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Three-Alpha Model for C^{12}

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A model is proposed in which the C^{12} nucleus is composed of three rigid alpha particles. Using separable alpha-alpha interactions the homogeneous Faddeev equations for the bound states reduce to a set of coupled nonsingular integral equations in one variable. Specialized to the case of only S-wave interactions, this set reduces to a single equation which is solved numerically. A ground-state solution is found, with a reasonable value for the binding energy, but no excited states. Further calculations are suggested, and an appendix containing the general angular-momentum analysis is included.

I. INTRODUCTION

THE cluster model[1–4] has had a long history and some success in explaining the properties of nuclei. As usually applied it amounts to approximating the nuclear wave function by a product of cluster wave functions, choosing a reasonable two-nucleon potential, and performing a variational calculation. In this paper we wish to introduce a somewhat different kind of cluster model for C^{12}. We shall treat the clusters (alpha particles in this case) as rigid entities without any internal structure, interacting with each other via a potential determined by alpha-alpha scattering experiments. With this simplification, and by further choosing the potentials to have a separable form, it is possible to solve the resulting three-body problem exactly.

Our original motivation for studying this model was that it involves the simplest possible three-body system[5]: three identical spinless particles for which a nonrelativistic theory should be adequate. We hoped, of course, that in spite of our rather drastic rigidity assumption the model would have some relevance to the real C^{12} nucleus, and the preliminary results to be presented here have been encouraging. If this should continue to be true the model should be an ideal "proving-ground" for three-particle calculations. This is especially true because the C^{12} system has several excited states[6] below or near its breakup threshold where numerical calculations are relatively simple.

The foundation of the model is the set of coupled integral equations from the three-body theory which has recently been developed by Faddeev,[7] Lovelace,[8] and others[9,10] and applied by many workers.[11–15] These are introduced in Sec. II. Since the three-alpha system is so simple we hope to eventually be able to include two-particle interaction in the higher angular-momentum channels and have therefore included the general angular-momentum analysis (with the details in an appendix). In this paper, however, we present, in Sec. III, the results of numerical calculations using only S-wave alpha-alpha interactions. These are chosen to fit the experimentally determined scattering length and effective range with Coulomb effects removed. The results are discussed and a program for further calculations presented in the concluding Sec. IV.

II. INTEGRAL EQUATIONS

Faddeev[7] has shown that a well-behaved set of three-body equations, involving the two-body T matrix rather than the potential, can be obtained by rearranging the Lippmann–Schwinger equation. For a bound state of three identical spinless particles of mass m there is a single homogeneous integral equation:

$$\psi(\mathbf{k},\mathbf{q})=2[E-E(k,q)]^{-1}(2\pi)^{-3}\int d^3q' \times t(E-\omega(q);\,\mathbf{k},\,\mathbf{q}'+\tfrac{1}{2}\mathbf{q})\psi(\mathbf{q}+\tfrac{1}{2}\mathbf{q}',\,\mathbf{q}'). \quad (1)$$

The function $\psi(\mathbf{k},\mathbf{q})$ is related to Ψ, the symmetrized momentum-space wave function in the center-of-mass system, by

$$\Psi=\psi(\mathbf{k}_1,\mathbf{q}_1)+\psi(\mathbf{k}_2,\mathbf{q}_2)+\psi(\mathbf{k}_3,\mathbf{q}_3)\,, \quad (2)$$

where, for example,

$$\mathbf{k}_1=\tfrac{1}{2}(\mathbf{p}_2-\mathbf{p}_3)\,,\qquad \mathbf{q}_1=\mathbf{p}_2+\mathbf{p}_3=-\mathbf{p}_1\,, \quad (3)$$

