

Quantum Theory of a Laser Model*

CHARLES R. WILLIS

Boston University, Boston, Massachusetts

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We derive the kinetic equations for the coupled single-particle density matrix ρ and the electromagnetic density matrix R to lowest order in the dimensionless coupling constant $\beta^2 \equiv (\omega_L/\omega_D)^2$. The laser frequency ω_L is $(4\pi)^{-1/2}(\mathfrak{N}r_0\lambda^2)^{1/2}\omega_0$ where \mathfrak{N} is the number of two-level systems per unit volume, r_0 is the classical electron radius, λ is the wavelength of the radiation, and $\hbar\omega_0$ is the two-level energy difference. The Doppler frequency ω_D characterizes the center-of-mass motion. For gas lasers β^2 is much less than 1 and, consequently, we generalize and use the Bogoliubov derivation of kinetic equations for weak interactions. We find solutions when the average field vanishes and which include spontaneous emission correctly. The single-particle density matrix and the radiation density matrix are coupled through their second moments. When we substitute the solution of the second-moment equations into the density-matrix equations, we find that each density matrix satisfies an uncoupled linear equation with known time-dependent coefficients. We introduce and discuss dissipation from the density-matrix point of view. With the use of the density-matrix formalism we indicate that the correct expansion parameter for higher order kinetic equations is β^2 .

I. INTRODUCTION

IN this paper we derive to order β^2 (the dimensionless coupling constant defined below) and partially solve the kinetic equations for the single-particle density matrix and the electromagnetic-field density matrix. Our system consists of N two-level systems interacting with radiation in a cavity. We include dissipation, pumping, and center-of-mass motion but we do not include noise. In a previous paper¹ we showed that the center-of-mass motion of the atoms in a gas laser is so rapid that the nonlinear differential-integral equations for the average electromagnetic fields are adequately represented by nonlinear differential equations. We then were able to solve these equations exactly in the self-consistent-field approximation which we refer to as the SCFA. The intuitive reason for the success of the slowly varying condition on the average electromagnetic fields (with the fast unperturbed time dependence removed) is that there are two dynamical processes that compete with each other. These are a rapid frequency shift $\omega_\alpha = k \cdot v_\alpha$ produced by the velocity of the center of mass and a change in the average electromagnetic field due to the interaction with matter. In gas lasers the measure of the center-of-mass motion ω_D is greater than the frequency ω_L which measures the electromagnetic-matter interaction. The definition of ω_L is $(4\pi)^{-1/2}(\mathfrak{N}r_0\lambda^2)^{1/2}\omega_0$, where \mathfrak{N} is the number of two-level atoms per unit volume, r_0 is the classical electron radius, and λ is the wavelength of the radiation. Consequently, $\beta \equiv (\omega_L/\omega_D)$ is much less than 1 and the rapid motion of the center of mass causes a fast oscillation in phase and a cancellation of the nonlocal time dependence during the time that it takes for the interaction to produce a change in the average field.

In this paper we do not directly apply the slowly

varying condition, but use instead the essentially equivalent Bogoliubov² asymptotic condition and expansion procedure based on the smallness of β^2 . We are thus able to go beyond the results of I in several different ways. First, we treat the full density matrix, not just the lowest moments of the density matrix, and thus we can solve the coherence problem for our model. Second, this paper includes spontaneous emission which does not affect quantitative values appreciably, but provides the trigger for the approach to the steady state and maintains the commutation relations. Third, the density-matrix method allows us to carry out a consistent treatment of dissipation. This helps to avoid errors that are sometimes made when phenomenological decay constants are introduced in nonlinear operator equations. Fourth, previous laser theories^{3,4} have been based on the existence of a nonvanishing electric field. We show that the steady state is almost the same whether the average field is zero or nonzero. There is a slight difference in the dependence of the steady state on the relaxation times. The fifth purpose of this paper is to provide the foundation for the calculation of higher order terms in β which we will present in a second paper. In this paper and in the second paper there is no requirement that the system must be near threshold. As long as $\beta^2 \ll 1$ we can be as much as a thousand times above threshold. Previous theories including those of the author have missed the largest term to order β^4 because the early introduction of average values wiped out the quantum interference terms.

In Sec. II we generalize the Bogoliubov² expansion procedure and use the generalization to derive the equations of motion for the coupled single-particle density matrix and radiation-density matrix to order β^2 . We solve the coupled moment equations in Sec. III.

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¹ C. R. Willis, *J. Math. Phys.* **6**, 1984 (1965); hereafter referred to as I.

² N. N. Bogoliubov, in *Studies in Statistical Mechanics*, edited by J. De Boer and G. E. Uhlenbeck (North-Holland Publishing Company, Amsterdam, 1962), pp. 5-118.

