Vector and Tensor Coulomb Energies*

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Vector and tensor Coulomb energies have been extracted from the known experimental Coulomb energy differences between isobaric doublets and triplets in light nuclei. The vector Coulomb energy essentially depends linearly on A within a given shell and shows discontinuities at major shell closures and superimposed weak oscillations. The tensor Coulomb energy exhibits a pronounced oscillatory structure. The characteristic A dependence of both Coulomb energies is discussed and the oscillations in particular are related to Coulomb pairing effects.

I. INTRODUCTION

PRECISE values for the masses of proton-rich light nuclei with $T_z=-1$ have recently been measured[1] by various nuclear reactions. In addition, excitation energies for states with $T=1$ in several self-conjugate nuclei have also been measured[1] using high-resolution γ-ray detectors. These new data together with the older data can be used to derive Coulomb energy differences for the isobaric doublets and triplets up to $A=43$. The Coulomb energy differences are related to the vector and tensor Coulomb energies and also to the coefficients in the isobaric mass formula.[2–4] The present study was undertaken to obtain a better understanding of the A and T dependence of these quantities. In intermediate and heavy nuclei many Coulomb energy differences have also been measured[5] very accurately. Vector and tensor Coulomb energies, however, cannot be extracted in these cases because at least two Coulomb energy differences for a given A and T must be known.

A detailed knowledge of the A dependence of the tensor Coulomb energy can possibly yield information on a violation of charge independence of nuclear forces or other effects which may affect the energetic position of the $T_z=0$ members of the isobaric triplets.[6–8]

II. THE COULOMB ENERGY DIFFERENCES

Table I gives the experimental Coulomb energy differences for the isobaric doublets (column 3) and isobaric triplets (columns 2 and 4). The energy differences Δ_{10} between the lowest states with $T=1$ and $T=0$ in the $T_z=0$ nuclei are indicated in column 5. The data result from measurements of Q values of various types of nuclear reactions, from measurements of maximum β^+ end-point energies and from measurements of the energies of associated γ rays.

Table II lists the few cases in light nuclei with $T>1$, where two or more Coulomb energy differences between ground-state isobaric analog states for a given A and T are known experimentally.

III. THE VECTOR AND TENSOR COULOMB ENERGIES

The Coulomb interaction between nucleons leads to an energy shift between the members of an isobaric multiplet. The Coulomb interaction

$$H_c=e^2\sum_{i<j}(\tfrac{1}{2}-t_z^{(i)})(\tfrac{1}{2}-t_z^{(j)})r_{ij}^{-1} \tag{1}$$

can be written as

$$H_c=T^{(0)}+T^{(1)}+T^{(2)}, \tag{2}$$

that is as a sum of an isoscalar, -vector, and -tensor operator.[3] Since this expansion does not go beyond the second rank, one obtains in the first-order perturbation theory the expression[2,3]

$$E_{\mathrm{Coul}}(A,T,T_z)=E_{\mathrm{Coul}}^{(0)}(A,T)-T_zE_{\mathrm{Coul}}^{(1)}(A,T)+[3T^2-T(T+1)]E_{\mathrm{Coul}}^{(2)}(A,T). \tag{3}$$

The scalar, vector, and tensor Coulomb energies $E_{\mathrm{Coul}}^{(0)}(A,T)$, $E_{\mathrm{Coul}}^{(1)}(A,T)$, and $E_{\mathrm{Coul}}^{(2)}(A,T)$ are independent of T_z. They depend only on the physical properties of the system, i.e., on the other quantum numbers which describe the system considered. By inverting Eq. (3) one obtains

$$E_{\mathrm{Coul}}^{(0)}(A,T)=\frac{1}{2T+1}\sum_{T_z=-T}^{+T}E_{\mathrm{Coul}}(A,T,T_z), \tag{4}$$

$$E_{\mathrm{Coul}}^{(1)}(A,T)=\frac{3}{T(T+1)(2T+1)}\times\sum_{T_z=-T}^{+T}(-T_z)E_{\mathrm{Coul}}(A,T,T_z), \tag{5}$$

* This work was supported in part by the U. S. Atomic Energy Commission.

[1] See references given in Table I.

[2] E. P. Wigner, Proc. Robert A. Welch Found. Conf. Chem. Res., 1st; Houston, Texas (1957) 1, 88 (1958); S. Weinberg and S. B. Treiman, Phys. Rev. **116**, 465 (1959); D. H. Wilkinson, Phys. Letters **11**, 243 (1964); **12**, 348 (1964).

[3] W. M. MacDonald, Phys. Rev. **98**, 60 (1955); **100**, 51 (1955); **101**, 271 (1956).

[4] D. H. Wilkinson, Phys. Rev. Letters **13**, 571 (1964).

[5] M. Harchol, S. Cochavi, A. A. Jaffe, and Ch. Drory, Nucl. Phys. **79**, 165 (1966). A compilation of references is given in Table 2 of the paper of Harchol *et al.*

[6] D. H. Wilkinson, Phil. Mag. **1**, 1031 (1956); R. Wallace and J. A. Welch, Phys. Rev. **117**, 1297 (1960); W. M. Fairbairn, Proc. Phys. Soc. (London) **77**, 599 (1961); A. Altman and W. M. MacDonald, Nucl. Phys. **35**, 593 (1962).

[7] S. Sengupta, Nucl. Phys. **21**, 542 (1960).

[8] J. Jänecke, Z. Physik **160**, 171 (1960).

TABLE I. Coulomb energy differences ΔE_{Coul}, vector Coulomb energies $E_{\text{Coul}}^{(1)}$, tensor Coulomb energies $E_{\text{Coul}}^{(2)}$, and ratios $E_{\text{Coul}}^{(2)}/E_{\text{Coul}}^{(1)}$ for the isobaric doublets and triplets. The Δ_{10} denote the excitation energies of the energetically lowest states with $T=1$ in the self-conjugate nuclei. Data are taken from Ref. a unless indicated otherwise.