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and the $\mathbf{p}_i$ are the individual particle momenta. The function $t(z,\mathbf{k},\mathbf{k}')$ in (1) is the off-shell two-particle T matrix, while $E<0$ is the bound-state energy and

$$E(k,q)=m^{-1}k^2+\omega(q)\,, \tag{4}$$

where

$$\omega(q)=\tfrac{3}{4}m^{-1}q^2\,. \tag{5}$$

If we approximate $t(z,\mathbf{k},\mathbf{k}')$ by a sum of separable functions[16] of the sort that would result from separable potentials,

$$t(z;\,\mathbf{k},\mathbf{k}')=\sum_\lambda(2\lambda+1)P_\lambda(\hat{k}\cdot\hat{k}')g_\lambda(k)\tau_\lambda(z)g_\lambda(k')\,, \tag{6}$$

where

$$\tau_\lambda(z)=\left[C^{-1}+(2\pi^2)^{-1}m\int_0^\infty\frac{p^2dp\,g_\lambda^2(p)}{p^2-mz}\right]^{-1}, \tag{7}$$

the integral equation greatly simplifies. Making the decomposition

$$\psi(\mathbf{k},\mathbf{q})=\sum_{JM\lambda l}\psi_{\lambda l}{}^J(k,q)\mathcal{Y}_{\lambda,l}{}^{J,M}(\hat{k},\hat{q})\,, \tag{8}$$

where

$$\mathcal{Y}_{\lambda,l}{}^{J,M}(\hat{k},\hat{q})=\sum_{\mu m}\langle\lambda l\mu m|JM\rangle Y_{\lambda,\mu}(\hat{k})Y_{l,m}(\hat{q})\,, \tag{9}$$

we can write $\psi_{\lambda l}{}^J(k,q)$ as the product

$$\psi_{\lambda l}{}^J(k,q)=[E-E(k,q)]^{-1}g_\lambda(k)\tau(E-\omega(q))f_{\lambda l}{}^J(q)\,. \tag{10}$$

The "reduced wave functions" $f_{\lambda l}{}^J(q)$ are solutions of a set of coupled equations in one variable:

$$f_{\lambda l}{}^J(q)=2\sum_{\lambda' l'}(2\pi)^{-3}\int_0^\infty q'^2dq'\,V_{\lambda l,\lambda' l'}{}^J(E;\,q,q')\times\tau_{\lambda'}(E-\omega(q'))f_{\lambda' l'}(q')\,, \tag{11}$$

where

$$V_{\lambda l,\lambda' l'}{}^J(E;\,q,q')=\int d\Omega_q d\Omega_{q'}\mathcal{Y}_{\lambda l}{}^{JM*}(\hat{k}',\hat{q})g_\lambda(k')\times[E-E(k'',q')]^{-1}g_{\lambda l}(k'')\mathcal{Y}_{\lambda' l'}{}^{JM}(\hat{k}'',\hat{q}')\,, \tag{12}$$

with

$$\mathbf{k}'=\mathbf{q}'+\tfrac{1}{2}\mathbf{q}\,,\qquad \mathbf{k}''=\mathbf{q}+\tfrac{1}{2}\mathbf{q}'\,. \tag{13}$$

In the Appendix to this paper we show that we can write

$$V_{\lambda l,\lambda' l'}{}^J(E;\,q,q')=\sum_{\mathcal{L}=0}^{J+\lambda+\lambda'}(2\mathcal{L}+1)M_{\lambda l,\lambda' l'}{}^{J,\mathcal{L}}(q,q')\times V_{\lambda\lambda'}{}^{\mathcal{L}}(E;\,q,q')\,, \tag{14}$$

where

$$V_{\lambda\lambda'}{}^{\mathcal{L}}(E;\,q,q')=\frac{1}{2}\int_{-1}^1 d(\hat{q}\cdot\hat{q}')P_{\mathcal{L}}(\hat{q}\cdot\hat{q}')(k')^{-\lambda}g_\lambda(k')\times[E-E(k'',q')]^{-1}(k'')^{-\lambda'}g_{\lambda'}(k'')\,. \tag{15}$$

The functions $M_{\lambda l,\lambda' l'}{}^{J,\mathcal{L}}(q,q')$ are the homogeneous polynomials of degree $\lambda+\lambda'$ in q and q', with coefficients which can be expressed in terms of 3-j and 6-j symbols.