³ W. E. Lamb, *Phys. Rev.* **134**, A1429 (1964).

⁴ H. Haken and H. Saueremann, *Z. Physik* **173**, 261 (1963); **176**, 58 (1963).

With the solution of the moment equations we uncouple the equations of motion of the matter and radiation density matrices and show each matrix satisfies a linear equation with known coefficients. We do not introduce dissipation in Secs. II and III in order to see what role exact dynamics plays as compared with dissipation. In Sec. IV we introduce dissipation and solve the moment equations approximately but quite accurately. We discuss some important features of the higher order terms in the solution of the density matrix of the system in Sec. V.

II. KINETIC EQUATIONS FOR THE RADIATION AND SINGLE-PARTICLE DENSITY MATRICES

Our Hamiltonian for N two-level systems interacting with the electromagnetic field is

$$H(N) = h(N) + H_f + H_i + H_{c.m.}, \quad (2.1)$$

where

$$h(N) = (\hbar\omega_0/2) \sum_{\alpha} \hat{\sigma}_{\alpha}; \quad H_f = \hbar\Omega(a^\dagger a + \frac{1}{2}),$$

$$H_i = \hbar\omega_0 \sum_{\alpha} \tilde{\gamma} \Gamma(X_{\alpha}) \{a^\dagger \sigma_{\alpha} + a \sigma_{\alpha}^\dagger\},$$

$$H_{c.m.} = \sum_{\alpha} \frac{P_{\alpha}^2}{2m} + \frac{1}{2} \sum_{\alpha} \sum_{\beta} V(X_{\alpha} - X_{\beta}) + \sum_{\alpha} \sum_i^M U(X_{\alpha} - \eta_i).$$

The a^\dagger and a are the usual creation and annihilation operators for a single mode of the electromagnetic field in the cavity. To minimize the number of subscripts we consider only a single mode. The operators for the internal degrees of freedom of the α th atom are

$$\sigma_{\alpha}^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_{\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We drop the nonresonant terms from H_i as we did in I. The definitions of $\tilde{\gamma}$ and $\Gamma(X_{\alpha})$ are

$$\tilde{\gamma} = (\hbar\omega_0)^{-1} (\hbar\Omega)^{1/2} e \langle \mathbf{e} \cdot \mathbf{r} | b \rangle (4\pi/V)^{1/2},$$

$$\Gamma(X_{\alpha}) = E(X_{\alpha}) V^{1/2},$$

where $E(X_{\alpha})$ is the normalized eigenfunction of the cavity corresponding to the frequency Ω , and V is the volume. We treat the center of mass classically as N atoms interacting with each other and with M pump atoms through two-body forces, $V(X_{\alpha} - X_{\beta})$ and $U(X_{\alpha} - \eta_i)$, respectively. The X_{α} is the coordinate of the center of mass of the α th atom and η_i is the center of mass of the i th atom.

The equation of motion for the density matrix of the total system is

$$i\hbar(\partial F_N/\partial t) + [F_N, H(N)] = 0, \quad (2.2)$$

where $[A, B]$ is the commutator of A and B . We remove the time dependence generated by the center-of-mass

motion, $H_{c.m.}$, from the density matrix by the following transformation:

$$f_N = e^{-iL_{c.m.}t} F_N,$$

where

$$-iL_{c.m.} \equiv \sum_{\alpha} \left(\frac{\partial H_{c.m.}}{\partial X_{\alpha}} \frac{\partial}{\partial P_{\alpha}} - \frac{\partial H_{c.m.}}{\partial P_{\alpha}} \frac{\partial}{\partial X_{\alpha}} \right)$$

and

$$i\hbar \frac{\partial f_N}{\partial t} + [f_N, h(N) + H_f + H_i(t)] = 0. \quad (2.3)$$

The definition of $H_i(t)$ is

$$H_i(t) = \hbar\omega_0 \sum_{\alpha} \Gamma[X_{\alpha}(t)] (a^\dagger \sigma_{\alpha} + a \sigma_{\alpha}^\dagger),$$

where $X_{\alpha}(t)$ is the solution of Hamilton's equation of motion for the center of mass with the Hamiltonian $H_{c.m.}$.

When we take the trace of Eq. (2.3) over all variables except the radiation variables, we obtain the equation of motion for the radiation density matrix R

$$i\hbar \frac{\partial R}{\partial t} + [R, H_f] = \text{tr}_{1,2,\dots,N} \int dX_1 \cdots dP_N [H_i(t), f_N], \quad (2.4)$$

where

$$R \equiv \text{tr}_{1,2,\dots,N} \int dX_1 \cdots dP_N f_N.$$

The symbol $\text{tr}_{1,2,\dots,N}$ represents a trace over a complete set of variables for the internal degrees of freedom 1, 2, \dots , N .