A	$\Delta E_{\text{Coul}}(A,1,0\|+1)$ (keV)	$\Delta E_{\text{Coul}}(A,\frac{1}{2},-\frac{1}{2}\|+\frac{1}{2})$ (keV)	$\Delta E_{\text{Coul}}(A,1,-1\|0)$ (keV)	Δ_{10} (keV)	$E_{\text{Coul}}^{(1)}(A,\frac{1}{2},\text{and } 1)$ (keV)	$E_{\text{Coul}}^{(2)}(A,1)$ (keV)	$E_{\text{Coul}}^{(2)}(A,1)/E_{\text{Coul}}^{(1)}(A,1)$
1		0.0 ± 0.0			0.0 ± 0.0		
2				2300 approx.			
3		763.84 ± 0.07			763.84 ± 0.07		
4				>21230			
5		1007 ± 41			1007 ± 41		
6	834.7 ± 6.0		1507 ± 7[a,b]	3562 ± 4[c]	1171 ± 4	112.1 ± 1.5	0.0957 ± 0.0013
7		1644.1 ± 0.2			1644.1 ± 0.2		
8	1559 approx.		1982 approx.	16779 approx.[c]	1771 ± 1	70.5 approx.	0.0398 approx.
9		1850.6 ± 0.9			1850.6 ± 0.9		
10	1967 ± 3		2649 ± 13	1740 ± 2[c]	2308 ± 7	113.7 ± 2.2	0.0493 ± 0.0010
11		2763.2 ± 1.1			2763.2 ± 1.1		
12	2522 ± 10		3036 ± 12	15110 ± 10[d]	2779 ± 4	85.7 ± 2.6	0.0308 ± 0.0009
13		3003.1 ± 0.8			3003.1 ± 0.8		
14	2939.0 ± 1.2		3614.0 ± 1.2	2312.0 ± 1.2[d]	3276.5 ± 0.3	112.5 ± 0.3	0.0343 ± 0.0001
15		3542.1 ± 0.9			3542.1 ± 0.9		
16	3031 ± 11		3414 ± 41[e]	12790 ± 10[d]	3223 ± 20	63.9 ± 7.1	0.0198 ± 0.0022
17		3542.0 ± 1.0			3542.0 ± 1.0		
18	3522 ± 3		4144 ± 6	1042 ± 3[f]	3833.3 ± 2.4	103.7 ± 1.0	0.0270 ± 0.0003
19		4020.6 ± 1.4			4020.6 ± 1.4		
20	4033 ± 11		4374 ± 60[g]	10280 ± 10[h]	4203 ± 30	56.8 ± 10.2	0.0135 ± 0.0024
21		4327 ± 8			4327 ± 8		
22	4285 ± 5		4901 ± 36[i,j]	660 ± 4[k]	4582 ± 18	102.2 ± 6.0	0.0223 ± 0.0013
23		4838.4 ± 2.2			4838.4 ± 2.2		
24	4780 ± 9		5114 ± 45[l,j]	9512 ± 8[m]	4947 ± 22	52.7 ± 7.5	0.0107 ± 0.0015
25		5041 ± 6			5041 ± 6		
26	5015 ± 4		5623 ± 12[a,n]	229 ± 3[k]	5319 ± 6	101.3 ± 2.1	0.0190 ± 0.0004
27		5592.6 ± 1.8			5592.6 ± 1.8		
28	5472 ± 12		5818 ± 61[o,j]	9324 ± 11[m]	5645 ± 30	57.7 ± 10.4	0.0102 ± 0.0018
29		5730 ± 6			5730 ± 6		
30	5708 ± 9		6222 ± 18[a,n]	684 ± 3[k]	5965 ± 10	85.7 ± 3.3	0.0144 ± 0.0006
31		6227 ± 11			6227 ± 11		
32	6078 ± 7		6778 ± 300[k]	7006 ± 7[m,p]	6426 ± 150	116.0 ± 50.0	0.0181 ± 0.0078
33		6351 ± 12			6351 ± 12		
34	6265 ± 6		6862 ± 26[n]	−143 ± 2[k]	6563 ± 13	99.5 ± 4.5	0.0152 ± 0.0007
35		6746 ± 16			6746 ± 16		
36							
37		6923 ± 45			6923 ± 45		
38	6836 ± 12		7391 ± 52[j]	121 ± 7[k,q]	7114 ± 25	92.6 ± 9.0	0.0130 ± 0.0013
39		7285 ± 23			7285 ± 23		
40	7123 ± 7		7597 ± 60[r]	7655 ± 6[m,p]	7360 ± 30	79.0 ± 10.1	0.0107 ± 0.0014
41		7277 ± 11			7277 ± 11		
42	7215.7 ± 2.3[s]		7766 ± 8[n]	−600 approx.[t]	7491 ± 4	91.7 ± 1.4	0.0122 ± 0.0002
43		7612 ± 150			7612 ± 150		
44							
45							
46	7838.9 ± 2.2[s]			(−550 ± 20)[u]			
47							
48							
49							
50	8415.8 ± 2.6[s]			−100 approx.[f]			
51							
52							
53							
54	9033 ± 5			−180 approx.[f]			
55							
56							
57							
58	9552 ± 6[v]			202 ± 2[v]			