Since we are looking for bound states the kernels of the integral equations will have no singularities in the domain of integration. This imples, among other things, that the $f_{\lambda l}{}^J(q)$ will be relatively smooth functions. Equation (10) therefore carries quite a bit of information about the bound-state wave functions in our model. Their asymptotic behavior in configuration space, for example, is determined by the singularities of $[E-E(k,q)]^{-1}\tau(E-\omega(q))$, and this can be shown to be consistent with the limits established by Slaggie and Wichmann.[17]

III. SOLUTION WITH S-WAVE INTERACTION

If we keep only the $\lambda=0$ terms the set of Eqs. (11) reduces to the single equation (suppressing the subscripts $\lambda=0$ and $l=J$)

$$f^J(q)=\pi^{-2}\int_0^\infty q'^2dq'\,V^J(E;\,q,q')\tau(E-\omega(q'))f^J(q')\,, \tag{16}$$

where $V^J(E;\,q,q')$ is given by (15) with $\lambda=\lambda'=0$. Taking $g(k)=(k^2+\beta^2)^{-1}$ we choose the strength parameter C and the range parameter β such that two-particle T matrix gives the experimentally determined alpha-alpha scattering length and effective range[18,19] with the Coulomb effects removed.[20] We find

$$\beta=0.736\text{ F}^{-1}\,,\qquad (8\pi\beta^3)^{-1}mC=-2.95\,. \tag{17}$$

These values produce a stable Be^8 bound state at -2.91 MeV, while the observed ground state is unstable by about 0.1 MeV. The difference is not an unreasonable value for the energy of Coulomb repulsion between two alphas, but is probably a bit large since the Hulthén wave function which results from our choice of $g(k)$ is almost certainly too large at small distances.

Equation (16) has certain features which simplify its solution. Since the "potential" $V^J(E;\,q,q')$ is a sym-

[16] This approximation should be good if the channel is dominated by a bound state or resonance. See Refs. 8 and 9.

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metric function of its two arguments and nonsingular in the domain of integration we can apply the well-developed Schmidt-Hilbert theory[21] of integral equations. Furthermore, if, as is the case with our choice, the function $g(k)$ has its nth derivative with sign $(-1)^n$, then the kernel has everywhere the sign $(-1)^J$. Roughly speaking, then, an attractive S-wave alpha-alpha force produces attraction only in the even J channels of the three-alpha system. Also, the extremum properties derived in the Schmidt-Hilbert theory require that the eigenfunction $f^J(q)$ corresponding to the lowest energy eigenvalue for each even J have no nodes.

Making use of these general features we have solved Eq. (16) numerically using three-point Gaussian integration, with abscissas and weights appropriate to the rapid convergence at infinity, to convert the integral equation into a matrix eigenvalue equation. The eigenvalues of the matrix were traced as a function of E; the bound-state energies are those values of E at which a matrix eigenvalue takes the value one. In this way we found the $J=0$ ground-state energy to be -12.8 MeV (relative to the three-alpha breakup threshold), with the reduced wave function shown in Fig. 1. Experimentally, excited states of C^{12} are found at 4.4 MeV ($J=2$) and 7.6 MeV ($J=0$) above the ground state.[6] We do not find any excited states below threshold; the behavior of the second matrix eigenvalue for $J=0$, however, seems to indicate that a bit more attraction would produce an excited state in this channel.

In comparing our calculated value for the C^{12} ground-state energy with the observed value of $M(C^{12}) = 3M(He^4) = -7.28$ MeV, we must remember to add in the energy of Coulomb repulsion among the alphas. This is difficult to calculate accurately but should be of the order of several MeV. The formula[22] for a uniform spherical charge distribution, $E_c(Z^A) = (0.584 \text{ MeV}) \times Z(Z-1)A^{-1/3}$, for example, gives $E_c(C^{12}) - 3E_c(He^4) = 5.44$ MeV. Our calculated value for the ground-state energy is therefore quite reasonable, indicating that our model may not be completely unphysical.