Similarly the equation of motion for the single-particle density matrix is

$$i\hbar \frac{\partial \rho}{\partial t} + [\rho, h(1)] = \text{tr}_{2,\dots,N} \text{tr}_q \int dX_1 \cdots dP_N [H_i(t), f_N], \quad (2.5)$$

where

$$\rho \equiv \text{tr}_{2,\dots,N} \text{tr}_q \int dX_1 \cdots dP_N f_N$$

and tr_q represents a trace over a complete set of variables for the radiation oscillator.

The equation of motion for the center-of-mass distribution function in the interaction representation is

$$\frac{\partial \mathcal{F}_N}{\partial t}(X_1, \dots, P_N, t) = \text{tr}_{1,2,\dots,N} \text{tr}_q [H_i(t), f_N].$$

In this paper we assume only free-particle motion with a Maxwellian distribution of velocities; consequently, we do not need to consider the center-of-mass distribution function further. However, if we wish to treat the effects of collisions carefully we have to include

a careful analysis of the center-of-mass distribution function.

We use Bogoliubov's² approximation scheme to solve Eqs. (2.4) and (2.5) to order $\tilde{\gamma}^2$. Bogoliubov developed a systematic approximation scheme to solve for the distribution function of many-body systems. He found a class of solutions where the time dependence of many-body distribution functions were functions of the single-particle distribution function and then found a nonlinear kinetic equation for the single-particle distribution function. We need to make two generalizations of Bogoliubov's original derivation. First, we need the quantum-mechanical generalization which consists simply of replacing the Poisson brackets of A and B by (i/\hbar) times the commutator of A and B . Since we are interested in two sets of variables, the internal variables and the radiation variables, we must have two different single variable density matrices R and ρ . This second generalization to two sets of variables is also straightforward.

Bogoliubov considered the problem of N particles in a volume Ω interacting classically through the potential energy $\psi(q_i - q_j)$ where q_i is the position of the i th particle. He showed that if asymptotically the correlations between particles vanish, then the equation of motion for the single particle distribution function F_1 to order ϵ^2 is

$$\begin{aligned} \frac{\partial F_1}{\partial t}(t, \mathbf{X}_1) &= [H_0; F_1] \\ &+ \frac{\epsilon}{v} \int_{\Omega} [\psi(q_1 - q_2); F_1(t, \mathbf{X}_1) F_1(t, \mathbf{X}_2)] d\mathbf{X}_2 \\ &+ \frac{\epsilon^2}{v} \int_{\Omega} [\psi(q_1 - q_2); \left[\int_0^{\infty} \psi \left((q_1 - q_2) - \frac{1}{m} (\hat{p}_1 - \hat{p}_2) \tau \right) d\tau; \right. \\ &\quad \left. F_1(t, \mathbf{X}_1) F_1(t, \mathbf{X}_2) \right]] d\mathbf{X}_2, \quad (2.6) \end{aligned}$$

where $X \equiv (q, \hat{p})$, $v \equiv \Omega/N$, ϵ is the small dimensionless interaction constant and $F_1(t, \mathbf{X})$ is the single-particle distribution function. The notation in this paragraph is Bogoliubov's² notation and Eq. (2.6) above is his Eq. (10.18). In the classical problem the term linear in ϵ vanishes when the single-particle distribution function is independent of position. The term in Eq. (2.6) proportional to ϵ^2 consists of a double commutator of the interaction potential evaluated with a time displacement τ . The propagation in the time τ is by means of the unperturbed Hamiltonian.

Here we consider solutions in which the center of mass distribution is spatially homogeneous. Consequently when we perform the spatial integration over the potential H_i in the term corresponding to the term linear in ϵ in Eq. (2.6) we obtain zero. This means that the

term corresponding to the linear term in Eq. (2.6) is zero.

Since the generalizations of the Bogoliubov derivation of Eq. (2.6) are straightforward, we give only the results here:

$$\begin{aligned} \frac{\partial R}{\partial t} &= -\frac{i}{\hbar} [H_f, R] - \tilde{\gamma}^2 N \omega_0^2 \text{tr}_1 \int dX dP \\ &\quad \times \int_0^{\infty} d\tau [H_i, [H_i(\tau), R \rho \mathfrak{F}_1]], \quad (2.7a) \end{aligned}$$

