Nuclear Spectroscopy in the α Particle*

A. DE-SHALIT†
Stanford Linear Accelerator Center, Stanford University, Stanford, California
and
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California
AND
J. D. WALECKA‡
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California
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We discuss the properties of the 15-dimensional $SU(4)$ supermultiplet of negative-parity excited states of the $A=4$ system. This multiplet consists of the electric-dipole resonance and its spin-isospin analogs. The position of the "center of gravity" of this supermultiplet is determined by the Wigner and Majorana components of the nucleon-nucleon force, while the splittings within the supermultiplet are due to the spin-dependent parts of the force. We calculate the energies and state vectors of these levels within the framework of the shell model after isolating the center-of-mass motion and assuring ourselves that we are dealing with proper intrinsic excitations of the system. We compute the position of the "center of gravity" of the supermultiplet and the splittings within it,u sing both a Kurath force and a force taken from low-energy nucleon-nucleon scattering. The results are compared with some recent experimental findings concerning these levels.

AMONG the very light nuclei, He^4 is the one with the highest symmetry. This reflects itself not only in the quantum numbers characterizing the ground state of He^4 but also in its exceptional stability: The lowest ionization energy for He^4 is the threshold energy 19.813 MeV for the reaction $He^4 \rightarrow H^3+p$, and no bound states below this energy are known.

In the past few years, several unbound excited states in the $A=4$ nuclei have been identified. These include a 0^+ $T=0$ state at 20.1 MeV[1-3] and a group of negative-parity states in H^4, He^4, and Li^4 at energies between 20 and 30 MeV[4-6] above the ground state of He^4. The latter turn out to have a particularly simple structure and their analysis is the subject of the present paper.

The double magic ground state of He^4 is assigned the quantum numbers $T=0$, $S=0$, $J^\pi=0^+$. To the extent that the nuclear forces are dominated by central (Wigner) and space-exchange (Majorana) forces, this state belongs to the identity (i.e., [1]) representation of $SU(4)$ in Wigner's supermultiplet theory.[7] The negative parity excited states belong then[7] to the [15] dimensional representation of $SU(4)$, and they include among them also the giant dipole resonance. The $[(2T+1)\otimes(2S+1)]$ content of this representation is $[3\otimes1]\oplus[1\otimes3]\oplus[3\otimes3]$ and therefore the [15] representation contains the states in Table I. To the extent that one can neglect the spin-dependent forces, these states should appear as a degenerate supermultiplet at the energy of the giant electric-dipole resonance. Just as the giant electric-dipole resonance may be thought of as the oscillation of the protons against the neutrons, the other states may be regarded roughly as oscillations of protons with spin up and neutrons with spin down against protons with spin down and neutrons with spin up, etc.[7]

The spin-dependent forces which will break the degeneracy of this supermultiplet are the spin-orbit force, the difference in the triplet and singlet central forces, and the tensor force. The $p_{3/2}-p_{1/2}$ spin-orbit splitting is known from n-α and p-α scattering to be of the order of 3 to 4 MeV. The difference in the triplet and singlet potentials gives rise to only a 2-MeV difference in the binding energy of the two-nucleon system. The spin-dependent splittings are therefore expected to produce a "fine structure" of a few MeV in the supermultiplet. The beauty of the α particle is that the system is so simple that it depends on only a few parameters of the nucleon-nucleon force, yet the spectrum is rich enough that the contribution of all the different spin components of that force can be separated by looking at the right combination of level splittings.

One of the problems with doing spectroscopy in the

TABLE I. States in the [15] $SU(4)$ supermultiplet [$L=1$].

L	S	J^π	T
1	0	1^-	1
1	1	$0^-, 1^-, 2^-$	0
1	1	$0^-, 1^-, 2^-$	1

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† On leave from The Weizmann Institute of Science, Rehovoth, Israel.
‡ A. P. Sloan Foundation Fellow.
[1] C. Werntz, Phys. Rev. 133, B19 (1964).
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[5] N. A. Vlasov and L. M. Samoilov, At. Energ. (USSR) 17, 3 (1964) (UCRL-transl., 1183, 1965).
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[7] L. L. Foldy and J. D. Walecka, Nuovo Cimento 34, 1026 (1964).

few-nucleon system is that of treating the center of mass correctly. Failure to take proper account of the center-of-mass motion may mix spurious center-of-mass excitations into internal excitations thus preventing direct meaningful comparison with experiment.

If a nucleus is described by a shell model and its ground state has $T=0$ and $S=0$, then all states which do not affect the intrinsic structure and involve only center-of-mass excitations must have the same values of $T=0$ and $S=0$. They all belong, therefore, to the [1] representation of $SU(4)$. To the extent that the center-of-mass motion is separable, as is the case for a shell model with harmonic-oscillator central potential, we are therefore sure to be dealing with pure intrinsic excitations if we concentrate on the lowest states belonging for instance to the [15] representation of $SU(4)$. Put more simply, we can say that if we study the states ${}^3P_{0,1,2}$, $T=0$ together with the $T=1$ states which are members of the [15] supermultiplet as indicated by Table I, we can be sure we have no spurious center-of-mass excitation mixed in. The latter will show up in the ${}^1P_1$ $T=0$ state.

To make a detailed calculation of the spectrum we shall assign shell-model configurations to the states involved. Instead of dealing with a free He^4 nucleus, we bind it with a potential $U(X)$ around the point $X=0$ where $\mathbf{X}\equiv\frac{1}{4}[\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3+\mathbf{x}_4]$ is the center-of-mass coordinate. This does not change the spectrum of *intrinsic* excitations, but superimposes on it a superfluous discrete spectrum of center-of-mass excitations in the potential $U(X)$.

The handling of this modified Hamiltonian is simplified greatly, as was pointed out by Lipkin,[8] if $U(X)$ is a harmonic-oscillator potential. Because of the identity

$$(\sum_i \mathbf{x}_i/A)^2=\frac{1}{A}\sum_i \mathbf{x}_i^2-\tfrac{1}{2}\sum_{i\neq j}\left(\frac{\mathbf{x}_i-\mathbf{x}_j}{A}\right)^2$$

the effect of binding the *center of mass* with a harmonic-oscillator potential $U(X)=\frac{1}{2}AM\omega^2\mathbf{X}^2$ is equivalent to that of binding *each one of the particles* with a harmonic potential $\frac{1}{2}M\omega^2\mathbf{x}_i^2$ and modifying the interparticle interaction by the addition of $\frac{1}{2}(M\omega^2/A)(\mathbf{x}_i-\mathbf{x}_j)^2$. From now on we shall therefore confine our considerations to He^4 whose center of mass is bound by the harmonic potential. Our Hamiltonian is therefore

$$\tilde{H}=\sum T(i)+\tfrac{1}{2}\sum_{i\neq j}V(ij)+U(X)=H_0+H_1, \quad (1)$$

where

$$H_0=\textstyle\sum_i T(i)+\frac{1}{2}M\omega^2X_i^2, \quad (2)$$

$$H_1=\tfrac{1}{2}\sum_{i\neq j}V(ij)-\frac{M\omega^2}{2A}(\mathbf{x}_i-\mathbf{x}_j)^2=\tfrac{1}{2}\sum_{i\neq j}\tilde{V}(ij). \quad (3)$$

The ground state of He^4 is represented by the complete occupation of the lowest states in the harmonic oscillator, i.e., by $|(1s_{1/2})^4T=0, J=0\rangle$, or in $SU(4)$ notation by $|(1s)^4[1]\rangle$. To obtain a negative parity excited state we have to excite the configuration $(1s)^3(1p)$ at a zeroth-order excitation energy of $\hbar\omega$. Other configurations of the same parity lie at an excitation of at least $3\hbar\omega$ and will be neglected. To classify the spin-isospin states we must combine a particle which belongs to the representation [4] of $SU(4)$ with a hole which belongs to the representation $[\bar{4}]$. This product can be reduced according to the rule

$$4\otimes\bar{4}=1\oplus15.$$

By our previous discussion, we can be sure we are dealing with intrinsic excitations if we confine ourselves to the [15] representation. If we look at the $T=0$ states in Table I, we see that the states belonging to [15] are pure L-S configurations:

$$|(s^3p);{}^3P_{0,1,2}\rangle, \quad T=0.$$

If we look at the $T=1$ states, then the 0^- and 2^- are simultaneously pure L-S and pure j-j since the wave functions are identical in the two coupling schemes in this case:

$$|(s^3p);{}^3P_2\rangle=|(s_{1/2})^{-1}p_{3/2};2\rangle,$$

and

$$|(s^3p);{}^3P_0\rangle=|(s_{1/2})^{-1}p_{1/2};0\rangle. \quad (4)$$

Thus, the only case where the intrinsic states of (s^3p) are not uniquely determined by T and J is in the $T=1$ $J^\pi=1^-$ states where there are two independent states available:

$$|(s^3p);{}^3P_1\rangle \text{ and } |(s^3p);{}^1P_1\rangle$$

or

$$|(s_{1/2})^{-1}p_{3/2};1\rangle \text{ and } |(s_{1/2})^{-1}p_{1/2};1\rangle, \quad T=1.$$

