Proof of the Impossibility of a Classical Action Principle for Magnetic Monopoles and Charges without Subsidiary Conditions*

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A proof is given that no action principle exists for the classical electromagnetic field when its sources are both charged particles and magnetic monopoles unless an extra condition, not derivable from the action principle, is assumed. The extra condition is that charges never touch magnetic monopoles. The Lorentz force law predicts that a charge and a magnetic monopole approaching each other along a straight line will collide. Since the Lorentz force law can only be gotten from an action principle with the aid of this extra condition, the necessary extra condition is in contradiction to the law derived with its help. Thus there is no satisfactory action principle for the classical electromagnetic field if both charges and magnetic monopoles exist. The foregoing provides an aesthetic argument against the existence of magnetic monopoles. An action principle has then been constructed using the extra condition. It is analogous to the usual action principle for (charge-only) electromagnetic theory, but $J^\nu A_\nu$ is replaced by $J^\nu R_\nu - K^\nu T_\nu$, where J^ν and K_ν are the 4-dimensional current and magnetic monopole densities, respectively, and R_ν and T_ν are gaugeless "effective potentials." Electromagnetic theory is formulated in a gaugeless way in terms of these effective potentials in the Appendix.

I. INTRODUCTION

In recent years, a good deal of work has been done on the theory of magnetic monopoles.[1] Much of the interest in magnetic monopoles seems to be motivated by aesthetic considerations, in particular the symmetry induced in Maxwell's equations and the fact that if magnetic monopoles exist they would provide a reason for the quantization of electric charge.[2] In view of the lack of success of a number of experimenters in finding magnetic monopoles,[1,3–7] it seems reasonable to see if there might be aesthetically unappealing features about magnetic monopoles. This paper is about one such feature.

We define the action of a physical system dependent on N independent variables as a scalar which, when varied with respect to each of the variables independently, gives the N equations of motion of these variables, thus completely specifying the dynamics of the system.

The purpose of this paper is to show that if special relativity and Coulomb's law are valid at all distances, then no classical action principle exists for the electromagnetic-field–particle system when the particles present include both charges and magnetic monopoles. The problem is much more difficult to formulate meaningfully in the quantum-mechanical case as the equations of motion are initially given in terms of a Hamiltonian. No proper quantum-mechanical action principle has yet been found and the impossibility of a classical action principle suggests that the quantum-mechanical case may also be impossible, especially since a classical action principle is so natural and easy to formulate in the case of the electromagnetic-field–particle system when the particles are only charges.

Dirac,[8] in his 1948 paper, has given a classical action principle, but in order to make it work he has had to impose the extra condition that "a (nodal line) must never pass through a charged particle." This condition is not derived from the action principle, but imposed on it and is a serious constraint on the equations of motion. It is shown later in this paper that this constraint is, in the case where the particles are a spinless charge and a spinless monopole, inconsistent with the Lorentz force law which is derived from the action with its help. In any case, the use of a constraint not derived from the action principle puts Dirac's action outside of our definition of action.

Cabibbo and Ferrari attempted to find a second-quantized action principle using Mandelstam's[9] formulation of quantum electrodynamics. Their lack of success[10] inspired this proof.

II. PROOF THAT NO CONVENTIONAL ACTION PRINCIPLE EXISTS WITHOUT EXTRA CONSTRAINTS

In this paper the Einstein summation convention will be adhered to regarding Greek letters, i.e., if a Greek letter appears as an index more than once in a term, a summation of that index over all four values is to be understood.