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\frac{i}{\hbar} [h(1), \rho] - \tilde{\gamma}^2 \omega_0^2 \text{tr}_q \int dX dP \\ &\quad \times \int_0^{\infty} d\tau [H_i, [H_i(\tau), R \rho \mathfrak{F}_1]], \quad (2.7b) \end{aligned}$$

where $\mathfrak{F}_1(X, P)$ is the single-particle center-of-mass distribution function which we take to be Maxwellian. The definition of $H_i(\tau)$ is

$$H_i(\tau) = \sum_{\alpha} \Gamma[X_{\alpha}(\tau)] (a^{\dagger} \sigma_{\alpha} e^{i\Delta\tau} + a \sigma_{\alpha}^{\dagger} e^{-i\Delta\tau}), \quad (2.8)$$

where $\Delta \equiv \omega_0 - \Omega$. In $H_i(\tau)$ we use

$$\begin{aligned} e^{iH_f\tau} a^{\dagger} e^{-iH_f\tau} &= a^{\dagger} e^{-i\Omega\tau}, \\ e^{ih(1)\tau} \sigma e^{-ih(1)\tau} &= \sigma e^{i\omega_0\tau}, \end{aligned}$$

and their complex conjugates.

The time integrals in Eqs. (2.7a) and (2.7b) are

$$\omega_D^{-1} C = \int_0^{\infty} e^{i\Delta\tau} \{ \Gamma[X(\tau)] \Gamma(X) \}_{\text{av}} d\tau \quad (2.9)$$

and its complex conjugate. We introduce a factor ω_D^{-1} so that C is dimensionless. The average in Eq. (2.9) is over the center-of-mass distribution function \mathfrak{F}_1 . For free-particle motion $X(\tau) = X(0) - V\tau$. For definiteness we consider a rectangular cavity, thus $\Gamma(X) = \sin kX$. Consequently, the average in Eq. (2.9) is

$$\begin{aligned} &\{ \Gamma[X(\tau)] \Gamma(X) \}_{\text{av}} \\ &= \int \sin k(X - V\tau) \sin kX \mathfrak{F}_1(X, V) dX dV \\ &= (\omega_0/k) \{ \sin^2 kX \}_{\text{av}} \int_0^{\infty} \cos kV \exp\left(\frac{mV^2}{2k_B T}\right) dV \\ &= \{ \sin^2 kX \}_{\text{av}} \exp\left(-\frac{\omega_D^2 \tau^2}{2}\right), \quad (2.10) \end{aligned}$$

where $\omega_D \equiv k(k_B T/m)^{1/2}$, k_B is Boltzmann's constant, and T is the temperature. The Doppler width ω_D is the

largest frequency in the problem excluding Ω and ω_0 . We consider a homogeneous system so that $\{\sin^2 kX\}_{av} = \frac{1}{2}$. We substitute Eq. (2.10) in Eq. (2.9) and obtain

$$C \equiv C_R + iC_I = \frac{1}{2} \int_0^\infty \exp(i\Delta\zeta - \frac{1}{2}\zeta^2) d\zeta \\ = \frac{1}{2} \left(\sqrt{\frac{\pi}{2}} \right) \exp\left(-\frac{\bar{\Delta}^2}{2}\right) + \frac{1}{2} i \int_0^\infty \sin \bar{\Delta}\zeta \\ \times \exp\left(-\frac{\zeta^2}{2}\right) d\zeta, \quad (2.11)$$

where $\bar{\Delta} \equiv (\omega_0 - \Omega)/\omega_D$ and a bar over a frequency or relaxation time indicates it has been made dimensionless with ω_D .

When we substitute Eq. (2.11) in Eqs. (2.7a) and

(2.7b), we obtain

$$\frac{\partial R}{\partial t} = -\frac{i}{\hbar} [H_f, R] \\ - \frac{\tilde{\gamma}^2 N \omega_0^2}{\omega_D} \text{tr}_1([a^\dagger \sigma + a \sigma^\dagger, [Ca^\dagger \sigma + C^* a \sigma^\dagger, R\rho]]) \quad (2.12a)$$

and

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [h(1), \rho] \\ - \frac{\tilde{\gamma}^2 \omega_0^2}{\omega_D} \text{tr}_q([a^\dagger \sigma + a \sigma^\dagger, [Ca^\dagger \sigma + C^* a \sigma^\dagger, R\rho]]). \quad (2.12b)$$