Suppose we concentrate on the Wigner and Majorana parts of the two-nucleon interaction and try to determine the position of the "center of gravity" of the [15] supermultiplet. We note that the extra term in the two-nucleon potential in Eq. (3), $-\sum\frac{1}{2}(M\omega^2/A)(\mathbf{x}_i-\mathbf{x}_j)^2$, coming from binding the center of mass in an oscillator potential, is a pure Wigner force. It cannot therefore contribute to the splittings within the supermultiplet, but only to the position of its "center of gravity." To determine its effect on the intrinsic excitation energy of the supermultiplet we must calculate

$$\tilde{\epsilon}^{[15]}\equiv E^{[15]}-E^{[1]}=\langle(s)^3p[15]|\tilde{H}|(s)^3p[15]\rangle-\langle(s)^4[1]|\tilde{H}|(s)^4(1]\rangle. \quad (5)$$

If the wave functions in (5) were exact eigenfunctions of the Hamiltonian (1), then the energy difference $\tilde{\epsilon}^{[15]}$ in Eq. (5) would have been independent of ω, since ω affects only the center-of-mass motion, and by our previous argument the energies $E^{[1]}$ and $E^{[15]}$ both correspond to the lowest center-of-mass energy $\frac{3}{2}\hbar\omega$. They differ from each other only in intrinsic excitation.

Since, however, we shall be taking for the states in (5) only the zeroth-order, shell-model, wave functions [i.e.,

[8] H. L. Lipkin, Nuovo Cimento Suppl. 4, 1147 (1956).

eigenfunctions of H_0 as defined in Eq. (7) below], it is no longer obvious that $\tilde{\epsilon}^{[15]}$ still has this property, and it is worthwhile to devote a few lines to the nature of the dependence of $\tilde{\epsilon}^{[15]}$ on ω in this approximation. To make this discussion clearer we shall consider the Hamiltonian (1), i.e.,

$$\tilde{H}=\sum[T(i)+\tfrac{1}{2}M\omega^2x_i^2]+\tfrac{1}{2}\sum_{i\neq j}\left[V(ij)-\frac{M\omega^2}{2A}(\mathbf{x}_i-\mathbf{x}_j)^2\right]$$
$$=\sum T_i+\tfrac{1}{2}\sum_{i\neq j}V(ij)+\tfrac{1}{2}AM\omega^2X^2 \quad (6)$$

and take its matrix elements with wave functions generated by the Hamiltonian

$$H_0=\sum[T(i)+\tfrac{1}{2}M\omega_0^2x_i^2], \quad (7)$$

where ω_0 may be different from ω. We can also write H_0 in the form

$$H_0=\sum T(i)+\tfrac{1}{2}\sum_{i\neq j}\frac{M\omega_0^2}{2A}(\mathbf{x}_i-\mathbf{x}_j)^2+\tfrac{1}{2}AM\omega_0^2X^2 \quad (8)$$

and its eigenfunctions thus separate into a product of an intrinsic wave function and a center-of-mass wave function. Since the states $|s^4[1]\rangle$ and $|(s)^3p[15]\rangle_0$ both involve the center-of-mass motion in its lowest state, we have [the subscript 0 indicates that the expectation value is to be taken with eigenstates of (8)]:

$$\langle s^4[1]|\tfrac{1}{2}AM\omega_0^2X^2|s^4[1]\rangle_0$$
$$=\langle(s)^3p[15]|\tfrac{1}{2}AM\omega_0^2X^2|(s)^3p[15]\rangle_0=\tfrac{1}{2}\times\tfrac{3}{2}\hbar\omega_0. \quad (9)$$

Using Eq. (9) we now find easily that

$$\langle(s)^3p[15]|\tilde{H}|(s)^3p[15]\rangle_0$$
$$=\langle(s)^3p[15]|H|(s)^3p[15]\rangle_0+\frac{\omega^2}{\omega_0^2}\times\tfrac{3}{4}\hbar\omega_0, \quad (10)$$

and

$$\langle s^4[1]|\tilde{H}|s^4[1]\rangle_0=\langle s^4[1]|H|s^4[1]\rangle_0+\frac{\omega^2}{\omega_0^2}\times\tfrac{3}{4}\hbar\omega_0; \quad (11)$$

hence,

$$\tilde{\epsilon}^{[15]}=\epsilon^{[15]}=\langle(s)^3p[15]|H|(s)^3p[15]\rangle_0$$
$$-\langle s^4[1]|H|s^4[1]\rangle_0. \quad (12)$$

The expression (12) *for* $\tilde{\epsilon}^{[15]}$ *is therefore independent of the potential* $U(X)$ which was introduced to "tie" the nuclear center of mass. This is, of course, due to the fact that the wave functions we have used are separable into center-of-mass and intrinsic coordinates, and their center-of-mass part is the same in both $|s^4[1]\rangle_0$ and $|(s)^3p[15]\rangle_0$.

Our result (12) also indicates the value that should be chosen for ω_0. The essence of our approximation lies in limiting ourselves to only one configuration for the ground state and one for the excited supermultiplet. Both configurations should be derived from the same central potential in order for (12) to be valid. The state $|s^4[1]\rangle_0$ should thus be chosen to describe as well as possible the ground state of He^4. It is therefore reasonable to assume that ω_0 should be chosen so as to reproduce as well as possible some integral property of the ground state of He^4. This can be chosen to be the Coulomb energy or the electron-scattering form factor. The corresponding values of ω_0 are given in Table II,[9,10] where we define $\hbar^2/M\bar{b}^2=\frac{1}{2}\hbar\omega_{\rm osc}$. (The usual oscillator parameter is defined as $\hbar^2/Mb^2=\hbar\omega_{\rm osc}$; therefore $\bar{b}=\sqrt{2}b$.)

TABLE II. Oscillator parameters for He^4. $\hbar^2/M\bar{b}^2=\frac{1}{2}\hbar\omega_0$.

	$\bar{b}$	$\hbar\omega_{\rm osc}$
Coulomb energy[a]	2.15 F	18.0 MeV
Electron scattering[b]	1.95 F	21.8 MeV

[a] Reference 9.
[b] Reference 10.

We shall now derive (12) using the standard shell-model methods. This will give us an opportunity to develop also the formalism required for the evaluation of the "fine structure" within the [15] supermultiplet.

Let us first construct the expressions for the energy keeping only the Wigner and Majorana parts of the force. We first define a particle-hole creation operator

$$\hat{\xi}^\dagger(LST)\equiv\sum_{m_lm_sm_t}(1m_{l_1}0m_{l_2}|10LM_L)(\tfrac{1}{2}m_{s_1}\tfrac{1}{2}m_{s_2}|\tfrac{1}{2}\tfrac{1}{2}SM_s)$$
$$\times(\tfrac{1}{2}m_{t_1}\tfrac{1}{2}m_{t_2}|\tfrac{1}{2}\tfrac{1}{2}TM_T)a^\dagger{}_{1\frac{1}{2}\frac{1}{2};m_{l_1}m_{s_1}m_{t_1}}b^\dagger{}_{0\frac{1}{2}\frac{1}{2};m_{l_2}m_{s_2}m_{t_2}}, \quad (13)$$

where $a^\dagger$ creates a particle in the $1p$ shell and $b^\dagger$ creates a hole in the $1s$ shell.[11] We now construct an LS coupling excited state by defining $|LST\rangle=\hat{\xi}^\dagger(LST)|G\rangle$, where $|G\rangle$ is the closed $(1s)^4$ shell ground state. If we take matrix elements of $[\tilde{H},\hat{\xi}^\dagger(LST)]$ between $|LST\rangle$ and $|G\rangle$, we get the excitation energy of the state $|LST\rangle$. We can also explicitly evaluate the commutator and keep those terms which will contribute to the matrix element. This is just the Tamm-Dancoff approximation and leads to the eigenvalue equation[12]

$$[\tilde{\epsilon}_p-\tilde{\epsilon}_s-\epsilon^{LST}+\tilde{v}_{ps;\,ps}{}^{LST}]=0, \quad (14)$$

where the Hartree-Fock single-particle energies are given by[13]

$$\hat{\epsilon}_p=\tfrac{5}{2}\hbar\omega_0\tfrac{1}{2}(1+\omega^2/\omega_0^2)+\sum_{L'S'T'}\frac{(2L'+1)(2S'+1)(2T'+1)}{3\times2\times2}$$
$$\times[\langle1p1sL'|\tilde{V}|1p1sL'\rangle_0-(-1)^{S'+T'}$$
$$\times\langle1p1sL'|\tilde{V}|1s1pL'\rangle_0], \quad (15)$$