[a] J. H. E. Mattauch, W. Thiele, and A. H. Wapstra, Nucl. Phys. **67**, 1 (1965); **67**, 32 (1965).
[b] J. L. Honsaker, Bull. Am. Phys. Soc. **9**, 627 (1964); W. Whaling, *ibid.* **9**, 627 (1964); H. Gulyamov, B. V. Rybakov, and V. A. Sidorov, Zh. Eksperim. i Teor. Fiz. **44**, 1829 (1963) [English transl.: Soviet Phys.—JETP **17**, 1230 (1963)].
[c] T. Lauritsen and F. Ajzenberg-Selove, Nucl. Phys. **78**, 1 (1966).
[d] F. Ajzenberg-Selove and T. Lauritsen, Nucl. Phys. **11**, 1 (1959); T. Lauritsen and F. Ajzenberg-Selove, in *Nuclear Data Sheets*, compiled by K. Way *et al.* (National Academy of Sciences—National Research Council, Washington 25 D. C.), NRC 61-5, 6-133, 185 and 229.
[e] C. D. Zafiratos, F. Ajzenberg-Selove, and F. S. Dietrich, Phys. Rev. **137**, B1479 (1965).
[f] *Nuclear Data Sheets*, compiled by K. Way *et al.* (National Academy of Sciences—National Research Council, Washington 25, D. C.), NRC 61-1-26, 60-6-13, and 61-3-52.
[g] P. F. Donovan and P. D. Parker, Phys. Rev. Letters **14**, 147 (1965).
[h] R. D. MacFarlane and A. Siivola, Nucl. Phys. **59**, 168 (1964).
[i] J. Jänecke and H. W. Fulbright (unpublished).
[j] M. E. Rickey (private communication).
[k] P. M. Endt and C. Van der Leun, Nucl. Phys. **34**, 1 (1962).
[l] A. Scott and R. Polichar, Bull. Am. Phys. Soc. **8**, 85 (1963); E. L. Haase, H. A. Hill, and D. B. Knudsen, Phys. Letters **4**, 338 (1963); S. D. Bloom L. G. Mann, R. Polichar, J. R. Richardson, and A. Scott, Phys. Rev. **134**, B481 (1964).
[m] M. Rickey, E. Kashy, and D. Knudsen, Bull. Am. Phys. Soc. **10**, 550 (1965).
[n] R. G. Miller and R. W. Kavanagh, Bull. Am. Phys. Soc. **10**, 10 (1965).
[o] M. E. Rickey and R. G. Matlock, Bull. Am. Phys. Soc. **10**, 462 (1965).
[p] W. L. Anderson, R. L. Hutson, and J. J. Kraushaar, Bull. Am. Phys. Soc. **10**, 424 (1965).
[q] J. Jänecke, Nucl. Phys. **48**, 129 (1963).
[r] M. E. Rickey, P. D. Kunz, J. J. Kraushaar, and W. G. Anderson, Phys. Letters **17**, 296 (1965).
[s] J. M. Freeman, G. Murray, and W. E. Burcham, Phys. Letters **17**, 317 (1965).
[t] See J. W. Nelson, J. D. Oberholtzer, and H. S. Plendl, Nucl. Phys. **62**, 434 (1965) and references cited therein. The quoted excitation energies for the lowest level with $T=0$ in ^{42}Sc range from 526 keV to 672 keV.
[u] J. Jänecke, Phys. Letters **6**, 69 (1963).
[v] M. Harchol, A. A. Jaffe, Ch. Drory, and J. Zioni, Phys. Letters **20**, 303 (1966).

$$E_{\mathrm{Coul}}^{(2)}(A,T)=\frac{5}{T(T+1)(2T-1)(2T+1)(2T+3)}$$
$$\times\sum_{T_z=-T}^{+T}[3T_z^2-T(T+1)]E_{\mathrm{Coul}}(A,T,T_z).\quad(6)$$

For $T>1$ these equations are not unique because, in principle, any combination of 3 or more members of a multiplet can be used to extract $E_{\mathrm{Coul}}^{(0)}$, $E_{\mathrm{Coul}}^{(1)}$, and $E_{\mathrm{Coul}}^{(2)}$. For $T=\frac{1}{2}$ and $T=1$, Eqs. (4), (5), and (6) become

$$E_{\mathrm{Coul}}^{(0)}(A,\tfrac{1}{2})=\tfrac{1}{2}[E_{\mathrm{Coul}}(A,\tfrac{1}{2},-\tfrac{1}{2})+E_{\mathrm{Coul}}(A,\tfrac{1}{2},+\tfrac{1}{2})],\quad(7)$$

$$E_{\mathrm{Coul}}^{(1)}(A,\tfrac{1}{2})=E_{\mathrm{Coul}}(A,\tfrac{1}{2},-\tfrac{1}{2})-E_{\mathrm{Coul}}(A,\tfrac{1}{2},+\tfrac{1}{2}),\quad(8)$$

and

$$E_{\mathrm{Coul}}^{(0)}(A,1)=\tfrac{1}{3}[E_{\mathrm{Coul}}(A,1,-1)+E_{\mathrm{Coul}}(A,1,0)+E_{\mathrm{Coul}}(A,1,+1)],\quad(9)$$

$$E_{\mathrm{Coul}}^{(1)}(A,1)=\tfrac{1}{2}[E_{\mathrm{Coul}}(A,1,-1)-E_{\mathrm{Coul}}(A,1,+1)],\quad(10)$$

$$E_{\mathrm{Coul}}^{(2)}(A,1)=\tfrac{1}{6}[E_{\mathrm{Coul}}(A,1,-1)-2E_{\mathrm{Coul}}(A,1,0)+E_{\mathrm{Coul}}(A,1,+1)].\quad(11)$$

Equations (8), (10), and (11) can be rewritten as

$$E_{\mathrm{Coul}}^{(1)}(A,\tfrac{1}{2})=\Delta E_{\mathrm{Coul}}(A,\tfrac{1}{2},-\tfrac{1}{2}|+\tfrac{1}{2}),\quad(12)$$

$$E_{\mathrm{Coul}}^{(1)}(A,1)=\tfrac{1}{2}[\Delta E_{\mathrm{Coul}}(A,1,-1|0)+\Delta E_{\mathrm{Coul}}(A,1,0|+1)],\quad(13)$$

$$E_{\mathrm{Coul}}^{(2)}(A,1)=\tfrac{1}{6}[\Delta E_{\mathrm{Coul}}(A,1,-1|0)-\Delta E_{\mathrm{Coul}}(A,1,0|+1)],\quad(14)$$

where

$$\Delta E_{\mathrm{Coul}}(A,T,T_z|T_z')\equiv E_{\mathrm{Coul}}(A,T,T_z)-E_{\mathrm{Coul}}(A,T,T_z')$$

is the Coulomb energy difference between neighboring isobars.

These expressions immediately make it possible to calculate $E_{\mathrm{Coul}}^{(1)}$ and $E_{\mathrm{Coul}}^{(2)}$ from the experimental Coulomb energy differences given in Table I, columns 2, 3, and 4. The corresponding values for $E_{\mathrm{Coul}}^{(1)}$ and $E_{\mathrm{Coul}}^{(2)}$ and also $E_{\mathrm{Coul}}^{(2)}/E_{\mathrm{Coul}}^{(1)}$ are shown in Table I in columns 6, 7, and 8.

In Figs. 1 and 2 the quantity $E_{\mathrm{Coul}}^{(1)}$ is plotted as a function of A and as a function of $A^{2/3}$. In Figs. 3 and 4 the quantity $E_{\mathrm{Coul}}^{(2)}$ is plotted as a function of A and as a function of $A^{-1/3}$. In Fig. 5 the quantity $AE_{\mathrm{Coul}}^{(2)}/E_{\mathrm{Coul}}^{(1)}$ is plotted as function of A. The curves exhibit oscillations and discontinuities at closed shells. The significance of the lines shown in these figures will be discussed in the following sections.