With the help of the commutation relations for the radiation and matter operators we evaluate the double commutator in Eqs. (2.12a) and (2.12b). After a little algebra, the result is

$$[a^\dagger \sigma + a \sigma^\dagger, [Ca^\dagger \sigma + C^* a \sigma^\dagger, R\rho]] = C\{a^\dagger a^\dagger R[\sigma, [\sigma, \rho]] + a^\dagger [a^\dagger, R][\sigma, \rho\sigma] + [a^\dagger, a^\dagger R][\sigma, \rho\sigma] + aa^\dagger R[\sigma^\dagger, [\sigma, \rho]] \\ + a[a^\dagger, R][\sigma^\dagger, \rho\sigma] + [a, a^\dagger R][\sigma, \rho]\sigma^\dagger + [a, [a^\dagger, R]]\rho\sigma\sigma^\dagger \\ + C^*\{a^\dagger a R[\sigma, [\sigma^\dagger, \rho]] + a^\dagger [a, R][\sigma, \rho\sigma^\dagger] + [a^\dagger, a R][\sigma^\dagger, \rho]\sigma + [a^\dagger, [a, R]]\rho\sigma^\dagger \\ + aa R[\sigma^\dagger, [\sigma^\dagger, \rho]] + a[a, R][\sigma^\dagger, \rho\sigma^\dagger] + [a, a R][\sigma^\dagger, \rho]\sigma^\dagger\}. \quad (2.13)$$

When we take the trace of Eq. (2.13) over the matter variables and substitute the result in Eq. (2.12a), we obtain

$$\partial R/\partial t + (i/\hbar)[H_f, R] = -\frac{\tilde{\gamma}^2 N \omega_0^2}{\omega_D} (C\{n_+[a, a^\dagger R] + n_-([a, [a^\dagger, R]] - [a, a^\dagger R])\} \\ + C^*\{n_+[aR, a^\dagger] + [a^\dagger, [a, R]] - n_-[a, Ra^\dagger]\}), \quad (2.14)$$

where $n_+ \equiv \langle \sigma^\dagger \sigma \rangle$ and $n_- \equiv \langle \sigma \sigma^\dagger \rangle$. Since σ and σ^\dagger satisfy anticommutation relations, we have $n_+ + n_- = 1$. The symbol n_+ represents the average occupancy of the excited state per atom and n_- represents the average occupancy of the ground state per atom.

In order to obtain the equation of motion for ρ we take the trace of Eq. (2.13) over the radiation variables and substitute the result in Eq. (2.12b). This yields

$$\partial \rho/\partial t + (i/\hbar)[h(1), \rho] = -\frac{\tilde{\gamma}^2 \omega_0^2}{\omega_D} (C\{\langle a^\dagger a^\dagger \rangle [\sigma, [\sigma, \rho]] + [\sigma, \rho\sigma] \text{tr}_q a^\dagger [a^\dagger, R] + \langle aa^\dagger \rangle [\sigma^\dagger, [\sigma, \rho]] + [\sigma^\dagger, \rho\sigma] \text{tr}_q a [a^\dagger, R]\} \\ + C^*\{\langle aa^\dagger \rangle [\sigma, [\sigma^\dagger, \rho]] + [\sigma, \rho\sigma^\dagger] \text{tr}_q a^\dagger [a, R] + \langle aa \rangle [\sigma^\dagger, [\sigma^\dagger, \rho]] + [\sigma^\dagger, \rho\sigma^\dagger] \text{tr}_q a [a, R]\}). \quad (2.15)$$

The second and eighth terms on the right-hand side of Eq. (2.15) vanish. The first and seventh terms are proportional to $-2\langle a^\dagger a^\dagger \rangle \sigma \rho \sigma$ and $-2\langle aa \rangle \sigma^\dagger \rho \sigma^\dagger$, respectively. The equation of motion for $\langle a^\dagger a^\dagger \rangle$ is obtained by multiplying Eq. (2.14) by $a^\dagger a^\dagger$ and performing the trace. The resultant equation is linear and homogeneous in $\langle a^\dagger a^\dagger \rangle$. Consequently, when we assume $\langle a^\dagger a^\dagger \rangle$ vanishes at $t=0$, it remains zero for all time. The same is true for $\langle aa \rangle$. Thus, Eq. (2.15) reduces to the following form when we assume R is diagonal at $t=0$

$$\frac{\partial \rho}{\partial t} + \frac{i}{\hbar} [h(1), \rho] = -\frac{\tilde{\gamma}^2 \omega_0^2}{\omega_D} (C\{\langle aa^\dagger \rangle [\sigma^\dagger, [\sigma, \rho]] + [\sigma^\dagger, \rho]\} + C^*\{\langle a^\dagger a \rangle [\sigma, [\sigma^\dagger, \rho]] - [\sigma, \rho\sigma^\dagger]\}). \quad (2.16)$$

The Eqs. (2.14) and (2.16) constitute a complete, closed set of equations whose solutions represent the solution of Eq. (2.2) correct to order $\tilde{\gamma}^2$.