[9] B. C. Carlson and I. Talmi, Phys. Rev. **96**, 436 (1954).
[10] B. Goulard, G. Goulard, and H. Primakoff, Phys. Rev. **133**, B186 (1964).
[11] Note: $b^\dagger{}_{l1/2\,1/2;\,m_lm_sm_t}\equiv(-1)^{l-m_l+1/2-m_s+1/2-m_t}a_{l1/2\,1/2;\,-m_l-m_s-m_t}$.
[12] J. D. Walecka, in *Preludes in Theoretical Physics*, edited by A. de-Shalit, H. Feshbach, and L. Van Hove (North-Holland Publishing Company, Inc., Amsterdam, 1965, p. 59).
[13] Note that for a single-particle harmonic oscillator the potential energy and the kinetic energy are equal.

$$\tilde{\epsilon}_s=\tfrac{3}{2}\hbar\omega_0\tfrac{1}{2}(1+\omega^2/\omega_0^2)+\sum_{L'S'T'}\frac{(2L'+1)(2S'+1)(2T'+1)}{1\times2\times2}$$
$$\times\langle 1s1sL'|\,\tilde{V}\,|1s1sL'\rangle_0(1-(-1)^{S'+T'}),\quad(16)$$

and the particle-hole interaction is

$$\tilde{v}_{ps;\,ps}{}^{LST}=-\sum_{L'}(2L'+1)\begin{Bmatrix}10L'\\10L\end{Bmatrix}[\langle 1p1sL'|\,\tilde{V}\,|1p1sL'\rangle_0$$
$$-4\delta_{S0}\delta_{T0}(-1)^{1+0-L'}\langle 1p1sL'|\,\tilde{V}\,|1s1pL'\rangle_0].\quad(17)$$

If we confine ourselves just to the states in the [15] representation, then $\delta_{S0}\delta_{T0}\equiv0$ and the last term in (17) does not contribute.

It is a simple matter to write the matrix elements of $\tilde{V}$ in terms of Talmi integrals and we arrive at

$$\begin{aligned}&\langle 1p1sL'|\,\tilde{V}\,|1p1sL'\rangle_0=\tfrac{1}{2}(\tilde{I}_0+\tilde{I}_1),\\&\langle 1p1sL'|\,\tilde{V}\,|1s1pL'\rangle_0=\tfrac{1}{2}(\tilde{I}_0-\tilde{I}_1),\\&\langle 1s1sL'|\,\tilde{V}\,|1s1sL'\rangle_0=\tilde{I}_0,\end{aligned}\quad(18)$$

where the Talmi integrals are defined by

$$I_p=\frac{2}{\Gamma(p+\frac{3}{2})}\int_0^\infty r^{2p+2}V(\beta r)e^{-r^2}dr.$$

We note that only relative s and p states in the two-nucleon interaction contribute to our problem. Combining these results, we find

$$\begin{aligned}&\tilde{v}_{ps;\,ps}{}^{[15]}=-\tfrac{1}{2}(\tilde{I}_0+\tilde{I}_1),\\&\tilde{\epsilon}_p=\tfrac{5}{2}\hbar\omega_0\tfrac{1}{2}(1+\omega^2/\omega_0^2)+\tfrac{1}{2}(3\tilde{I}_0+5\tilde{I}_1),\\&\tilde{\epsilon}_s=\tfrac{3}{2}\hbar\omega_0\tfrac{1}{2}(1+\omega^2/\omega_0^2)+3\tilde{I}_0.\end{aligned}\quad(19)$$

Let us now concentrate on the contribution of $U(X)$ to $\epsilon^{[15]}$. Let us write

$$\tilde{I}_p\equiv I_p+I_p{}^{\text{c.m.}},$$

where the second contribution comes from the additional potential in Eq. (3). We find

$$I_0{}^{\text{c.m.}}=-\tfrac{3}{8}\hbar\omega_0(\omega^2/\omega_0^2),\quad I_1{}^{\text{c.m.}}=-\tfrac{5}{8}\hbar\omega_0(\omega^2/\omega_0^2),\quad(20)$$

and therefore

$$\epsilon_{\text{c.m.}}{}^{[15]}=\tfrac{1}{2}\hbar\omega_0(\omega^2/\omega_0^2)+\tfrac{1}{2}(3I_0{}^{\text{c.m.}}+5I_1{}^{\text{c.m.}})$$
$$-3I_0{}^{\text{c.m.}}-\tfrac{1}{2}(I_0{}^{\text{c.m.}}+I_1{}^{\text{c.m.}})\equiv0.$$

This is equivalent to our result (12). We can therefore finally write

$$(\epsilon_p-\epsilon_s-\epsilon^{[15]}+v_{ps;\,ps}{}^{[15]})=0,\quad(21)$$

where now

$$\epsilon_p=\langle 1p|\,T\,|1p\rangle_0+\sum_{L'S'T'}\frac{(2L'+1)(2S'+1)(2T'+1)}{3\times2\times2}$$
$$\times[\langle 1p1sL'|\,V\,|1p1sL'\rangle_0$$
$$-(-1)^{S'+T'}\langle 1p1sL'|\,V\,|1s1pL'\rangle_0],\quad(22)$$

$$\epsilon_s=\langle 1s|\,T\,|1s\rangle_0+\sum_{L'S'T'}\frac{(2L'+1)(2S'+1)(2T'+1)}{1\times2\times2}$$
$$\times[\langle 1s1sL'|\,V\,|1s1sL'\rangle_0(1-(-1)^{S'+T'})],\quad(23)$$

$$v_{ps;\,ps}{}^{[15]}=-\langle 1p1sL|\,V\,|1p1sL\rangle_0.\quad(24)$$

These are just the equations one would write down in the shell model without ever worrying about the center-of-mass problem. They are true, however, only if we use harmonic-oscillator wave functions and consider states from whose symmetry we can conclude that they involve the center-of-mass motion in its lowest state.

The next question is how to determine $\epsilon_p-\epsilon_s$. The usual method is to take the particle-hole energies from neighboring nuclei. If we do this we have[14]

$$\begin{aligned}\epsilon_p-\epsilon_s&=[M_{1p}(\text{He}^5)-(M(\text{He}^4)+M(n))]\\&\quad-[M(\text{He}^4)-(M(\text{He}^3)+M(n))]\\&\cong+2\text{ MeV}-[-20.6\text{ MeV}]=+22.6\text{ MeV},\end{aligned}\quad(25)$$

where the first term involves an estimate of where the $1p$ state lies between the $1p_{3/2}$ and $1p_{1/2}$ states in He^5. The result (25) agrees fairly well with that one would get by merely taking the harmonic-oscillator spacing ignoring v (see Table II):

$$\begin{aligned}\epsilon_p-\epsilon_s&\cong\hbar\omega_0=18\text{ MeV}\quad&&\text{(Coulomb energies)}\\&\quad 21.8\text{ MeV}\quad&&\text{(electron scattering)}\end{aligned}$$

but we see there is some ambiguity about what we mean by particle-hole energies in such a light system. We also see that $v_{ps;\,ps}{}^{[15]}$—the particle-hole interaction—is repulsive and moves the "center of gravity" of the supermultiplet $\epsilon^{[15]}$ to higher energies.