Vector Coulomb energies $E_{\mathrm{Coul}}^{(1)}$ and tensor Coulomb energies $E_{\mathrm{Coul}}^{(2)}$ have also been calculated for multiplets

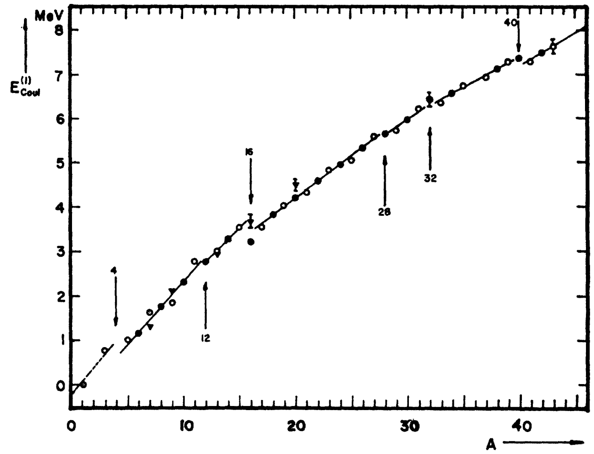

FIG. 1. Vector Coulomb energies $E_{\mathrm{Coul}}^{(1)}$ as a function of A. The filled and open circles correspond to the isobaric triplets and doublets, respectively. Major shells and subshells are indicated. The significance of the lines is discussed in Sec. IVC. The few experimental data for $T=\frac{3}{2}$ (indicated by ∇) and $T=2$ (indicated by $\blacktriangledown$) are also shown and discussed in Sec. IVE.

TABLE II. Vector and tensor Coulomb energies $E_{\mathrm{Coul}}^{(1)}$ and $E_{\mathrm{Coul}}^{(2)}$ for ground-state isobaric multiplets with $T>1$. References are given in column 5.

A	T	$E_{\mathrm{Coul}}^{(1)}$	$E_{\mathrm{Coul}}^{(2)}$	Ref.
7	$\frac{3}{2}$	1304±65		a
9	$\frac{3}{2}$	2128±17	94.6±4.5	b, c, d, e
13	$\frac{3}{2}$	2965±13	85.6±2.8	f, g
16	2	3684±140	111 ±29	h
20	2	4484±140	120 ±29	h

[a] C. Detraz, J. Cerny, and R. H. Pehl, Phys. Rev. Letters 14, 708 (1965).
[b] R. Middleton and D. J. Pullen, Nucl. Phys. 51, 50 (1964).
[c] T. Lauritsen, B. Lynch, and G. Griffiths, Bull. Am. Phys. Soc. 8, 597 (1963).
[d] F. S. Dietrich and J. W. Davies, Bull. Am. Phys. Soc. 8, 598 (1963).
[e] J. Cerny, R. H. Pehl, F. S. Goulding, and D. A. Landis, Phys. Rev. Letters 13, 726 (1964).
[f] D. C. Henley and C. A. Barnes, Bull. Am. Phys. Soc. 10, 1194 (1965).
[g] E. Adelberger and C. A. Barnes, Bull. Am. Phys. Soc. 10, 1195 (1965).
[h] J. Cerny, R. H. Pehl, and G. T. Garvey, Phys. Letters 12, 234 (1964).

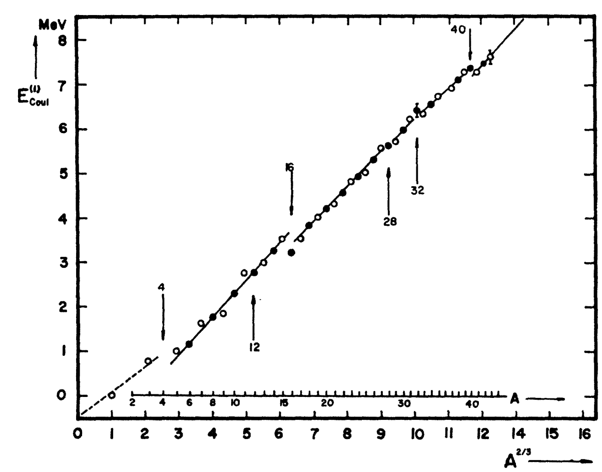

FIG. 2. Vector Coulomb energies $E_{\mathrm{Coul}}^{(1)}$ as a function of $A^{2/3}$. The filled and open circles correspond to the isobaric triplets and doublets, respectively. Major shells and subshells are indicated. The significance of the lines is discussed in Secs. IVB and IVC.

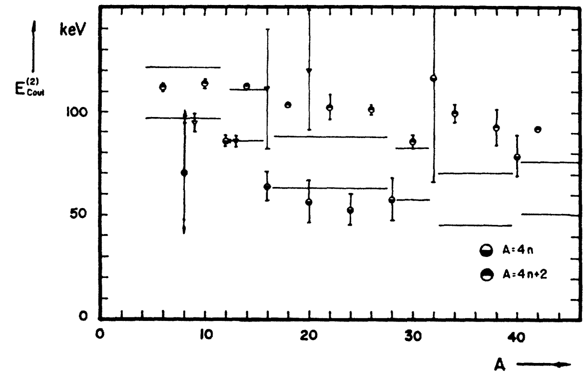

FIG. 3. Tensor Coulomb energies $E_{\mathrm{Coul}}^{(2)}$ as a function of A. The experimental points for the nuclei with $A=4n$ and $A=4n+2$ are indicated. The significance of the lines is discussed in Sec. IV C. A few experimental data for $T=\frac{3}{2}$ (indicated by $\triangledown$) and $T=2$ (indicated by $\blacktriangledown$) are also shown and discussed in Sec. IVE.

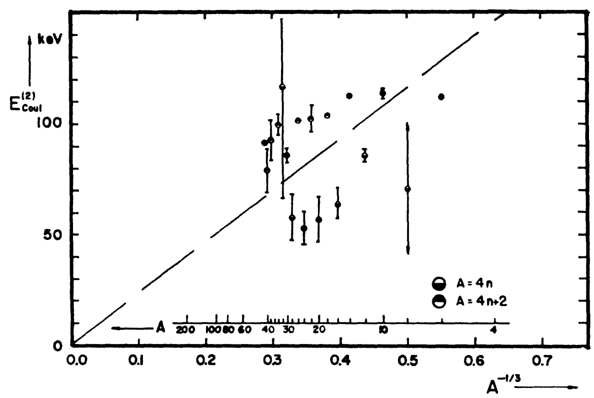

FIG. 4. Tensor Coulomb energies $E_{\mathrm{Coul}}^{(2)}$ as a function of $A^{-1/3}$. The experimental points for the nuclei with $A=4n$ and $A=4n+2$ are indicated. The significance of the line is discussed in Sec. IV B.