III. MOMENT EQUATIONS AND LINE SHIFTS

The right-hand sides of Eqs. (2.14) and (2.16) consist of real and imaginary parts. The imaginary parts are commutators and represent time-dependent shifts of the unperturbed Hamiltonian which reduce to frequency shifts

in the steady state. In order to see this we take the imaginary part of the right-hand side of Eq. (2.14) which is

$$-\frac{\tilde{\gamma}^2 N \omega_0^2}{\omega_D} 2iC_I \langle \hat{\sigma} \rangle \{ [a, a^\dagger R] - [aR, a^\dagger] - [a, [a^\dagger, R]] \} + n_+ \{ [a, [a^\dagger, R]] - [a^\dagger, [a, R]] \} \\ = -\frac{\tilde{\gamma}^2 N \omega_0^2}{\omega_D} 2iC_I \langle \hat{\sigma} \rangle [a^\dagger a, R], \quad (3.1)$$

where we use $[a, [a^\dagger, R]] = [a^\dagger, [a, R]]$ and $\langle \sigma^\dagger \sigma - \sigma \sigma^\dagger \rangle = \langle \hat{\sigma} \rangle$. We combine Eq. (3.1) with Eq. (2.14), and the result is

$$\frac{\partial R}{\partial t} + i \left[\left(\Omega + 2C_I \frac{\omega_L^2}{\omega_D} \langle \hat{\sigma}(t) \rangle \right) a^\dagger a, R \right] \\ = -\frac{\omega_L^2}{\omega_D} C_R \{ n_+ \{ [a, a^\dagger R] + [aR, a^\dagger] + [a^\dagger, [a, R]] \} + n_- \{ [a, [a^\dagger, R]] - [a, a^\dagger R] - [aR, a^\dagger] \} \} \equiv \delta R / \delta t, \quad (3.2)$$

where $\omega_L^2 \equiv \tilde{\gamma}^2 N \omega_0^2$. By repeating the same procedure for the particle density matrix, we obtain

$$\frac{\partial \rho}{\partial t} + i \left[\left(\omega_0 + \frac{\omega_L^2}{\omega_D} C_I \frac{\langle a^\dagger a(t) \rangle}{N} \right) \hat{\sigma}, \rho \right] = -\frac{\omega_L^2}{N \omega_D} C_R \{ \langle a a^\dagger \rangle [\sigma^\dagger, [\sigma, \rho]] + \langle a^\dagger a \rangle [\sigma, [\sigma^\dagger, \rho]] + [\sigma^\dagger, \rho \sigma] - [\sigma, \rho \sigma^\dagger] \} \equiv \delta \rho / \delta t. \quad (3.3)$$

Thus, in the steady state the imaginary part of C gives rise to a shift in the frequency of the electromagnetic field and a shift in the matter frequency.

The kinetic equations, Eqs. (3.2) and (3.3), for R and ρ are coupled to each other through second moments. The equation for the radiation density matrix R depends on the density matrix ρ through the second moments $n_+ \equiv \langle \sigma^\dagger \sigma \rangle$ and $n_- \equiv \langle \sigma \sigma^\dagger \rangle$. The equation for the matter density matrix depends on R through the second moments $\langle a^\dagger a \rangle$ and $\langle a a^\dagger \rangle$.

We obtain the moment equations when we multiply Eq. (3.2) by $a^\dagger a$ and Eq. (3.3) by $\sigma^\dagger \sigma$ and take the appropriate traces

$$\frac{\partial \mathcal{E}}{\partial t} = -\frac{\omega_L^2}{\omega_D} C_R \{ n_+ \text{tr}_q \{ a^\dagger a [a, a^\dagger R] + a^\dagger a [aR, a^\dagger] + a^\dagger a [a^\dagger, [a, R]] \} \\ + n_- \text{tr}_q \{ a^\dagger a [a, [a^\dagger, R]] - a^\dagger a [a, a^\dagger R] - a^\dagger a [aR, a^\dagger] \} \}, \\ = \frac{2\omega_L^2}{\omega_D} C_R \{ n_+ (\mathcal{E} + 1) - n_- \mathcal{E} \}, \quad (3.4)$$

$$\frac{\partial n_+}{\partial t} = -\frac{2\omega_L^2}{N \omega_D} C_R \{ n_+ (\mathcal{E} + 1) - n_- \mathcal{E} \}, \quad (3.5)$$

where $\mathcal{E} \equiv \langle a^\dagger a \rangle$. Thus, the coupled second moment equations are rate equations with spontaneous emission included properly.