Before attempting to calculate this, let us formulate the problem of determining the splittings within the supermultiplets. To get the splitting we can simply replace

$$\tilde{V}(ij)\rightarrow V(ij)$$

since they differ by a Wigner force and we know from the supermultiplet theory that this cannot contribute to the splittings. We introduce for convenience the jj-coupling particle-hole creation operators

$$\hat{\xi}_{JT}{}^\dagger(1p_j,1s_{1/2})\equiv\hat{\xi}_{JT}{}^\dagger(j)\equiv\sum_{m_jm_t}(jm_{j_1}\tfrac{1}{2}m_{j_2}|\,j\tfrac{1}{2}JM_J)$$
$$\times(\tfrac{1}{2}m_{t_1}\tfrac{1}{2}m_{t_2}|\,\tfrac{1}{2}\tfrac{1}{2}TM_T)a_{1j;\,m_{j1}m_{t1}}{}^\dagger b_{01;\,m_{j2}m_{t2}}{}^\dagger$$

and look for excited states of the form

$$|JT\rangle=\textstyle\sum_j\alpha_j{}^{JT}\hat{\xi}_{JT}{}^\dagger(j)\,|G\rangle.\quad(26)$$

Except for the $T=1$, $J^\pi=1^-$ states there will only be one term in the sum (26) for any given value of J and T, since for the $T=0$, $J^\pi=1^-$ state we must take the correct combination of states to have a pure $^3P_1$. The

[14] F. H. Lewis, Jr., and J. D. Walecka, Phys. Rev. 133, B849 (1964).

transformation coefficients in this case are just the 9-j symbols [we couple $(sl)j$]

$$|^3P_1\rangle=(\sqrt{\tfrac{2}{3}})|p_{1/2}s_{1/2}1^-\rangle-(\sqrt{\tfrac{1}{3}})|p_{3/2}s_{1/2}1^-\rangle,$$
$$|^1P_1\rangle=(\sqrt{\tfrac{1}{3}})|p_{1/2}s_{1/2}1^-\rangle+(\sqrt{\tfrac{2}{3}})|p_{3/2}s_{1/2}1^-\rangle. \quad (27)$$

Linearizing the equations of motion in the same way as before, we arrive at the equation

$$\sum_{j'}\{[(\epsilon_{p_j}-\epsilon_{s_{1/2}})-\epsilon^{JT}]\delta_{jj'}+v_{jj'}{}^{JT}\}\alpha_{j'}{}^{JT}=0, \quad (28)$$

where ϵ_{p_j} and $\epsilon_{s_{1/2}}$ are again the single-particle single-hole energies. The particle-hole interaction is given by[14]

$$v_{jj'}{}^{JT}=-\sum_{J'}\sum_{T'}(2J'+1)(2T'+1)\begin{Bmatrix}\frac{1}{2} & j' & J\\ \frac{1}{2} & j & J'\end{Bmatrix}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & T\\ \frac{1}{2} & \frac{1}{2} & T'\end{Bmatrix}$$
$$\times[\langle p_{j'}s_{1/2}J'T'|V|p_js_{1/2}J'T'\rangle_0-(-1)^{\frac{1}{2}+j+J'}$$
$$\times(-1)^{\frac{1}{2}+\frac{1}{2}+T'}\langle p_{j'}s_{1/2}J'T'|V|s_{1/2}p_jJ'T'\rangle_0]. \quad (29)$$

Equation (29) allows us to express the particle-hole interaction in terms of matrix elements of the particle-particle interactions. We give the matrix of coefficients

$$-(2J'+1)(2T'+1)\begin{Bmatrix}\frac{1}{2} & j' & J\\ \frac{1}{2} & j & J'\end{Bmatrix}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & T\\ \frac{1}{2} & \frac{1}{2} & T'\end{Bmatrix}$$

in Table III.

These relations can also be derived using fractional-parentage coefficients.[15] If we are only interested in the splittings we obtain

$$E(jTJ^\pi)=\tfrac{3}{4}\sum_{T'}(2T'+1)\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ T' & T & \frac{1}{2}\end{Bmatrix}^2V_j(T'J)$$
$$+\sum_{J'}(2J'+1)\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ J' & J & j\end{Bmatrix}^2V_j(TJ'), \quad (30)$$

where the particle-particle matrix elements $V_j(T'J')$ are the same as in Eq. (29). Equation (30) again expresses the particle-hole energies in terms of the particle-particle energies. The coefficients giving one in terms of the other (Table IV) differ from those of Table III, because the energies are calculated with respect to a different zero point. If we include the Hartree-Fock energy of the p_j particle in the interaction matrix rather than in the configuration energies, we obviously change nothing. This energy is given by

$$v(p_j)^{\text{H-F}}=\sum_{J'T'}\frac{(2J'+1)(2T'+1)}{(2j+1)(2)}$$
$$\times[\langle p_js_{1/2}J'T'|V|p_js_{1/2}J'T'\rangle_0-(-1)^{j+\frac{1}{2}-J'}$$
$$\times(-1)^{\frac{1}{2}+\frac{1}{2}-T'}\langle p_js_{1/2}J'T'|V|s_{1/2}p_jJ'T'\rangle_0], \quad (31)$$

or writing it out explicitly, we have

$$v(p_{1/2})^{\text{H-F}}=\tfrac{1}{4}\{v(0^-,0)+3v(1^-,0)+3v(0^-,1)+9v(1^-,1)\},$$
$$v(p_{3/2})^{\text{H-F}}=\tfrac{1}{8}\{3v(1^-,0)+5v(2^-,0)$$
$$+9v(1^-,1)+15v(2^-,1)\}.$$

Adding this matrix of coefficients to those of Table III, we get Table IV.

We proceed now to write the particle-particle matrix elements in terms of Talmi integrals. Let us assume a nucleon-nucleon interaction of the form

$$V(r)=(a_w+a_MP_M+a_sP_\sigma+a_tP_\tau)V(r_{12})$$
$$+J(r_{12})(S\odot L)(b_0+b_tP_\tau), \quad (32)$$

where P_M, P_σ, and P_τ are the space-, spin-, and isospin-exchange operators and $S\odot L$ is the tensor operator, $S_{12}=3(\boldsymbol{\sigma}_1\cdot\hat{r})(\boldsymbol{\sigma}_2\cdot\hat{r})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$, written as the scalar product of two irreducible tensor operators of rank 2, where $S=[\sigma(1)\odot\sigma(2)]_{2M}$ and $L\equiv[r_{12}\odot r_{12}]_{2M}$. [We can easily generalize our results to include different radial dependences for the forces by letting $(a_w+a_M+\cdots)V\rightarrow(a_wV^W+a_MV^M+\cdots)$ at the end.] We first go over to L-S coupling. We have

$$\langle p_{j'}s_{1/2}J'T'|V|p_js_{1/2}J'T'\rangle_0-(-1)^{\frac{1}{2}+j+J'}(-1)^{\frac{1}{2}+\frac{1}{2}+T'}\langle p_{j'}s_{1/2}J'T'|V|s_{1/2}p_jJ'T'\rangle_0$$
$$=6[(2j+1)(2j'+1)]^{1/2}\sum_{S=0}^{1}(2S+1)\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & S\\ 1 & 0 & L\\ j & \frac{1}{2} & J'\end{Bmatrix}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & S\\ 1 & 0 & L\\ j' & \frac{1}{2} & J'\end{Bmatrix}(a_w+a_M(-1)^{1+S+T'}+a_s(-1)^{1+S}+a_t(-1)^{1+T'})$$
$$\times(\tfrac{1}{2}(I^0+I^1)-(-1)^{S+T'}\tfrac{1}{2}(I^0-I^1))+6\cdot3\cdot[(2j+1)(2j'+1)]^{1/2}$$
$$\times\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ 1 & 0 & L\\ j & \frac{1}{2} & J'\end{Bmatrix}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ 1 & 0 & L\\ j' & \frac{1}{2} & J'\end{Bmatrix}(b_0+b_t)(-1)^{J'}\begin{Bmatrix}J' & 1 & 1\\ 2 & 1 & 1\end{Bmatrix}(2\sqrt{5})\delta_{T'1}I_T, \quad (33)$$

where $I_T\equiv\langle 1p\|J(r)[r\odot r]_2\|1p\rangle$ is the matrix element of the radial part of the tensor force in a relative $1p$ state. We note that since the tensor force vanishes both for $S=0$ and in a relative $l=0$ state, we can only have a contribution from the tensor force in an $l=1$, $S=1$, $T=1$ state. Writing these equations out we find for the *particle-*

[15] A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic Press Inc., New York, 1963), p. 538.