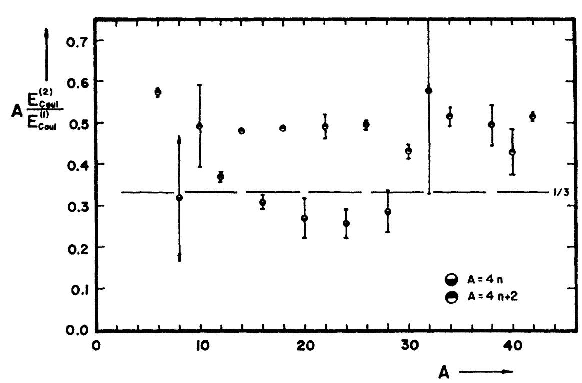

FIG. 5. Plot of A times the ratio $E_{\mathrm{Coul}}^{(2)}/E_{\mathrm{Coul}}^{(1)}$ as a function of A. The significance of the horizontal line is discussed in Sec. IV B.

with $T>1$ whenever two or more Coulomb energy differences of the particular multiplet are known experimentally. The values are shown in Table II in columns 3 and 4.

IV. DISCUSSION

A. General Remarks

Since we are concerned primarily with energy differences, the scalar Coulomb energy $E_{\mathrm{Coul}}^{(0)}(A,T)$ which represents an average Coulomb energy for a given multiplet is of secondary interest only. Moreover, $E_{\mathrm{Coul}}^{(0)}(A,T)$ cannot be separated from the contributions from nuclear forces because both should exhibit the same A and T dependence. We shall, therefore, restrict our discussion to the vector and tensor Coulomb energies.

Certain properties of the Coulomb energy differences between isobaric doublets and triplets have been discussed before in the literature.[6–10] A list of additional references is given in Ref. 8.

The masses of the members of an isobaric multiplet are given by

$$M(A,T,T_z)=M_0(A,T)+E_{\mathrm{Coul}}(A,T,T_z)+T_z\Delta m\,, \quad (15)$$

where $\Delta m=0.782$ MeV is the $n-\mathrm{H}$ mass difference. The isobaric mass formula[2–4]

$$M(A,T,T_z)=\alpha(A,T)+\beta(A,T)T_z+\gamma(A,T)T_z^2 \quad (16)$$

is obtained by combining Eqs. (3) and (15). The coefficients in this formula are therefore related to the scalar, vector, and tensor Coulomb energies by

$$\alpha(A,T)=M_0(A,T)+E_{\mathrm{Coul}}^{(0)}(A,T) -T(T+1)E_{\mathrm{Coul}}^{(2)}(A,T)\,, \quad (17)$$

$$\beta(A,T)=\Delta m-E_{\mathrm{Coul}}^{(1)}(A,T)\,, \quad (18)$$

$$\gamma(A,T)=3E_{\mathrm{Coul}}^{(2)}(A,T)\,. \quad (19)$$

B. Comparison with a Homogeneous Charge Distribution

As an extreme simplification one may consider the atomic nucleus as a homogeneously charged sphere of radius $R=r_0A^{1/3}$. On this assumption one obtains immediately the following expressions:

$$E_{\mathrm{Coul}}=\frac{3}{5}\frac{e^2}{r_0}\frac{Z^2}{A^{1/3}}\,, \quad (20)$$

$$\Delta E_{\mathrm{Coul}}=\frac{6}{5}\frac{e^2}{r_0}\frac{\bar{Z}}{A^{1/3}}\,, \quad (21)$$

$$E_{\mathrm{Coul}}^{(1)}=\frac{3}{5}\frac{e^2}{r_0}A^{2/3}\,, \quad (22)$$

$$E_{\mathrm{Coul}}^{(2)}=\frac{1}{5}\frac{e^2}{r_0}\frac{1}{A^{1/3}}\,, \quad (23)$$

$$\frac{E_{\mathrm{Coul}}^{(2)}}{E_{\mathrm{Coul}}^{(1)}}=\frac{1}{3A}\,. \quad (24)$$

[9] N. V. V. J. Swamy, V. K. Kembhavi, and D. G. Galgali, Phys. Rev. **120**, 2069 (1960).
[10] S. Sengupta, Nucl. Phys. **30**, 300 (1962).

Here, E_{Coul} and ΔE_{Coul} are the Coulomb energy and Coulomb energy difference between neighboring isobars, and $E_{\text{Coul}}^{(1)}$ and $E_{\text{Coul}}^{(2)}$ are the vector and tensor Coulomb energies. The quantity $\bar{Z}$ in Eq. (21) is $\bar{Z}=\frac{1}{2}(Z_1+Z_2)$. The equations have to be slightly modified when E_{Coul} is taken as proportional to $Z(Z-1)$, or when an additive Coulomb self-energy or exchange term proportional to Z is taken into account. A $Z(Z-1)$ dependence of the Coulomb energy is obtained directly from Eq. (1) with the assumption $r_{ij}=\text{const}$.

From Eqs. (22) and (23) it follows that both the vector and tensor Coulomb energy should be T-independent, provided one can consider the atomic nucleus as a homogeneously charged sphere (the same conclusion holds also under similar simplifying assumptions). The experimental points in Figs. 2, 4, and 5 show that the above assumptions can indeed by used as a first but very crude approximation. The quantity $E_{\text{Coul}}^{(1)}$ *is* roughly proportional to $A^{2/3}$, $E_{\text{Coul}}^{(2)}$ *is* very roughly proportional to $A^{-1/3}$, and $AE_{\text{Coul}}^{(2)}/E_{\text{Coul}}$ *is* approximately constant and roughly equal to $\frac{1}{3}$ in accordance with Eqs. (22), (23), and (24).

C. Comparison with the Coulomb Energy Formula of Carlson and Talmi

A more refined analysis has to take account of the structure of the nuclei under consideration, which apparently leads to the breaks in the vector Coulomb energy at closed shells, to the superimposed oscillations, and also to the pronounced oscillatory behavior of the tensor Coulomb energy.