The equations of motion for a charge in an electromagnetic field are

$$m dU_\mu/d\tau = eF_{\mu\nu}U^\nu\,, \qquad (1)$$

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[1] For a fairly complete list, see the bibliography in the paper by E. Goto, H. H. Kolm, and K. W. Ford, Phys. Rev. **132**, 387 (1963).
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where U_μ is the four-velocity, τ the proper time, and $F_{\mu\nu}$ the electromagnetic field tensor. The left-hand term in (1) is trivially obtained from the variation with respect to X^μ, the particle coordinates, of the free-particle term, S_F, in the action

$$S_F \equiv -mc\int_{-\infty}^{\infty} d\tau. \tag{2}$$

If S is the total action, we wish to find all S_I,

$$S_I \equiv S - S_F, \tag{3}$$

such that the variation of S_I with respect to X^μ gives the right-hand side of (1) for any given $F_{\mu\nu}$. Thus we demand

$$\delta_X S_I = e\int_{-\infty}^{\infty} d\tau\, F_{\mu\nu}(X)U^\nu \delta X^\mu. \tag{4}$$

Since the $\int_{-\infty}^{\infty} d\tau$ appearing in the answer (4) cannot arise as a result of the variation, S_I must be of the form

$$S_I = \int_{-\infty}^{\infty} d\tau\, M_I. \tag{5}$$

In general we wish to let M_I be any function of the X^μ and any of the derivatives, $d^nX^\mu/d\tau^n$, of X^μ with respect to the proper time τ. Thus if we write

$$\delta_X S_I \equiv \int_{-\infty}^{\infty} d\tau \frac{\Delta M_I}{\Delta X^\mu}\delta X^\mu(\tau), \tag{6}$$

which is to be taken as the definition of $\Delta M_I/\Delta X^\mu$, then

$$\frac{\Delta M_I}{\Delta X^\mu} = \sum_{n=0}^{\infty} (-1)^n \frac{d^n}{d\tau^n}\frac{\partial M_I}{\partial(d^nX^\mu/d\tau^n)}. \tag{7}$$

Equating (6) and (4), we get

$$\int_{-\infty}^{\infty} d\tau\, \delta X^\mu(\tau)\left[\frac{\Delta M_I}{\Delta X^\mu} - eF_{\mu\nu}(X)U^\nu\right] = 0. \tag{8}$$

Since the $\delta X^\mu(\tau)$ are arbitrary functions of τ, Eq. (8) implies

$$\Delta M_I/\Delta X^\mu = eF_{\mu\nu}U^\nu \tag{9}$$

for each choice of μ separately.

The general solution to (9) is given, since $\Delta/\Delta X^\mu$ is a linear operator, by a particular solution plus the general solution to the homogeneous equation

$$\Delta N_I/\Delta X^a = 0. \tag{10}$$

In what follows we adopt the convention that repeated small *italic letters* are not summed over. Repeated capital *italic letters* will be summed over the three indices not equal to their smaller counterparts; i.e., a repeated A will be summed over the three indices not equal to a.

We assume that for some value of X^a, call it $X_0{}^a$, all of the fields vanish, i.e.,

$$F_{\mu\nu}(X_0{}^a, X^A) = 0 \tag{11}$$

for any value of X^A.

Such a value of X^a might be, for instance, $\pm\infty$. A particular solution to (9) is

$$M_I = eU^\nu \int_{X_0{}^a}^{X^a} F_{a\nu}(X^{a\prime}, X^A)dX^{a\prime}, \tag{12}$$

as can be verified by direct calculation. The asymmetry of $F_{a\nu}$ means that U^a does not appear in (12) and thus that $\Delta/\Delta X^a$ is just d/dX^a. In (12), as in similar integrals to follow, the integral is meant to be taken along the X^a axis only, the other three coordinates X^A being fixed. We now consider the general solution to (10). Consider the following action:

$$S_N = \int_{-\infty}^{\infty} d\tau\, N_I. \tag{13}$$

The variation of S_N with respect to X^a is, by definition,

$$\delta_{X^a} S_N = \int_{-\infty}^{\infty} d\tau \frac{\Delta N_I}{\Delta X^a}\delta X^a. \tag{14}$$