particle matrix elements

$$v_{\frac{1}{2}\frac{1}{2}}(0^-0)=(a_w+a_M+a_s-a_t)I_0,$$

$$v_{\frac{1}{2}\frac{1}{2}}(0^-1)=(a_w-a_M+a_s+a_t)I_1+\frac{2\sqrt{5}}{3}(b_0+b_t)I_T,$$

$$v_{\frac{1}{2}\frac{1}{2}}(1^-0)=\tfrac{2}{3}(a_w+a_M+a_s-a_t)I_0+\tfrac{1}{3}(a_w-a_M-a_s-a_t)I_1,$$

$$v_{\frac{1}{2}\frac{1}{2}}(1^-1)=\tfrac{1}{3}(a_w+a_M-a_s+a_t)I_0+\tfrac{2}{3}(a_w-a_M+a_s+a_t)I_1-\frac{2\sqrt{5}}{9}(b_0+b_t)I_T,$$

$$v_{\frac{3}{2}\frac{1}{2}}(1^-0)=\tfrac{1}{3}(a_w+a_M+a_s-a_t)I_0+\tfrac{2}{3}(a_w-a_M-a_s-a_t)I_1,$$

$$v_{\frac{3}{2}\frac{3}{2}}(2^-0)=(a_w+a_M+a_s-a_t)I_0,$$

$$v_{\frac{3}{2}\frac{3}{2}}(1^-1)=\tfrac{2}{3}(a_w+a_M-a_s+a_t)I_0+\tfrac{1}{3}(a_w-a_M+a_s+a_t)I_1-\frac{\sqrt{5}}{9}(b_0+b_t)I_T,$$

$$v_{\frac{3}{2}\frac{3}{2}}(2^-1)=(a_w-a_M+a_s+a_t)I_1+\frac{\sqrt{5}}{15}(b_0+b_t)I_T,$$

$$v_{\frac{1}{2}\frac{3}{2}}(1^-0)=-\frac{\sqrt{2}}{3}[(a_w+a_M+a_s-a_t)I_0-(a_w-a_M-a_s-a_t)I_1],$$

$$v_{\frac{1}{2}\frac{3}{2}}(1^-1)=\frac{\sqrt{2}}{3}[(a_w+a_M-a_s+a_t)I_0-(a_w-a_M+a_s+a_t)I_1]+\frac{\sqrt{10}}{9}(b_0+b_t)I_T.$$

We can now use our coefficients to get the *particle-hole* energies (we are only interested here in *splittings* within the [15] supermultiplet so we use the coefficients in Table IV).

Particle-hole energies (we define $b_0+b_t\equiv b$ since only this combination enters our results):

$$E_{1/2}(0^-0)=(a_w+a_M+a_s-a_t)I_0+(2a_w-2a_M+a_s+a_t)I_1+(\sqrt{5})bI_T,$$

$$E_{1/2}(1^-0)=\tfrac{1}{3}(5a_w+5a_M+2a_s-2a_t)I_0+\tfrac{2}{3}(2a_w-2a_M+a_s+a_t)I_1-\frac{\sqrt{5}}{3}bI_T,$$

$$E_{1/2}(0^-1)=(a_w+a_M)I_0+2(a_w-a_M+a_s+a_t)I_1+\frac{\sqrt{5}}{3}bI_T,$$

$$E_{1/2}(1^-1)=(3a_w+3a_M-a_s+a_t)\tfrac{1}{3}I_0+\tfrac{1}{3}(6a_w-6a_M+5a_s+5a_t)I_1-\frac{\sqrt{5}}{9}bI_T,$$

$$E_{3/2}(1^-0)=\tfrac{1}{3}(7a_w+7a_M+a_s-a_t)I_0+\tfrac{1}{3}(2a_w-2a_M+a_s+a_t)I_1-\frac{\sqrt{5}}{6}bI_T,$$

$$E_{3/2}(2^-0)=(a_w+a_M+a_s-a_t)I_0+(2a_w-2a_M+a_s+a_t)I_1+\frac{\sqrt{5}}{10}bI_T,$$

$$E_{3/2}(1^-1)=\tfrac{1}{3}(3a_w+3a_M-2a_s+2a_t)I_0+\tfrac{1}{3}(6a_w-6a_M+4a_s+4a_t)I_1-\frac{\sqrt{5}}{18}bI_T,$$

$$E_{3/2}(2^-1)=(a_w+a_M)I_0+2(a_w-a_M+a_s+a_t)I_1+\frac{\sqrt{5}}{30}bI_T,$$

(34)

$$V_{\frac{1}{2}\frac{1}{2}}(1^-0)=\frac{\sqrt{2}}{6}(4a_w+4a_M-2a_s+2a_t)I_0+\frac{\sqrt{2}}{6}(-4a_w+4a_M-2a_s-2a_t)I_1+\frac{\sqrt{10}}{6}bI_T,$$

$$V_{\frac{1}{2}\frac{1}{2}}(1^-1)=\frac{\sqrt{2}}{3}(a_t-a_s)I_0-\frac{\sqrt{2}}{3}(a_t+a_s)I_1+\frac{\sqrt{10}}{18}bI_T,$$

TABLE III. Matrix of coefficients to go from particle-particle ($J'T'$) to particle-hole (JT) interactions in He^4:

$$-(2J'+1)(2T'+1)\begin{Bmatrix}\frac{1}{2}j'J\\ \frac{1}{2}jJ'\end{Bmatrix}\begin{Bmatrix}\frac{1}{2}\frac{1}{2}T\\ \frac{1}{2}\frac{1}{2}T'\end{Bmatrix}.$$

	JT \ $J'T'$	$(0^-,0)$	$(1^-,0)$	$(0^-,1)$	$(1^-,1)$
$p_j=p_{1/2}$, $p_{j'}=p_{1/2}$	$(0^-,0)$	$-1/4$	$3/4$	$3/4$	$-9/4$
	$(1^-,0)$	$1/4$	$1/4$	$-3/4$	$-3/4$
	$(0^-,1)$	$1/4$	$-3/4$	$1/4$	$-3/4$
	$(1^-,1)$	$-1/4$	$-1/4$	$-1/4$	$-1/4$
	JT \ $J'T'$	$(1^-,0)$	$(2^-,0)$	$(1^-,1)$	$(2^-,1)$
$p_j=p_{3/2}$, $p_{j'}=p_{3/2}$	$(1^-,0)$	$-1/8$	$5/8$	$3/8$	$-15/8$
	$(2^-,0)$	$3/8$	$1/8$	$-9/8$	$-3/8$
	$(1^-,1)$	$1/8$	$-5/8$	$1/8$	$-5/8$
	$(2^-,1)$	$-3/8$	$-1/8$	$-3/8$	$-1/8$
	JT \ $J'T'$	$(1^-,0)$	$(1^-,1)$		
$p_j=p_{1/2}$, $p_{j'}=p_{3/2}$	$(1^-,0)$	$-1/2$	$3/2$		
	$(1^-,1)$	$1/2$	$1/2$		

where $V_{\frac{1}{2}\frac{1}{2}}$ are the off-diagonal elements of v in the particle-hole configuration.

We still have to pick out the correct combination for the $J^\pi=1^-$, $T=0$ state corresponding to the $^3P_1$. We use

$$|{}^3P_1\rangle=(\sqrt{\tfrac{2}{3}})\,|p_{1/2}(s_{1/2})^{-1}1^-\rangle-(\sqrt{\tfrac{1}{3}})\,|p_{3/2}(s_{1/2})^{-1}1^-\rangle$$

and after a little algebra arrive at

$$E_{^3P_1}(1^-,0)=(a_w+a_M+a_s-a_t)I_0+(2a_w-2a_M+a_s+a_t)I_1-\frac{\sqrt{5}}{2}bI_T. \quad (35)$$

We are now in a position to draw several interesting conclusions about the splittings within the supermultiplet.

(i) If we look at the splitting of the 0^- and 2^- states for both $T=0$ and $T=1$, we find

$$E(0^-,0)-E(2^-,0)=(9/10)(\sqrt{5})bI_T,$$
$$E(0^-,1)-E(2^-,1)=(3/10)(\sqrt{5})bI_T. \quad (36)$$

Therefore, *none of the central forces* contribute to this splitting. This can be understood very easily as this

TABLE IV. Same as Table III only including the single-particle energy of Eq. (31).