IV. CONCLUSIONS

We have proposed a model for C^{12} in which this nucleus is composed of three "rigid" alpha particles. Specializing to the case of separable interactions in S waves only, we have solved the Faddeev equations numerically, obtaining a reasonable value for the ground-state energy. We do not, however, find the observed excited states.

We hope to extend these calculations in the near future. First of all it will be of interest to see whether the inclusion of D-wave forces will give the additional

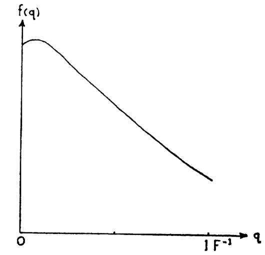

FIG. 1. The reduced wave function for the model C^{12} ground state. Since the wave function has not been normalized the ordinate scale is arbitrary.

attraction necessary to produce the excited states. This will involve solving sets of coupled integral equations with somewhat complicated kernels, but should be straightforward numerical work. We should also like to investigate the effect of changing the form of $g(k)$ and, although this is not very well justified, using two-term separable potentials to approximate the repulsion which must be present in the alpha-alpha potential at small distances.

A second kind of further calculation would apply and test the wave functions found in the model. Perhaps the simplest of these would be a calculation of the C^{12} charge form factor, following the work of Amado[12] on the triton. One might also be able to estimate the Coulomb energy, but we suspect that a first-order perturbation calculation would not be accurate. A better scheme might be to use the Coulomb-modified g functions of Ref. 20, treating the pure Coulomb T matrix as a perturbation on our integral equations. Even this might be futile, however, if our g functions do not have the correct asymptotic behavior, since the Coulomb energy will be sensitive to the form of the Be^8 wave function at small distances.

We can easily extend our calculations to energies above threshold by adding an inhomogeneous term to the integral equations. Since there is no stable Be^8 nucleus, and three-particle scattering seems impractical, there is no possibility of a direct confrontation with experiment. The best we could do would be to search for the positions and "wave functions" of the three-alpha resonances, possibly testing the latter in experiments in which a C^{12} compound-nucleus description seems to be valid.

Even if these further calculations reveal that our model does not provide a particularly good description for C^{12}, we feel it may still serve as a useful proving ground for methods of attacking the three-particle problem. The distraction of "inessential complications," such as spin and complicated kinematics, is reduced to a minimum and, at least in certain cases, it may be possible to penetrate the veil of numerical calculation and see what is really going on.

ACKNOWLEDGMENTS

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[21] See, for example, R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers Inc., New York, 1953), p. 122.

[22] I. Kaplan, *Nuclear Physics* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1963), p. 540.

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APPENDIX

To reduce (12) to the form given in (14) we begin with the expansion

$$[E-E(k'',q')]^{-1}(k')^{-\lambda}g_\lambda(k')(k'')^{-\lambda'}g_{\lambda'}(k'')=\sum_{\mathcal{L}}(2\mathcal{L}+1)V_{\lambda\lambda'}{}^{\mathcal{L}}(E;q,q')P^{\mathcal{L}}(\hat{q}\cdot\hat{q}'),\quad\text{(A1)}$$

where $V_{\lambda\lambda'}{}^{\mathcal{L}}(E;q,q')$ is given by (15). Then

$$M_{\lambda l,\lambda' l'}{}^{J\mathcal{L}}=\int d\Omega_q d\Omega_{q'}\,(k')^{\lambda}\mathcal{Y}_{\lambda l}{}^{JM*}(\hat{k}',\hat{q})P_{\mathcal{L}}(\hat{q}\cdot\hat{q}')\times(k'')^{\lambda'}\mathcal{Y}_{\lambda' l'}{}^{JM}(\hat{k}'',\hat{q}').\quad\text{(A2)}$$