Since the $\delta X^a(\tau)$ are arbitrary functions of τ, the statement that $\Delta N_I/\Delta X^a = 0$ is equivalent to the statement that $\delta_{X^a}S_N = 0$. Therefore, if $\Delta N_I/\Delta X^a = 0$, S_N is not a functional of X^a, but only of the X^A; i.e., varying the path in the X^a direction produces no variation in the value of S_N. The statement that S_N is not path-dependent at all is equivalent to the statement that N_I is a total τ derivative. If N_I is not a total derivative, but S_N does not vary with X^a, N_I is a function only of the X^A and not of X^a. Thus, in general, if $\delta_{X^a}S_N = 0$,

$$N_I = N_a[X^A] + dH_a/d\tau, \tag{15}$$

where the square brackets indicate that $N_a[X^A]$ is a function of the X^A and their time derivatives to arbitrary order.

We see then that the general solution to (9) is

$$M_I = eU^\nu \int_{X_0{}^a}^{X^a} F_{a\nu}(X^{a\prime}, X^A)dX^{a\prime} + N_a[X^A] + \frac{d}{d\tau}H_a(X), \tag{16}$$

for each choice of a. Thus if any S_I is to give the right-hand side of (1), it must be of the form

$$S_I = \int_{-\infty}^{\infty} d\tau \left\{ eU^\nu \int_{X_0{}^a}^{X^a} F_{a\nu}dX^{a\prime} + N_a[X^A] \right\}, \tag{17}$$

for each choice of a separately.

We wish now to display in more accessible form the condition that S_I must give the right-hand side of (1) for each choice of a separately. To do this we take the variation of S_I, $\delta_X S_I$, and set it equal to (4).

$$\delta_X S_I=\int_{-\infty}^{\infty} d\tau\left\{e\frac{d}{d\tau}(\delta X^\nu)\int_{X_0{}^a}^{X^a} F_{a\nu}dX^{a\prime}+eU^\nu\delta X^\alpha\int_{X_0{}^a}^{X^a}\partial_\alpha F_{a\nu}dX^{a\prime}+\frac{\Delta N_a}{\Delta X^A}\delta X^A\right\}$$

$$=\int_{-\infty}^{\infty} d\tau\left\{\delta X^\nu U^\alpha e\int_{X_0{}^a}^{X^a} dX^{a\prime}(\partial_\nu F_{a\alpha}-\partial_\alpha F_{a\nu})+\frac{\Delta N_a[X^A]}{\Delta X^A}\delta X^A\right\}$$

$$=e\int_{-\infty}^{\infty} d\tau\, F_{\nu\mu}(X)U^\mu\delta X^\nu, \tag{18}$$

where we have written $\partial/\partial X^\mu$ as ∂_μ. Equating the second and third lines, we get

$$\int_{-\infty}^{\infty} d\tau\left\{\delta X^\nu e\left(U^\alpha\int_{X_0{}^a}^{X^a} dX^{a\prime}(\partial_\nu F_{a\alpha}-\partial_\alpha F_{a\nu})-F_{\nu\mu}U^\mu\right)+\frac{\Delta N_a}{\Delta X^A}\delta X^A\right\}=0. \tag{19}$$

Since $\delta X^\nu(\tau)$ is an arbitrary function of τ and since the $(\nu=a)$ term in (19) vanishes identically, (19) implies

$$\frac{\Delta N_a}{\Delta X^A}=eU^\alpha\left\{F_{A\alpha}-\int_{X_0{}^a}^{X^a} dX^{a\prime}(\partial_A F_{a\alpha}-\partial_\alpha F_{aA})\right\}. \tag{20}$$

The left-hand side of (20) is not a function of X^a. The term on the right-hand side of (20) that has $(\alpha=a)$ is identically zero. Taking ∂_a of (20) we get as a necessary condition on $F_{\mu\nu}$ for the existence of an action principle

$$U^\alpha(\partial_a F_{A\alpha}+\partial_A F_{\alpha a}+\partial_\alpha F_{aA})=0, \tag{21}$$

but since Maxwell's equations are

$$\partial^\nu F_{\mu\nu}=-J_\mu;\quad \partial^\nu(F^\dagger)_{\mu\nu}=K_\mu;\quad (F^\dagger)_{\mu\nu}\equiv\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta},$$

where J_μ and K_μ are the electric and magnetic current densities, respectively,

$$\partial_a F_{A\alpha}+\partial_A F_{\alpha a}+\partial_\alpha F_{aA}=-\epsilon_{aA\alpha\beta}K^\beta. \tag{22}$$

Equation (21) can be written, using (22), as

$$U^\mu K^\nu-U^\nu K^\mu=0, \tag{23}$$

at each point, for all μ and ν.