	JT \ $J'T'$	$(0\text{-}0)$	$(1\text{-}0)$	$(0\text{-}1)$	$(1\text{-}1)$
$p_j=p_{1/2}$, $p_{j'}=p_{1/2}$	$(0\text{-}0)$	0	3/2	3/2	0
	$(1\text{-}0)$	1/2	1	0	3/2
	$(0\text{-}1)$	1/2	0	1	3/2
	$(1\text{-}1)$	0	1/2	1/2	2
	JT \ $J'T'$	$(1\text{-}0)$	$(2\text{-}0)$	$(1\text{-}1)$	$(2\text{-}1)$
$p_j=p_{3/2}$, $p_{j'}=p_{3/2}$	$(1\text{-}0)$	1/4	5/4	3/2	0
	$(2\text{-}0)$	3/4	3/4	0	3/2
	$(1\text{-}1)$	1/2	0	5/4	5/4
	$(2\text{-}1)$	0	1/2	3/4	7/4

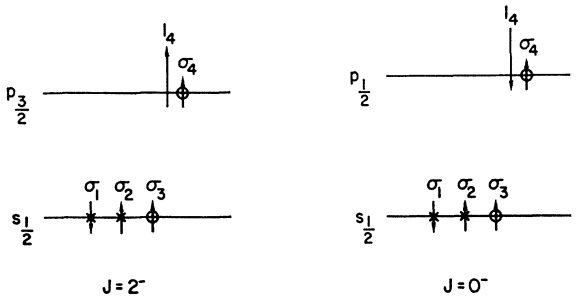

FIG. 1. Structure of the $J^\pi=0^-$ and $J^\pi=2^-$ levels.

splitting must be due to the *difference* between the interaction of the $p_{1/2}$ particle in the $J=0^-$ state with the $(s_{1/2})^3$ configuration and that of the $p_{3/2}$ particle in the $J=2^-$ state with the same configuration. Neglecting differences in the radial wave function for $p_{1/2}$ and $p_{3/2}$, such a difference in the interaction can result only from spin-dependent forces. However the spin $\boldsymbol{\sigma}_4$ of the p particle always sees a saturated pair of spins $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ of the unlike particles in the s orbit, and as far as the spin of the like particle $\boldsymbol{\sigma}_3$ is concerned, it is parallel to $\boldsymbol{\sigma}_4$ both for $J=2^-$ and for $J=0^-$, as can be seen from Fig. 1. Thus, the central spin-dependent interaction of the p particle with the $(s_{1/2})^3$ is the same for both $J=0^-$ and $J=2^-$ and does not lead to an additional splitting between them.

There is one effect we have not yet included, namely, that it is known that the $p_{3/2}$ and $p_{1/2}$ levels are split by a single-particle spin-orbit force. If we go back to our Hartree-Fock single-particle energies we see that with the two-particle force we have assumed in Eq. (32), we have

$$v(p_{1/2})^{\text{H-F}}=v(p_{3/2})^{\text{H-F}}=\tfrac{3}{2}(a_w+a_M)I_0+\tfrac{1}{2}(5a_w-5a_M+4a_s+4a_t)I_1 \quad (37)$$

and we cannot explain this splitting within the present approximation. We will therefore simply add an empirical single-particle spin-orbit splitting in our single-particle configuration energies, and define

$$\epsilon_{p_{1/2}}-\epsilon_{p_{3/2}}\equiv\epsilon. \quad (38)$$

We can either take this value from p-α and n-α scattering to be

$$\epsilon\cong 4\text{ MeV}\quad (p\text{-}\alpha\text{ and }n\text{-}\alpha\text{ scattering})$$

or use ϵ as a parameter to be determined from the splittings within the supermultiplet. The spin-orbit splitting may come from the spin-orbit part of the nucleon-nucleon interaction or from second-order effects of the tensor force. In any event, it is a quantity which we will not attempt to calculate in the present approximation. Since the 0^- and 2^- states are pure j-j configurations, we can immediately include ϵ and obtain,

instead of (36),

$$E(0^-,0)-E(2^-,0)=\epsilon+(9/10)(\sqrt{5})bI_T,$$
$$E(0^-,1)-E(2^-,1)=\epsilon+(3/10)(\sqrt{5})bI_T. \quad (39)$$

(ii) If we look at the $T=0$ splittings, we just have to compute the energy splittings within the triplet ^3P_J. From the above we get

$$E(0^-,0)-E(1^-,0)=\tfrac{1}{3}\epsilon+\tfrac{3}{2}(\sqrt{5})bI_T,$$
$$E(1^-,0)-E(2^-,0)=\tfrac{2}{3}\epsilon-\tfrac{3}{5}(\sqrt{5})bI_T. \quad (40)$$

Again the central forces do not contribute to these splittings, the effect being entirely due to the single-particle spin-orbit interaction and to the tensor force. The former can be evaluated directly from the fact that it involves the scalar product of two vectors,[16] leading, in the absence of tensor forces, to

$$\frac{E(2^-0)-E(1^-0)}{E(1^-0)-E(0^-0)}=\frac{6-2}{2-0}=2 \quad \text{if} \quad I_T=0. \quad (41)$$

Combining the results of (i) and (ii) we see that we have three independent splittings in terms of the two parameters ϵ and bI_T. The contributions of these two quantities can therefore be determined and the consistency of the whole picture can be checked, by comparing (39) and (40) with experiment.

(iii) The splitting between the isospin multiplets can be obtained from the above as

$$E(2^-,1)-E(2^-,0)=(a_t-a_s)I_0+(a_s+a_t)I_1-(1/15)(\sqrt{5})bI_T. \quad (42)$$

It is independent of the Wigner and Majorana forces, as it must be, and is also independent of ϵ since both states in (42) are pure $p_{3/2}$ configurations.

(iv) In the limit of pure j-j coupling, the $T=1$ spectrum splits into two $s_{1/2}$ doublets. It is known[17] that the lowest member of such doublets has even or odd J according to whether the parity of the configuration is negative or positive. The splittings of these doublets is evidently given by

$$E_{3/2}(1^-,1)-E_{3/2}(2^-,1)=\tfrac{2}{3}(a_t-a_s)I_0-\tfrac{2}{3}(a_s+a_t)I_1-(4/45)(\sqrt{5})bI_T,$$
$$E_{1/2}(1^-,1)-E_{1/2}(0^-,1)=\tfrac{1}{3}(a_t-a_s)I_0-\tfrac{1}{3}(a_s+a_t)I_1-(4/9)\sqrt{5})bI_T;$$
$$(j\text{-}j \text{ coupling}). \quad (43)$$

With no tensor force these splittings would simply be in the ratio of 2:1. For the $J^\pi=1^-$, $T=1$ states, however, we must actually diagonalize the interaction to find the correct state vectors. From Eq. (28) we see that we must solve the two equations

$$[\tfrac{2}{3}\epsilon-\epsilon_s+E_{1/2}(1^-,1)-\epsilon^{1^-,1}]\alpha_{1/2}{}^{1^-,1}+v_{\frac{1}{2}\frac{3}{2}}(1^-,1)\alpha_{3/2}{}^{1^-,1}=0, \quad (44)$$
$$v_{\frac{1}{2}\frac{3}{2}}(1^-,1)\alpha_{1/2}{}^{1^-,1}+[-\tfrac{1}{3}\epsilon-\epsilon_s+E_{3/2}(1^-,1)-\epsilon^{1^-,1}]\alpha_{3/2}{}^{1^-,1}=0,$$

in order to get the splittings and the coefficients of the state vectors $\alpha_{1/2}{}^{1^-,1}$ and $\alpha_{3/2}{}^{1^-,1}$. Note that $v_{\frac{3}{2}\frac{3}{2}}(1^-,1)=v_{\frac{1}{2}\frac{3}{2}}(1^-,1)$ since the matrix elements of the potential are real, and that again, only the spin-dependent parts of the interaction enter into this calculation.

If we denote the two new eigenvalues by $\epsilon_\pm{}^{1^-,1}$, we see that

$$\tfrac{1}{2}(\epsilon_+{}^{1^-,1}+\epsilon_-{}^{1^-,1})=\tfrac{1}{2}[E_{1/2}(1^-,1)+E_{3/2}(1^-,1)+\tfrac{1}{3}\epsilon-2\epsilon_s] \quad (45)$$

and the center of gravity of the two $J^\pi=1^-$, $T=1$ levels is not affected by the "mixing" introduced by $v_{\frac{1}{2}\frac{3}{2}}$.

(v) "Center-of-gravity" theorem.

From Eq. (21) we have

$$\epsilon^{[15]}=\epsilon_p-\epsilon_s+v_{ps;ps}{}^{[15]}. \quad (46)$$

This is the energy of the supermultiplet before the spin-dependent forces are turned on. We can ask how $\epsilon^{[15]}$ is related to the actual spectrum, that is, do the spin-dependent forces shift the position of the "center of gravity" of the supermultiplet? Let us define this new quantity by

$$\epsilon_{c.g.}\equiv\frac{\sum^{[15]}(2J+1)(2T+1)E^{JT}}{\sum^{[15]}(2J+1)(2T+1)}, \quad (47)$$

where E^{JT} is the actual energy of the level J, T. If the spin-orbit force is of the form $H_{s.o.}=-\xi(r)\mathbf{l}\cdot\mathbf{s}$, then it is a simple matter to see that (note that $\epsilon_{s1/2}{}^{s.o.}=0$):

$$\sum^{[15]}(2J+1)(2T+1)\epsilon_{s.o.}{}^{JT}\equiv 0. \quad (48)$$

If we now take the energy shifts obtained from Eq. (29) and sum them over the supermultiplet, we obtain

$$\epsilon_{c.g.}=\epsilon_p-\epsilon_s+\frac{\sum^{[15]}(2J+1)(2T+1)v_{jj}{}^{JT}}{\sum^{[15]}(2J+1)(2T+1)}. \quad (49)$$

In the sum we have to take the correct combination of matrix elements for the $^3P_1$, $T=0$ state [see Eq. (35)] and we can forget the configuration mixing in the 1^-, $T=1$ states since the "center of gravity" of these states is unchanged by the mixing as we saw in (45). Carrying out the sum we arrive at

$$\epsilon_{c.g.}=\epsilon_p-\epsilon_s-[\tfrac{1}{2}(a_w+a_M)I_0+\tfrac{1}{2}(a_w-a_M)I_1]-\tfrac{2}{5}(a_s+a_t)I_1$$
$$=\epsilon_p-\epsilon_s+v_{ps;ps}{}^{[15]}-\tfrac{2}{5}(a_s+a_t)I_1$$
$$=\epsilon^{[15]}-\tfrac{2}{5}(a_s+a_t)I_1. \quad (50)$$

[16] Reference 15, p. 279.