Based on an effective two-body interaction Carlson and Talmi[11] have derived an expression for the Coulomb energy of nuclei with Z' protons outside closed shells.

$$E_{\text{Coul}}=Z'C+\tfrac{1}{2}Z'(Z'-1)a+[\tfrac{1}{2}Z']b. \qquad (25)$$

Here, $[\frac{1}{2}Z']$ denotes the largest integer less or equal to $\frac{1}{2}Z'$. The quantities a and b are related to the electrostatic interaction of two protons in the j shell, and C represents the electrostatic interaction of a single proton with the core. Equation (25) holds approximately assuming there are *only protons* in the j shell, but it is probably not too bad for other cases, e.g., when the configurations are more complicated than j^n or when one has protons *and* neutrons in the same shell. The pairing term $[\frac{1}{2}Z']b$, however, has to be modified for the latter case to comply with Eq. (3). In excited isobaric analog states there is no independent pairing of protons and neutrons. Instead, the Coulomb pairing energy depends on the probability that in the states of good isobaric spin the members of a pair are protons.

Equation (3) or (16) suggest that $[\frac{1}{2}Z']b$ has to be replaced by a term which is quadratic in T_z (which is then quadratic in $Z=\frac{1}{2}A-T_z$ and also quadratic in $Z'=Z-Z_0$), i.e.,

$$[\tfrac{1}{2}Z']\rightarrow\lambda+\mu T_z+\nu T_z^2=\{\mu-\nu(A-2Z_0)\}Z'+\nu Z'^2. \qquad (26)$$

This equation can indeed be verified and explicit expressions for the coefficients μ and ν can be given in certain models. One possibility, for instance, is to adopt an independent-particle picture, assume that the states under consideration are the energetically lowest states possible for a given A and T, and also that each single-particle level is fourfold degenerate (at most two protons and two neutrons in one level). Thus, one has Nilsson-like or more general orbits in which the two proton and the two neutron single-particle states are related by the operation of time reversal. On this assumption the isobaric spin wave function for any A, T, and $T_z=\pm T$ can easily be expressed as a product of wave functions for each of the Nilsson-like orbits. By applying the step-up or step-down operator the relevant wave functions for all other $T_z\neq\pm T$ can be constructed. An inspection of these wave functions immediately gives the probability for the number of proton pairs in the *same* orbit which are expected to give an additional and approximately equal contribution to the Coulomb energy. These numbers indeed depend quadratically on T_z, and one obtains for the coefficients μ and ν in Eq. (26)

$$\mu=\begin{cases}\dfrac{1}{2} & \text{for even-}A\text{ nuclei,}\\[2ex] \dfrac{1}{2}\left(1-\dfrac{1}{2T}(-1)^{A/2-T}\right) & \text{for odd-}A\text{ nuclei,}\end{cases} \qquad (27)$$

and

$$\nu=\begin{cases}\dfrac{1}{4T}\left(1+\dfrac{1}{2T-1}(-1)^{A/2-T}\right) & \text{for even-}A\text{ nuclei,}\\[2ex] \dfrac{1}{4T} & \text{for odd-}A\text{ nuclei, } T>\tfrac{1}{2}.\end{cases} \qquad (28)$$

The above expressions for μ and ν are in agreement with corresponding expressions given by Wilkinson[4] for the p shell except for ν in even-A nuclei. Some explicit values for μ and ν are listed in Table III.

If it is assumed that the states under consideration have lowest seniority and that they can be described in terms of an extreme shell model, then one can in principle apply a similar procedure. One has to extract the probability for the number of $J=0$ coupled proton pairs which in this case are expected to give the additional small contribution to the Coulomb energy. The calculations may lead to expressions[12] similar to Eqs. (27) and (28).

[11] B. C. Carlson and I. Talmi, Phys. Rev. **96**, 436 (1954); A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic Press Inc., New York, 1963), p. 345.

[12] K. T. Hecht (private communication); Conference on Isobaric Spin in Nuclear Physics, Tallahassee, 1966 (unpublished).

TABLE III. Numerical values for the coefficients μ and ν in Eq. (26).

	μ				ν			
	$A=4n$	$A=4n+1$	$A=4n+2$	$A=4n+3$	$A=4n$	$A=4n+1$	$A=4n+2$	$A=4n+3$
$T=1/2$		0		1		0		0
$T=1$	1/2		1/2		0		1/2	
$T=3/2$		2/3		1/3		1/6		1/6
$T=2$	1/2		1/2		1/6		1/12	
$T=5/2$		2/5		3/5		1/10		1/10
$T=3$	1/2		1/2		1/15		1/10	
$T=7/2$		4/7		3/7		1/14		1/14
$T=4$	1/2		1/2		1/14		3/56	

The modified expression for the pairing energy of Eq. (26) together with Eq. (25) gives

$$E_{\text{Coul}}=Z'C+\tfrac{1}{2}Z'(Z'-1)a+[(\mu-\nu(A-2Z_0)Z'+\nu Z'^2]b \quad (29)$$

for the Coulomb energy and

$$\Delta E_{\text{Coul}}=a\bar{Z}+[C-\tfrac{1}{2}a(2Z_0+1)]+(\mu-2\nu\bar{T}_z)b \quad (30)$$

for the Coulomb energy difference between the analog states of neighboring isobars. Here, $\bar{Z}=\frac{1}{2}(Z_1+Z_2)$ and $\bar{T}_z=\frac{1}{2}(T_{z1}+T_{z2})=\frac{1}{2}A-\bar{Z}$. For convenience Eq. (30) will be written as

$$\Delta E_{\text{Coul}}=E_1\bar{Z}+E_2+(\mu-2\nu\bar{T}_z)E_3. \quad (31)$$

The energies E_1, E_2, and E_3 are defined in an obvious way. They generally vary from shell to shell. From Eqs. (30) or (31) it follows that the Coulomb energy differences ΔE_{Coul} are expected to exhibit superimposed oscillations as a function of $\bar{Z}$ (or A) for a constant $\bar{T}_z$, i.e., for nuclei with a constant neutron excess, due to the Coulomb pairing energy E_3 (see also Ref. 13). The amplitude of these oscillations is expected to be equal to $\frac{1}{2}E_3$ for the mirror nuclei with $T=\frac{1}{2}$ and $\bar{T}_z=0$ provided a description in terms of Nilsson-like orbits is applicable to these nuclei. On the other hand, the amplitude should be only $\frac{1}{4}E_3$ for the isobaric triplets with $T=1$ and both $\bar{T}_z=+\frac{1}{2}$ and $\bar{T}_z=-\frac{1}{2}$ (the peak-to-peak values are E_3 and $\frac{1}{2}E_3$, respectively). The experimental Coulomb energy differences (not plotted; see Table I, columns 2, 3, and 4 or Ref. 8, Fig. 1) indeed show these oscillations with a marked difference in amplitude for the isobaric triplets and doublets. This effect has been noticed before,[8] but was not explained.