By considering the equation of motion of a monopole in an electromagnetic field, we arrive by the same analysis at the analogous necessary condition (where $V^\mu\equiv dY^\mu/d\tau$; Y^μ are the pole coordinates)

$$V^\mu J^\nu-V^\nu J^\mu=0, \tag{24}$$

at each point, for all μ and ν.

We will show later that (23) and (24) are sufficient as well as necessary conditions for an action principle. Either (23) or (24) implies

$$J^\mu K^\nu-J^\nu K^\mu=0, \tag{25}$$

at each point, for all μ and ν.

Thus we see explicitly that we cannot make an arbitrary choice of J^μ and K^μ in Maxwell's equations and have an action principle for monopoles and charges together. There are only two ways in which (25) can be satisfied: Either

(a) J^μ or K^μ is zero, at each space-time point, i.e., charges and monopoles never overlap. (This is guaranteed by the previously given condition in Dirac's paper that "a nodal line must never pass through a charged particle.") Or,

(b) $J^\mu(X)=f(X)K^\mu(X)$, where $f(X)$ is any scalar function. Regardless of the choice of $f(X)$, for a system of point particles, this is true only if each particle has a particular fixed ratio of charge strength to pole strength. This can be seen by writing out $J^\mu=fK^\mu$ as $J^0=fK^0$ and $\mathbf{J}=f\mathbf{K}$, which gives $K^0\mathbf{J}=J^0\mathbf{K}$, and substituting the appropriate forms for the charge densities and currents.

The Lorentz force law is implied by Coulomb's law and special relativity. It says that if a charge is moved toward a pole along a radial line no force is exerted and they will at some time be at the same point, violating (a) and (b). Thus if special relativity and Coulomb's law are valid at all distances, there can be no action principle for monopoles and charges.

In Dirac's[2,8] treatment of monopoles, the assumption is made that monopoles lie on a line where the electron wave function is zero. Thus (23) and (24) are satisfied by assumption.

III. FORMAL CONSTRUCTION OF AN ACTION PRINCIPLE WITH EXTRA CONSTRAINTS

In spite of the above remarks, we will now formally construct an action principle assuming that the charges and poles are, for whatever reason, distributed according to (23) and (24). No other constraints than (23) and (24) will be necessary. First consider

$$S_{Ie}\equiv\tfrac{1}{4}e\int_{-\infty}^{\infty} d\tau\, U^\nu\int_{X_0{}^\mu}^{X^\mu} dX^{\mu\prime}F_{\mu\nu}. \tag{26}$$

Its variation is

$$\delta_X S_{Ie}=\tfrac{1}{4}e\int_{-\infty}^{\infty} d\tau\left\{\frac{d}{d\tau}(\delta X^\nu)\int_{X_0^\mu}^{X^\mu} F_{\mu\nu}dX^{\mu'} + U^\nu\delta X^\alpha\partial_\alpha\int_{X_0^\mu}^{X^\mu} dX^{\mu'}F_{\mu\nu}\right\}$$

$$=\tfrac{1}{4}e\int_{-\infty}^{\infty} d\tau\left\{-\delta X^\nu U^\alpha\partial_\alpha\int_{X_0^\mu}^{X^\mu} F_{\mu\nu}dX^{\mu'} + U^\nu\delta X^\alpha\partial_\alpha\int_{X_0^\mu}^{X^\mu} F_{\mu\nu}dX^{\mu'}\right\}$$

$$=\tfrac{1}{4}e\int_{-\infty}^{\infty} d\tau\,\delta X^\alpha U^\nu\int_{X_0^\mu}^{X^\mu} dX^{\mu'}(\partial_\alpha F_{\mu\nu}+\partial_\nu F_{\alpha\mu})$$

$$=\tfrac{1}{4}e\int_{-\infty}^{\infty} d\tau\,\delta X^\alpha\int_{X_0^\mu}^{X^\mu} dX^{\mu'}U^\nu(\partial_\alpha F_{\mu\nu}+\partial_\nu F_{\alpha\mu}). \quad (27)$$