An important property of Eqs. (3.4) and (3.5) which holds also when dissipation is present is that the only moments of R and ρ that appear are second moments, $\langle a^\dagger a \rangle$ and $\langle \sigma^\dagger \sigma \rangle$. Consequently, if we solve the coupled second moment Eqs. (3.4) and (3.5) and substitute the results into Eqs. (3.2) and (3.3), we obtain a linear operator equation for R with known time-dependent coefficients. Similarly, we obtain a linear operator equation for ρ with known time dependent coefficients. Thus, the solution of the second-moment equations uncouples the single particle and radiation density matrices.

As we show in the next section, the only effect that the introduction of dissipation has on the equations up

to this point is to add linear terms with constant coefficients. In the absence of dissipation, Eqs. (3.4) and (3.5) have a constant of the motion

$$\mathcal{E} + N n_+ = \text{const} = N. \quad (3.6)$$

We set the constant equal to N to include the usual gas case where at $t=0$ we have $\mathcal{E}=0$, and thus Eq. (3.6) actually defines the number of two-level systems. If we had permanent two-level systems, the constant could be any integer. The constant of the motion, Eq. (3.6) is generally not valid in the presence of dissipation.

When we substitute Eq. (3.6) into Eq. (3.4), we obtain

$$\mathcal{E}' = 2\beta^2 C_R \left\{ -\frac{2\mathcal{E}^2}{N} + \left(1 - \frac{1}{N} \right) \mathcal{E} + 1 \right\}, \quad (3.7)$$

where the prime indicates differentiation with respect to the dimensionless "time" $\omega_D t$ and β^2 is the dimensionless coupling constant $(\omega_L/\omega_D)^2$. The steady state solution of Eq. (3.7) is

$$\mathcal{E}_s = \frac{(N-1)}{4} \left[1 + \left\{ 1 + \frac{8N}{(N-1)^2} \right\}^{1/2} \right] \approx \frac{1}{2}(N+1), \quad (3.8)$$

$$n_+ \approx \frac{1}{2}(1-1/N).$$

The steady state consists of $\frac{1}{2}(N+1)$ units of energy in radiation and $\frac{1}{2}(N-1)$ units of energy in matter. The difference is due to spontaneous emission.

The time-dependent solution of Eq. (3.7) approaches the unique steady state Eq. (3.8) monotonically and is

$$\mathcal{E}(\tau) = \frac{\mathcal{E}_s [1 - \exp(-A\tau)]}{1 + \mathcal{E}_s \exp(-A\tau)} \rightarrow \mathcal{E}_s \approx \frac{(N+1)}{2},$$

where $\mathcal{E}(0) = 0$ and $A \equiv (4\beta^2 C_R) N^{-1} = 4C_R \gamma^2 \omega_0^2 \omega_D^{-2}$ is independent of N . That A is independent of N reflects the fact that A arises from spontaneous emission. When spontaneous emission is omitted as in average field theories,^{1,3,4} there has to be a nonvanishing initial energy density $\mathcal{E}(0)$ to get the laser started. We observe that in a time $t \gg (A\omega_D)^{-1}$ the second moments reach their stationary values. Consequently, for $t \gg (A\omega_D)^{-1}$ we have two uncoupled linear differential equations with constant coefficients for the density matrices R and ρ .

The steady-state frequency shift is

$$\Delta\Omega \equiv 2C_I \langle \hat{\sigma} \rangle_s \omega_L^2 \omega_D^{-1} = 2C_I \gamma^2 \omega_0^2 \omega_D^{-1}$$

which is less than 0.1 sec^{-1} for a He-Ne laser and is unobservable. This line shift is due to spontaneous emission. If we omit spontaneous emission the population inversion $\langle \hat{\sigma} \rangle_s$ is zero and the frequency shift vanishes.

In the next section we show that the introduction of dissipation does not appreciably affect the qualitative behavior derived in this section. The approach to equilibrium is more rapid, the line shift is observable because of the increase of $\langle \hat{\sigma} \rangle_s$ caused by the introduction of dissipation, and the density-matrix equations uncouple. The only qualitative change is that in the presence of dissipation a threshold inversion density has to be overcome in order to obtain laser action.

IV. INTRODUCTION OF DISSIPATION

The problem of dissipation in lasers has both a simple and a difficult aspect. In the moment equations the introduction of dissipation simply adds linear terms with constant coefficients. On the other hand, the introduction of dissipation in Eq. (2.2) for the density matrix for the entire system is a difficult problem. We do not attempt to give a solution of the problem here, but we content ourselves with a discussion which is sufficient to indicate what needs to be done to obtain a solution.