TABLE IV. Numerical values for the quantities E_1, E_2, and E_3 in Eq. (35) derived from the experimental vector Coulomb energies shown in Fig. 1.

Shell	E_1 (keV)	E_2 (keV)	E_3 (keV)
$1s_{1/2}$	(614)	(−233)	(150)
$1p_{3/2}$	577	−564	(342)
$1p_{1/2}$	515	−333	(24)
$1d_{5/2}$	376	+424	136
$2s_{1/2}$	347	+762	(150)
$1d_{3/2}$	273	+1906	106
$1f_{7/2}$	303	+1098	(66)

Using Eqs. (31), (5), and (6) one obtains for the vector and tensor Coulomb energies and for their ratio

$$E_{\text{Coul}}^{(1)}=\tfrac{1}{2}E_1A+E_2+\mu E_3, \quad (32)$$

$$E_{\text{Coul}}^{(2)}=\tfrac{1}{6}(E_1+2\nu E_3), \quad (33)$$

$$\frac{E_{\text{Coul}}^{(2)}}{E_{\text{Coul}}^{(1)}}=\frac{1+2\nu E_3/E_1}{3A+6(E_2+\mu E_3)/E_1}. \quad (34)$$

For $T=\frac{1}{2}$ and $T=1$, in particular, one has

$$E_{\text{Coul}}^{(1)}=\tfrac{1}{2}E_1A+E_2+\tfrac{1}{2}E_3+\begin{cases}(-1)^{(A+1)/2}E_3/2 & \text{for isobaric doublets}\\ 0 & \text{for isobaric triplets,}\end{cases} \quad (35)$$

$$E_{\text{Coul}}^{(2)}=\tfrac{1}{6}[E_1+\tfrac{1}{2}(1-(-1)^{A/2})E_3] \quad \text{for isobaric triplets,} \quad (36)$$

$$\frac{E_{\text{Coul}}^{(2)}}{E_{\text{Coul}}^{(1)}}=\frac{1+\frac{1}{2}[1-(-1)^{A/2}]E_3/E_1}{3A+6(E_2+\frac{1}{2}E_3)/E_1} \quad \text{for isobaric triplets.} \quad (37)$$

The vector Coulomb energies for the isobaric triplets are shown in Fig. 1 as a function of A as filled circles. A set of straight lines is shown drawn through the experimental points. Discontinuities appear at the major shells at $A=4$, 16, and 40, and less pronounced breaks appear at the subshells at $A=12$, 28, and 32. The vector Coulomb energies for the isobaric doublets (open circles) oscillate around the straight lines. For $A=4n+1$ the points always lie lower ($A=5$ is the only exception); for $A=4n+3$ the points lie higher. The above behavior is basically in accordance with Eq. (35).

When considering the deviations, one has to allow for the fact that for $A=5$, 9, and 16, at least one member of the isobaric doublet or triplet is unstable with regard to the emission of a nucleon which may lead to

[13] E. Feenberg and G. Goertzel, Phys. Rev. **70**, 597 (1946); B. C. Carlson and I. Talmi, *ibid.* **96**, 436 (1954); O. Kofoed-Hansen, Rev. Mod. Phys. **30**, 449 (1958); I. Unna, Nucl. Phys. **8**, 1243 (1959).

a Thomas-Ehrman shift.[14] Also, particularly for $A=16$ and 40, the configurations of the states under consideration (the 0^- states for $A=16$ and the 4^- states for $A=40$) are more complicated and involve excitations out of the preceding closed major shell. Based on simpleminded considerations one would expect that the vector Coulomb energy for these triplets is equal to the average vector Coulomb energy of the neighboring doublets, i.e., $E_{\text{Coul}}^{(1)}(A_0,1)=\frac{1}{2}[E_{\text{Coul}}^{(1)}(A_0-1,\frac{1}{2})+E_{\text{Coul}}^{(1)}(A_0+1,\frac{1}{2})]$. This appears not to be the case.

Values for the quantities E_1, E_2, and E_3 for the various shells were derived from the slope, from the intersection with the ordinate and from the amplitude of the oscillations of the vector Coulomb energies shown in Fig. 1. They are given in Table IV. The average Coulomb pairing energy E_3 is about 149 keV. Including only the data for $A\geq 17$ an average value of about 122 keV is obtained.

The tensor Coulomb energies are plotted in Fig. 3 as a function of A. Horizontal lines corresponding to Eq. (36) are shown. The respective energies E_1 for the various shells were taken from the preceding analysis of the vector Coulomb energies. For E_3 the average value of 149 keV was used. The horizontal lines exhibit discontinuities at all major shells and subshells. According to Eq. (36) the experimental tensor Coulomb energies for the nuclei with $A=4n+2$ should lie on the upper curve, and for $A=4n$ they should lie on the lower curve. The experimental points follow essentially the general trend of the calculated lines and they show the expected oscillations. Figure 3 will be discussed below in more detail.

The experimental values for the quantity $AE_{\text{Coul}}^{(2)}/E_{\text{Coul}}^{(1)}$ are plotted in Fig. 5 as a function of A. Most values are somewhat bigger than the calculated value $\frac{1}{3}$ for a homogeneously charged sphere [see Eq. (24)]. There is at least qualitative agreement, however, with Eq. (37) which gives for the above quantity an average value $>\frac{1}{3}$, a weak A dependence, and superimposed oscillations.

The preceding discussion has shown that the experimental Coulomb energies are in good agreement with Eq. (35). Numerical values for the quantities E_1, E_2, and E_3 or C, a, and b can be extracted from the vector energies. Using these quantities one can then predict the tensor Coulomb energy. Figure 3 shows that the experimental points exhibit small deviations from the calculated lines. These deviations possibly result from a violation of charge independence of nuclear forces[15] or other effects like the Thomas-Ehrman shift,[14] configuration mixing, or electromagnetic spin-orbit interaction. These effects *may* influence the energetic position of the $T_z=0$ members of the isobaric triplets only, i.e., the tensor energies only. The small deviations may also result from the fact that the theoretical equation (29) underlying the above considerations is based on too restrictive assumptions.