Inserting (21) into (27), we get

$$\delta_X S_{Ie}=\tfrac{1}{4}e\int_{-\infty}^{\infty} d\tau\,\delta X^\alpha U^\nu\int_{X_0^\mu}^{X^\mu} dX^{\mu'}\partial_{\mu'}F_{\alpha\nu} = e\int_{-\infty}^{\infty} d\tau\,\delta X^\alpha U^\nu F_{\alpha\nu}. \quad (28)$$

Thus the action (26) gives the correct answer (28) under the condition (21).

The equations of motion for a pole in an electromagnetic field are

$$m_g dV_\mu/ds=-g(F^\dagger)_{\mu\nu}V^\nu, \quad (29)$$

where s is the monopole proper time.

A similar analysis to that given above shows that (24) is a necessary and sufficient condition for an action to exist which gives (29). If (24) is true, then an action which gives the right-hand side of (29) is

$$S_{Im}=-\tfrac{1}{4}g\int_{-\infty}^{\infty} ds\, V^\nu\int_{Y_0^\mu}^{Y^\mu} dY^{\mu'}(F^\dagger)_{\mu\nu}. \quad (30)$$

Thus, if (23) and (24) are true, the total particle action can be written as

$$S_p=m_e c\int_{-\infty}^{\infty} d\tau+m_g c\int_{-\infty}^{\infty} ds -\tfrac{1}{4}e\int_{-\infty}^{\infty} d\tau\, U^\nu\int_{X_0^\mu}^{X^\mu} dX^{\mu'}F_{\mu\nu} +\tfrac{1}{4}g\int_{-\infty}^{\infty} ds\, V^\nu\int_{Y_0^\mu}^{Y^\mu} dY^{\mu'}(F^\dagger)_{\mu\nu}. \quad (31)$$

There is another form in which S_p can be written. Since

$$\int_{-\infty}^{\infty} d^4x\, J^\nu(x)=e\int_{-\infty}^{\infty} d\tau\, U^\nu(\tau),$$
$$\int_{-\infty}^{\infty} d^4x\, K^\nu(x)=g\int_{-\infty}^{\infty} ds\, V^\nu(s), \quad (32)$$

$$S_p=m_e c\int_{-\infty}^{\infty} d\tau+m_g c\int_{-\infty}^{\infty} ds -\frac{1}{4}\int_{-\infty}^{\infty} d^4x\, J^\nu\int_{X_0^\mu}^{X^\mu} dX^{\mu'}F_{\mu\nu} +\frac{1}{4}\int_{-\infty}^{\infty} d^4x\, K^\nu\int_{Y_0^\mu}^{Y^\mu} dY^{\mu'}(F^\dagger)_{\mu\nu}. \quad (33)$$

We now define two "effective potentials"

$$R_\nu(X)\equiv\frac{1}{4}\int_{X_0^\mu}^{X^\mu} dX^{\mu'}F_{\mu\nu};$$
$$T_\nu(Y)\equiv\frac{1}{4}\int_{Y_0^\mu}^{Y^\mu} dY^{\mu'}(F^\dagger)_{\mu\nu}. \quad (34)$$

Then (33) becomes

$$S_p=m_e c\int_{-\infty}^{\infty} d\tau+m_g c\int_{-\infty}^{\infty} ds -\int_{-\infty}^{\infty} d^4x\, J^\nu R_\nu+\int_{-\infty}^{\infty} d^4x\, K^\nu T_\nu. \quad (35)$$

In the case of charges alone, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$; $K_\nu=0$; $m_g=0$; and

$$S_p=m_e c\int_{-\infty}^{\infty} d\tau-\int_{-\infty}^{\infty} d^4x\, J^\nu A_\nu.$$

Equation (26) also becomes the usual S_{Ie}.