[17] A. de-Shalit and J. D. Walecka, Nucl. Phys. 22, 184 (1961).

TABLE V. Potential parameters.

Force	Shape	V_0 (MeV)	μ (F^{-1})
Kurath	$-V_0e^{-\mu r}/\mu r$	36	0.714
Free (Serber-Yukawa)	$-V_0e^{-\mu r}/\mu r$	Singlet 46.9	0.855
		Triplet 52.1	0.726
Free (Serber-exponential)	$-V_0e^{-\mu r}$	Singlet 108	1.409
		Triplet 193	1.506

We conclude therefore that the central spin-dependent forces give rise to an additional shift in position of the center of gravity beyond that given by the supermultiplet theory according to Eq. (46).

Numerical Results

We shall calculate the spectra of He^4 with three different nucleon-nucleon potentials. The first is a potential of the type used by Kurath[18] in his systematic attempt to fit the spectra of light nuclei in an intermediate coupling calculation. This potential is of the form

$$V(r)=-V_0[P_M+\tfrac{1}{4}P_\sigma](e^{-\mu r}/\mu r). \quad (51)$$

We give the numerical value of the parameters we use in Table V. The second and third potentials are taken from fitting low-energy nucleon-nucleon scattering.[19] We chose a Serber force (which fits the data up to about 90 MeV). Note that if we really have a Serber force then *the tensor force does not contribute to the splitting of the supermultiplet* since we have shown that it can only contribute through the odd relative angular-momentum states. We take

$$V=[{}^1V(r)\tfrac{1}{4}(1-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+{}^3V(r)\tfrac{1}{4}(3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]\tfrac{1}{2}(1+P_M) \quad (52)$$

which we can also write as

$$V=\tfrac{1}{4}\{({}^3V+{}^1V)(1+P_M)+({}^3V-{}^1V)(P_\sigma-P_\tau)\} \quad (53)$$

and calculate the energies using both a Yukawa and exponential radial dependence. The parameters are also summarized in Table V.

We can immediately evaluate the necessary Talmi integrals, and the results are given in Table VI. (We use $\bar{b}=1.95$ F.)

Let us first concentrate on the splittings within the supermultiplet since these are really the quantities

TABLE VI. Talmi integrals ($\bar{b}=1.95$ F) (in MeV).

Force	M_{I0}	M_{I1}	B_{I0}	B_{I1}	${}^3I_0$	${}^1I_0$
Kurath	−8.8	−3.2	−2.2	−0.8		
Free (Serber-Yukawa)					−14.5	−9.6
Free (Serber-exponential)					−15.2	−9.4

[18] D. Kurath, Phys. Rev. **101**, 216 (1956); G. J. Malosh, University of Pittsburgh Technical Report, 1965 (unpublished).
[19] L. Hulthén and M. Sugawara, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, pp. 52, 62.

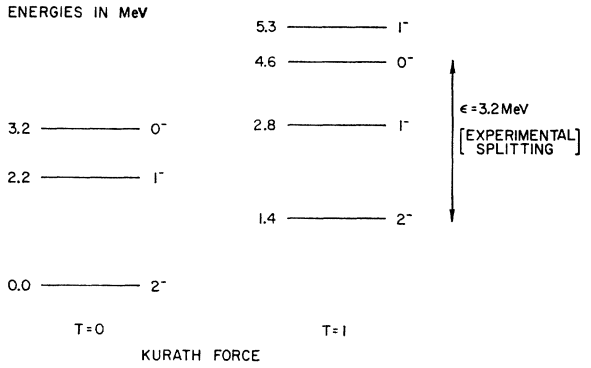

FIG. 2. Splitting of the [15] supermultiplet using the Kurath force.

which we can calculate most reliably. The resulting spectrum with the Kurath force is shown in Fig. 2. The spin-orbit splitting was taken as

$$\epsilon=+3.2 \text{ MeV}$$

to fit the $0^- - 2^-$ $T=1$ separation in Li^4.[4] We therefore predict five spacings. The experimental spectrum is shown in Fig. 3.[4–6] The agreement is very satisfactory. We note that the 1^- and 0^-, $T=1$ states have not been observed so far in He^4, and these energies were taken from H^4 and Li^4.

The spectrum with the free Serber force depends on two parameters chosen to be positive:

$$\epsilon\equiv\epsilon_{p_{1/2}}-\epsilon_{p_{3/2}}, \qquad a\equiv-\tfrac{1}{2}({}^3I_0-{}^1I_0).$$

We give the general spectrum in Fig. 4. (Recall that a Serber-tensor force does not contribute to the splittings.) The values of the parameter a which we get from Table VI are

$$a=+2.45 \text{ MeV} \quad \text{(Yukawa)}$$
$$=+2.90 \text{ MeV} \quad \text{(exponential)}.$$

The resulting spectra, again using

$$\epsilon=+3.2 \text{ MeV} \quad (Li^4)$$

from the $0^- - 2^-$ $T=1$ splitting in Li^4, is given in Fig. 5.

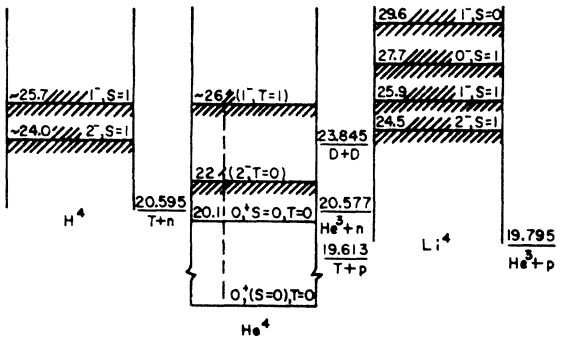

FIG. 3. Experimental spectrum in the $A=4$ system.

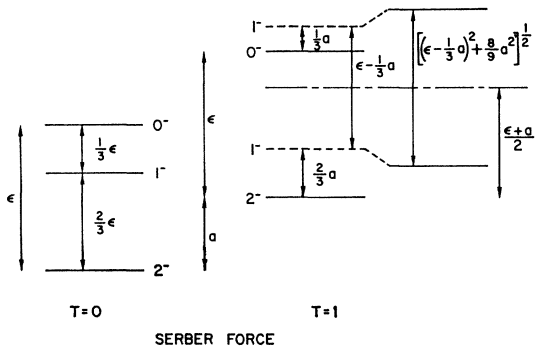

FIG. 4. General spectrum with a Serber force.

The Yukawa and exponential potentials give almost identical spectra and the results are very similar to those of the Kurath force, although somewhat closer to the experimental splittings.

We can next ask what the prediction is for the position of the center of gravity of the supermultiplet. This depends on the Wigner and Majorana parts of the force. There is some ambiguity in this calculation because of the difficulty of locating the unperturbed particle-hole energy $\epsilon_p-\epsilon_s$. We summarize the results in Table VII.

The experimental value, using the Landé interval rule to locate the missing 1^- and 0^-, $T=0$ states is

$$\epsilon_{c.g.}=+25.8 \text{ MeV} \quad \text{(experimental)}.$$

In all cases, $\hbar\omega=18.0=\epsilon_p-\epsilon_s$ gives too low a result and should probably be discounted. In the other cases the Kurath force does very well and the free force is 2 to 3 MeV too high. The free force also gives a result too high by 1 to 2 MeV in the giant resonance region[14] in C^{12} and O^{16}. It is difficult to draw any definite conclusions from this, however, since the levels we are discussing are not really bound states but broad continuum resonances.