By using the expansion theorem for solid harmonics[23] we can express $(k')^{\lambda}Y_{\lambda,\mu}(k')$ as a sum over L and M of terms proportional to $(q')^{L}Y_{L,M}(q')(q'')^{\lambda-L}Y_{\lambda-L,\mu-M}\times(q'')$, with L running from zero to λ. Then, by recoupling, we find

$$(k')^{\lambda}\mathcal{Y}_{\lambda l}{}^{JM}(\hat{k}',\hat{q})=\sum_{Lj}C_{\lambda l,Lj}{}^{J}(q',q)\mathcal{Y}_{Lj}{}^{JM}(\hat{q}',\hat{q}),\quad\text{(A3)}$$

where, using the shorthand notation $[n]=2n+1$,

$$C_{\lambda l,Lj}{}^{J}(q',q)=(-1)^{J+L}(q')^{L}(\tfrac{1}{2}q)^{\lambda-L}\left(\frac{[\lambda]![\lambda-L][l][\lambda][j]}{[L]![\lambda-L]!}\right)^{1/2}\times\begin{pmatrix}\lambda-L & l & j\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}L & \lambda-L & \lambda\\ l & J & j\end{Bmatrix}.\quad\text{(A4)}$$

Therefore

$$M_{\lambda l,\lambda' l'}{}^{J\mathcal{L}}=\sum_{LL'jj'}C_{\lambda l,Lj}{}^{J}(q',q)D_{Lj,L'j'}{}^{J\mathcal{L}}\times C_{\lambda' l',L'j'}{}^{J}(q,q'),\quad\text{(A5)}$$

where

$$D_{Lj,L'j'}{}^{J\mathcal{L}}=\int d\Omega_q d\Omega_{q'}\,\mathcal{Y}_{Lj}{}^{JM*}(\hat{q}',\hat{q})P_{\mathcal{L}}(\hat{q}\cdot\hat{q}')\times\mathcal{Y}_{L'j'}{}^{JM}(\hat{q},\hat{q}').\quad\text{(A6)}$$

It is a straightforward matter then to show that

$$D_{Lj,L'j'}{}^{J\mathcal{L}}=(-1)^{j+j'}([j][j'][L][L'])^{1/2}\times\begin{pmatrix}j' & L & \mathcal{L}\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}j & L' & \mathcal{L}\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}L' & j' & J\\ L & j & \mathcal{L}\end{Bmatrix}.\quad\text{(A7)}$$

The general expression for M is rather complicated but, especially when one of the angular momenta is zero, there is usually considerable simplification in particular cases.

Sum rules are always useful in checking numerical values. In this case the completeness of the spherical harmonics gives

$$\sum_{\mathcal{L}}(2\mathcal{L}+1)M_{\lambda l,\lambda' l'}{}^{J\mathcal{L}}=(q'+\tfrac{1}{2}q)^{\lambda}(q+\tfrac{1}{2}q')^{\lambda'}([\lambda][\lambda'][l][l'])^{1/2}\times\begin{pmatrix}\lambda & l & J\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\lambda' & l' & J\\ 0 & 0 & 0\end{pmatrix},\quad\text{(A8)}$$

and

$$\sum_{\mathcal{L}}(2\mathcal{L}+1)D_{\lambda l,\lambda' l'}{}^{J\mathcal{L}}=([l][l'][j][j'])^{1/2}\times\begin{pmatrix}L & j & J\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}L' & j' & J\\ 0 & 0 & 0\end{pmatrix},\quad\text{(A9)}$$

while taking $\hat{q}'=\hat{q}$ in (A3) gives

$$\sum_{Lj}C_{\lambda l,Lj}{}^{J}(q',q)([L][j])^{1/2}\begin{pmatrix}L & j & J\\ 0 & 0 & 0\end{pmatrix}=(q'+\tfrac{1}{2}q)^{\lambda}([\lambda][l])^{1/2}\begin{pmatrix}\lambda & l & J\\ 0 & 0 & 0\end{pmatrix}.\quad\text{(A10)}$$

[23] M. Danos and L. C. Maximon, J. Math. Phys. 6, 766 (1965).