In the Appendix we will derive Maxwell's equation in terms of the effective potentials.

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APPENDIX

In 1962, Mandelstam[9] showed a way in which quantum electrodynamics could be formulated without potentials appearing in the equations of motion. Cabibbo and Ferrari[11] extended Mandelstam's scheme

[11] N. Cabibbo and E. Ferrari, Nuovo Cimento **23**, 1147 (1962).

to include magnetic monopoles through the use of two four-potentials. In both these theories, while the final equations of motion are gauge-invariant, potentials are used in their development.

The formulation of electrodynamics in terms of the effective potentials is developed here without ever using potentials and thus there is no such thing as gauge or gauge transformations.

In (34) we have defined the effective potential in terms of $F_{\mu\nu}$, while normally $F_{\mu\nu}$ is given in terms of the potential. We will now "invert" (34) to do this. Consider

$$4R_0=\int_{X_0^1}^{X^1} F_{10}dX^{1\prime}+\int_{X_0^2}^{X^2} F_{20}dX^{2\prime}+\int_{X_0^3}^{X^3} F_{30}dX^{3\prime}; \quad (36)$$

$$4R_1=\int_{X_0^0}^{X^0} F_{01}dX^{0\prime}+\int_{X_0^2}^{X^2} F_{21}dX^{2\prime}+\int_{X_0^3}^{X^3} F_{31}dX^{3\prime}. \quad (37)$$

Thus

$$4\partial_1R_0=F_{10}+\int_{X_0^2}^{X^2} dX^{2\prime}\partial_1F_{20}+\int_{X_0^3}^{X^3} dX^{3\prime}\partial_1F_{30}. \quad (38)$$

From (22) we have

$$\partial_1F_{20}=-K^3+\partial_2F_{10}+\partial_0F_{21}; \quad (39)$$

$$\partial_1F_{30}=K^2+\partial_3F_{10}+\partial_0F_{31}. \quad (40)$$

Thus (38) becomes

$$4(\partial_1R_0)=3F_{10}+\left\{\int_{X_0^3}^{X^3} dX^{3\prime}K^2-\int_{X_0^2}^{X^2} dX^{2\prime}K^3\right\}$$
$$+\partial_0\left\{\int_{X_0^2}^{X^2} dX^{2\prime}F_{21}+\int_{X_0^3}^{X^3} dX^{3\prime}F_{31}\right\}, \quad (41)$$

but

$$\int_{X_0^2}^{X^2} dX^{2\prime}F_{21}+\int_{X_0^3}^{X^3} dX^{3\prime}F_{31}\equiv 4R_1-\int_{X_0^0}^{X^0} dX^{0\prime}F_{01}. \quad (42)$$

Thus

$$4(\partial_1R_0-\partial_0R_1)=4F_{10}$$
$$+\left\{\int_{X_0^3}^{X^3} dX^{3\prime}K^2-\int_{X_0^2}^{X^2} dX^{2\prime}K^3\right\} \quad (43)$$

and

$$F_{10}=(\partial_1R_0-\partial_0R_1)-\tfrac{1}{4}\epsilon_{10\alpha\beta}\int_{X_0^\alpha}^{X^\alpha} dX^{\alpha\prime}K^\beta \quad (44)$$

and, since there is nothing special about the choice (1,0),

$$F_{\mu\nu}=(\partial_\mu R_\nu-\partial_\nu R_\mu)-\tfrac{1}{4}\epsilon_{\mu\nu\alpha\beta}\int_{X_0^\alpha}^{X^\alpha} dX^{\alpha\prime}K^\beta. \quad (45)$$

