helpful discussion with Dr. Frank von Hippel who has completed a somewhat related study of the same problem.[15] I am grateful to Dr. von Hippel for his comments and for a copy of his work prior to publication.

[13] Richard D. Levee and Robert L. Pexton, University of California, Lawrence Radiation Laboratory Report UCRL-7155 Rev. I, (1963) (unpublished).

[14] R. H. Dalitz, Proc. Roy. Soc. (London) **A288**, 183 (1965). Dalitz obtains the $Y_0^*(1405)$ parameters from a vector-meson potential with an exchanged mass of about 1 GeV.

[15] Frank von Hippel and John H. Douglas, Phys. Rev. **146**, 1042 (1966).