Finally, we discuss electric dipole transitions from the $T=1$, 1^- states to the ground state. In the case of the Serber force the matrix equations (28) for the states

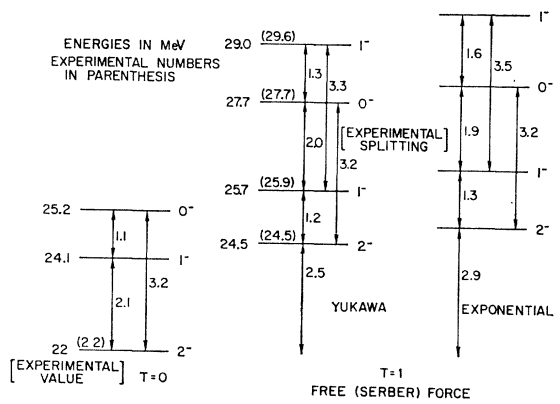

FIG. 5. Splitting of the [15] supermultiplet using the free Serber force.

TABLE VII. Predictions for $\epsilon_{c.g.}$ (all in MeV).

Force	$v_{ps;\,ps}$[15]	$-\frac{2}{5}(a_t+a_s)I_1$	$\epsilon_p-\epsilon_s$	$\epsilon_{c.g.}$
Kurath	+2.8	+0.3	22.6[a]	25.7
			21.8[b]	24.9
			18.0[c]	21.1
Free (Serber-Yukawa)	6.0	0	22.6[a]	28.6
			21.8[b]	27.8
			18.0[c]	24.0
Free (Serber-exponential)	6.1	0	22.6[a]	28.7
			21.8[b]	27.9
			18.0[c]	24.1

[a] From neighboring nuclei [Eq. (25)].
[b] From $\hbar\omega$ as determined from electron scattering (Table I).
[c] From $\hbar\omega$ as determined from Coulomb energies (Table I).

$T=1$ $J^\pi=1^-$ are of the form

$$\begin{pmatrix}\epsilon-\frac{1}{3}a-\lambda & \frac{1}{3}\sqrt{2}a\\ \frac{1}{3}\sqrt{2}a & -\lambda\end{pmatrix}\begin{pmatrix}\alpha_{1/2}\\ \alpha_{3/2}\end{pmatrix}=0. \tag{54}$$

(We have shifted our energy origin to simplify things. This does not change any results. λ is the new eigenvalue.)

The eigenvalues are

$$\lambda_\pm=\tfrac{1}{2}\{(\epsilon-\tfrac{1}{3}a)\pm[(\epsilon-\tfrac{1}{3}a)^2+(8/9)a^2]^{1/2}\} \tag{55}$$

and if we define the coupling parameter

$$x\equiv a/\epsilon, \tag{56}$$

then the ratio of coefficients is

$$(\alpha_{1/2}/\alpha_{3/2})_+=-\tfrac{2}{3}\sqrt{2}x/\{1-\tfrac{1}{3}x-[(1-\tfrac{1}{3}x)^2+(8/9)x^2]^{1/2}\},$$
$$(\alpha_{1/2}/\alpha_{3/2})_-=-\tfrac{2}{3}\sqrt{2}x/\{1-\tfrac{1}{3}x+[(1-\tfrac{1}{3}x)^2+(8/9)x^2]^{1/2}\}. \tag{57}$$

Now, electric dipole transitions to the ground state can only take place from the $|{}^1P_1\,T=1\rangle$ components of these states. Using the above and Eq. (27), we can compute the ratio of the probabilities for finding the upper and lower $J^\pi=1^-$, $T=1$ states in a ${}^1P_1$ configuration to be

$$\frac{|\langle{}^1P_1|\psi_+\rangle|^2}{|\langle{}^1P_1|\psi_-\rangle|^2}=\left[\frac{1-x-[(1-\frac{1}{3}x)^2+(8/9)x^2]^{1/2}}{1-x+[(1-\frac{1}{3}x)^2+(8/9)x^2]^{1/2}}\right]^2$$
$$\times\frac{[(1-\frac{1}{3}x)^2+(8/9)x^2]^{1/2}+(1-\frac{1}{3}x)}{[(1-\frac{1}{3}x)^2+(8/9)x^2]^{1/2}-(1-\frac{1}{3}x)}.$$

This ratio has the following limiting values for pure L-S or j-j coupling:

$$\frac{|\langle{}^1P_1|\psi_+\rangle|^2}{|\langle{}^1P_1|\psi_-\rangle|^2}\xrightarrow[x\to0]{}\tfrac{1}{2} \qquad (j\text{-}j \text{ coupling})$$
$$\xrightarrow[x\to\infty]{}\tfrac{9}{2}x^2\to\infty \qquad (L\text{-}S \text{ coupling}).$$

$|\psi_+\rangle$ is the upper state and $|\psi_-\rangle$ the lower state. As $x\to0$, the upper state becomes pure $|(1s_{1/2})^{-1}(1p_{1/2})1^-\rangle$, the lower state becomes pure $|(1s_{1/2})^{-1}(1p_{3/2})1^-\rangle$, and $\frac{1}{2}$

TABLE VIII. $|^1P_1\rangle$ content of the $J^\pi=1^-$, $T=1$ states.

Force	$x\equiv a/\epsilon$	$\lvert\langle^1P_1\lvert\psi_+\rangle\rvert^2/\lvert\langle^1P_1\lvert\psi_-\rangle\rvert^2$
Free (Serber-Yukawa)	0.77	2.4
Free (Serber-exponential)	0.91	3.1

is just the square of the ratio of the $|^1P_1\rangle$ content of these states. As $x\to\infty$ we have pure L-S coupling and the upper state becomes pure $|^1P_1\rangle$ while the lower state is $|^3P_1\rangle$. Therefore $E1$ transitions from the lower state are forbidden in this limit. The results for the free force are given in Table VIII which says that the *upper* $T=1$, 1^- state should have most of the $E1$ strength (phase space only increases this ratio). This appears to be in contradiction with the experimental results where the peak in the photoabsorption cross section is at the lower state. The only way out of this dilemma within the framework of the present calculation is through the use of more complicated forces. In the calculation which we have carried out, a and ϵ can be thought of as being determined experimentally from the observed spectrum and the above ratio is therefore fixed. (The Kurath force gives the same results.) One must go farther and include for example the two-body tensor and spin-orbit forces with the hope of improving the wave functions without drastically changing the spectrum. One would like to be closer to the j-j coupling limit for then $|\langle^1P_1|\psi_+\rangle|^2/|\langle^1P_1|\psi_-\rangle|^2=\frac{1}{2}$ which is in agreement with the experiments.[6] Since all the *splittings* within the supermultiplet come *only* from the spin-dependent parts of the nucleon-nucleon force, these other components of the force may well play a non-negligible role even though they enter only in relative p states.

CONCLUSIONS

The negative-parity excited states of He^4 at an excitation energy of about $\hbar\omega$ are described by exceptionally symmetric wave functions. Because of this symmetry their splitting can be due only to a part of the nucleon-nucleon interaction: Wigner and Majorana forces can lead to no splittings among these levels, and one has to resort to spin-dependent interactions to obtain the observed structure of the s^3p 15-dimensional supermultiplet. Furthermore, if we use harmonic-oscillator wave functions as described before, only two averages of the interaction come in—those taken in a relative s state and relative p state. If we believe that the He^4 structure is not too sensitive to off-energy-shell matrix elements of the interaction and that only low relative energies play an important role there, a further simplification results. The low-energy nucleon-nucleon scattering data are well accounted for by a Serber-type force, and this force leads to no interaction in relative p state. We are then left with just one integral of the interaction, in addition to the $p_{3/2}$-$p_{1/2}$ splitting, which determines the structure of the configuration s^3p. Hence the "selectivity" of these states as far as information on the nuclear force is concerned.

The position of the known levels could be calculated to an accuracy which is an order of magnitude better than their widths. This was achieved by a calculation that assumed no width whatsoever. One would expect that energy shifts are less affected than widths of levels by small modifications of the wave functions. The observed widths of the levels indicates that a resonating nucleon is reflected ten or more times before it overcomes the centrifugal barrier and manages to escape from the excited He^4; success of a bound-state approximation is perhaps not unexpected. Still the fact that levels of different widths turn out to have equal shifts is surprising and may have interesting physics behind it. One is reminded that a similar situation exists in other nuclei as well as in elementary-particle physics. The conclusion that the four I multiplets in the $SU(3)$ decuplet are equidistant in mass is basically a "bound-state" approximation. Experimentally they are indeed found to be equidistant although one of them—the Ω^-—has practically a zero width, and the widths of the others vary wildly. The understanding of these results may prove to be interesting, especially if their seemingly general validity could be explained.

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