A similar analysis gives

$$(F^\dagger)_{\mu\nu}=(\partial_\mu T_\nu-\partial_\nu T_\mu)-\tfrac{1}{4}\epsilon_{\mu\nu\alpha\beta}\int_{Y_0^\alpha}^{Y^\alpha} dY^{\alpha\prime}J^\beta. \quad (46)$$

Equations (45) and (46) clearly show the reduction to the usual $F_{\mu\nu}$ when only charges are present, since if $\partial^\nu(F^\dagger)_{\mu\nu}=0$, $F_{\mu\nu}$ can be written in the form $(\partial_\mu A_\nu-\partial_\nu A_\mu)$, which gives in Eq. (34) $R_\nu=A_\nu+\partial_\nu\Lambda$, and thus R_ν and A_ν differ only by a gauge transformation. Also if there are no monopoles the Y^μ coordinates are not defined. These results can be put in somewhat nicer form by making the definitions

$$R_{\mu\nu}\equiv\partial_\mu R_\nu-\partial_\nu R_\mu;\quad T_{\mu\nu}\equiv\partial_\mu T_\nu-\partial_\nu T_\mu;$$
$$K^{\mu\nu}(X)\equiv\frac{1}{4}\left[\int_{X_0^\mu}^{X^\mu} dX^{\mu\prime}K^\nu-\int_{X_0^\nu}^{X^\nu} dX^{\nu\prime}K^\mu\right];$$
$$J^{\mu\nu}(X)\equiv\frac{1}{4}\left[\int_{X_0^\mu}^{X} dX^{\mu\prime}J^\nu-\int_{X_0^\nu}^{X^\nu} dX^{\nu\prime}J^\mu\right]. \quad (47)$$

Note that all the above tensors are antisymmetric in their indices. With these definitions (45) and (46) become

$$F_{\mu\nu}=R_{\mu\nu}-(K^\dagger)_{\mu\nu}; \quad (48)$$

$$(F^\dagger)_{\mu\nu}=T_{\mu\nu}-(J^\dagger)_{\mu\nu}. \quad (49)$$

Taking the dual of (48) and putting it equal to (49), we find the identity

$$T_{\mu\nu}-(J^\dagger)_{\mu\nu}=(R^\dagger)_{\mu\nu}+K_{\mu\nu}. \quad (50)$$

Maxwell's equation in terms of the potentials are, therefore,

$$\partial^\nu R_{\mu\nu}=\partial^\nu(K^\dagger)_{\mu\nu}-J_\mu, \quad (51)$$

$$\partial^\nu T_{\mu\nu}=\partial^\nu(J^\dagger)_{\mu\nu}+K_\mu, \quad (52)$$

or,

$$\partial^\nu\partial_\nu R_\mu=J_\mu-\partial^\nu(K^\dagger)_{\mu\nu}+\partial_\mu\partial^\nu R_\nu, \quad (53)$$

$$\partial^\nu\partial_\nu T_\mu=-K_\mu-\partial^\nu(J^\dagger)_{\mu\nu}+\partial_\mu\partial^\nu T_\nu. \quad (54)$$

From (34) we get

$$\partial^\nu R_\nu=-\frac{1}{4}\int_{X_0^\alpha}^{X^\alpha} dX^{\alpha\prime}J_\alpha;\quad \partial^\nu T_\nu=\frac{1}{4}\int_{Y_0^\alpha}^{Y^\alpha} dY^{\alpha\prime}K_\alpha. \quad (55)$$

Thus

$$\Box R_\mu=J_\mu-\tfrac{1}{4}\partial_\mu\int_{X_0^\alpha}^{X^\alpha} dX^{\alpha\prime}J_\alpha-\partial^\nu(K^\dagger)_{\mu\nu}, \quad (56)$$

$$\Box T_\mu=-K_\mu+\tfrac{1}{4}\partial_\mu\int_{Y_0^\alpha}^{Y^\alpha} dY^{\alpha\prime}K_\alpha-\partial^\nu(J^\dagger)_{\mu\nu}. \quad (57)$$