Numerical values for the quantity $\Delta=E_{T_z=0}{}^{T=1}(\text{calc})-E_{T_z=0}{}^{T=1}(\text{exp})$ were obtained by several authors[6-8] based on different procedures. The reported values are small but finite with some preference for positive values. From the tensor Coulomb energies considered in the present work one can also extract values for Δ which are of the order of 50 keV and are also mostly positive. Such a "shift" is compatible with a departure from charge independence of nuclear forces by not more than a few percent. More quantitative statements concerning charge-dependent forces, however, cannot be extracted from the A dependence of the tensor Coulomb energies. Other effects can easily be responsible for the relatively small effects. Most important, however, the preceding analysis has been based completely on the assumption that Eq. (29) is exact. This is not the case, and even small refinements of this equation are sufficient, as will be shown below, to explain the difference in the observed and calculated tensor energies.

The Coulomb energy differences between neighboring Sc and Ca isotopes and between Sb and Sn isotopes, for instance, are known experimentally[5] to decrease slightly with increasing neutron number. This effect appears reasonable because one might expect that the average distance between the protons is increasing with volume. Equation (31) does *not* describe the observed behavior. Since the coefficients in Eq. (31) are related to the expectation value of $1/R$ it appears reasonable to substitute $\hat{E}_i/A^{1/3}$ for all E_i with constant values for $\hat{E}_i$. The vector Coulomb energy essentially becomes a linear function of $A^{2/3}$ which, as can be seen from Fig. 2, describes the experimental data as well as Eq. (35). The revised expression for the tensor Coulomb energy, on the other hand, does not describe the experimental values any better than Eq. (36). Values for the quantity Δ extracted with the use of the above substitution become more negative by about 100 keV compared with those based on Eq. (36).

Sengupta[7] has given a semiclassical expression for the Coulomb energy

$$E_{\text{Coul}}=(e^2/r_0A^{1/3})\times\{0.6Z^2-0.46Z^{4/3}-[1-(-1)^Z]0.15\}\,, \quad (38)$$

$$\Delta E_{\text{Coul}}=(e^2/r_0A^{1/3})\times\{0.6(2Z+1)-0.613Z^{1/3}-(-1)^Z 0.30\}\,. \quad (39)$$

The second and third terms (exchange and pairing energy) in these equations are not compatible with Eqs. (3) or (16). With the approximation $(Z/A)^{1/3}\approx\text{const}$, Eq. (39) is basically in agreement with the semi-

[14] R. G. Thomas, Phys. Rev. **81**, 148 (1951); J. B. Ehrman, *ibid.* **81**, 412 (1951); R. G. Thomas, *ibid.* **88**, 1109 (1952); A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 329 (1958).

[15] R. J. Blin-Stoyle and J. Le Tourneux, Phys. Rev. **123**, 627 (1961); R. J. Blin-Stoyle, in *Selected Topics in Nuclear Spectroscopy*, edited by B. J. Verhaar (North-Holland Publishing Company, Amsterdam, 1964), p. 213.

empirical equation[8]

$$\Delta E_{\mathrm{Coul}}=\bar{E}_1(\bar{Z}/A^{1/3})+\bar{E}_2+(\mu-2\nu\bar{T}_z)\bar{E}_3. \tag{40}$$

The above results have shown that even a small modification of the basic equation (29) leads to a considerable change in the apparent shift Δ in the energetic position of the $T_z=0$ member of the isobaric triplets. Therefore, all numerical values for Δ extracted under the assumption of a certain A dependence of the Coulomb energy must be considered with caution. In conclusion, the previous discussion has established that Eq. (29) works remarkably well in describing the experimental vector Coulomb energies despite the approximations used in the derivation, and that even the experimental and calculated tensor Coulomb energies are basically in agreement.

D. Preliminary Comparison with the Coulomb Energy Formulas of Hecht

A detailed study of the A and T dependence of the vector and tensor Coulomb energies is presently being carried out by Hecht.[12] The calculations are based on the seniority scheme in jj coupling with the use of the five-dimensional quasispin formalism.[16] His preliminary results indicate that even for states with j^n configurations of lowest seniority the specific A and T dependence of both, $E_{\mathrm{Coul}}^{(1)}$ and $E_{\mathrm{Coul}}^{(2)}$, is more complex than given by the approximate Eqs. (32) and (33). This fact supports the previous statement concerning the difficulty of obtaining quantitative information on a departure from charge independence of nuclear forces from the A dependence of the Coulomb energies.

[16] K. T. Hecht, Phys. Rev. **139**, B794 (1965).

E. Isobaric Multiplets with $T>1$

The very limited information on $E_{\mathrm{Coul}}^{(1)}$ and $E_{\mathrm{Coul}}^{(2)}$ for $T>1$ is given in Table II and included in Figs. 1 and 3. According to Eq. (32) the experimental values of $E_{\mathrm{Coul}}^{(1)}$ should essentially follow the lines indicated in Fig. 1. For $T=\frac{3}{2}$ there should be superimposed oscillations with an amplitude which is smaller than that for $T=\frac{1}{2}$. The oscillations for $T=\frac{1}{2}$ and $T=\frac{3}{2}$ should be out of phase. The three experimental points in the p shell indeed give some indication of these effects. For $T=2$ the experimental points[17] should lie directly on the lines. This is not the case. Both points are about 300 keV too high. As mentioned before, the case $A=16$ is not too conclusive because of the configurations of the states involved, while the case $A=20$ seems to establish a discrepancy. According to Eq. (33) the experimental values for $E_{\mathrm{Coul}}^{(2)}$ should lie about 8 keV above the respective lower line indicated in Fig. 3. The experimental points are essentially in agreement with this expectation. More experimental information is needed to draw more definite conclusions about the isobaric multiplets with $T>1$.

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[17] G. T. Garvey, J. Cerny, and R. H. Pehl, Phys. Rev. Letters **13**, 548 (1964); J. Cerny, R. H. Pehl, and G. T. Garvey, Phys. Letters **12**, 234 (1964).