In phenomenological treatments three relaxation

times are introduced into the moment equations. They are T_r , the radiation relaxation time; T_1 , the relaxation time of the average population inversion; and T_2 , the relaxation time of the average polarization. When we return to the original equation for the full density matrix, Eq. (2.2), we need to know the details of the physical mechanism that causes the dissipation to understand how these relaxation times arise. For definiteness we consider our system of matter plus radiation to be in weak interaction with reservoirs. Consequently, the new Hamiltonian of our system plus reservoirs is

$$H = H(N) + H_{tr} + H_{mr} + \epsilon_f V_f + \epsilon_m V_m, \quad (4.1)$$

where H_{tr} and H_{mr} are the Hamiltonians of the field reservoir and matter reservoir, respectively. The interaction potentials between our system and the field and matter reservoirs are $\epsilon_f V_f$ and $\epsilon_m V_m$ where ϵ_f and ϵ_m are dimensionless coupling constants. In most mechanisms V_f and V_m do not commute with the internal radiation matter interaction H_i .

Several authors⁵⁻⁷ have shown that the equation of motion for the density matrix of a system in contact with reservoirs to lowest order in the coupling constant is

$$\frac{\partial F_N}{\partial t} + \frac{i}{\hbar} [F_N, H(N) + \epsilon_f^2 U_f + \epsilon_m^2 U_m] = \epsilon_f^2 K_f F_N + \epsilon_m^2 K_m F_N, \quad (4.2)$$

where the quantities U_f and U_m are functions of system operators and reservoir parameters such as temperature. The U 's represent shifts in the system Hamiltonian due to the interaction with reservoirs. The operators⁸ K_f and K_m are relaxation operators which have the property of maintaining the Hermiticity, normalization, and positive definiteness of F_N . The operators K_f and K_m are four index operators in the sense that

$$\begin{aligned} \langle \alpha | \frac{\partial F_N}{\partial t} | \beta \rangle + \frac{i}{\hbar} \langle \alpha | [F_N, H(N) + \epsilon_f^2 U_f + \epsilon_m^2 U_m] | \beta \rangle \\ = \sum_{\gamma} \sum_{\delta} \{ \epsilon_f^2 \langle \alpha \beta | K_f | \gamma \delta \rangle + \epsilon_m^2 \langle \alpha \beta | K_m | \gamma \delta \rangle \} \\ \times \langle \gamma | F_N | \delta \rangle, \quad (4.3) \end{aligned}$$

where $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$ and $|\delta\rangle$ each stand for an eigenstate of the radiation-matter Hamiltonian $H(N)$. Although in some problems the operators K_f and K_m reduce to two index operators, the laser problem requires four index properties for two reasons. First, if average variables such as $\langle a \rangle$, $\langle a^\dagger \rangle$, $\langle \sigma \rangle$, and $\langle \sigma^\dagger \rangle$ are nonzero, their relaxation is caused by the off-diagonal parts of K_f and K_m

⁵ R. K. Wangsness and F. Bloch, Phys. Rev. **89**, 728 (1953).

⁶ C. R. Willis and P. G. Bergmann, Phys. Rev. **128**, 391 (1962).

⁷ W. Weidlich and F. Haake, Z. Physik **185**, 30 (1965); **186**, 203 (1965).

⁸ C. R. Willis, Phys. Rev. **127**, 1405 (1962).

because a , a^\dagger , σ , and σ^\dagger have only off-diagonal matrix elements. The second related reason is that the relaxation mechanism for nondiagonal operators may be different than for diagonal operators. The most common example of different relaxation times is the T_1 relaxation time for diagonal matrix elements and the T_2 relaxation time for off-diagonal matrix elements of spin systems. In lasers we have the same distinction with T_1 for diagonal matrix elements and T_2 for the off-diagonal polarization matrix elements of σ and σ^\dagger . Although the radiation relaxation time is usually taken to be T_r for both diagonal and off-diagonal matrix elements of the radiation operators, this need not be the case.

We do not have to solve Eq. (4.3) if we are only interested in moments of the density matrix which depends on the detailed structure of the operators K_f and K_m . We need only the following moments of K_f and K_m :

$$\frac{\Delta\langle a^\dagger a \rangle}{\Delta t} = \epsilon_f^2 \text{tra}^\dagger a K_f F_N = 2 \frac{\langle a^\dagger a \rangle}{T_r} = 2 \frac{\langle \mathcal{E} \rangle}{T_r}, \quad (4.4a)$$

$$\frac{\Delta\langle a \rangle}{\Delta t} = \epsilon_f^2 \text{tra} K_f F_N = \frac{\langle a \rangle}{T_r}, \quad (4.4b)$$

$$\frac{\Delta\langle \sigma^\dagger \sigma \rangle}{\Delta t} = \epsilon_m^2 \text{tr} \sigma^\dagger \sigma K_m F_N = \frac{\langle \sigma^\dagger \sigma \rangle}{T_1} = \frac{n_+}{T_1}, \quad (4.4c)$$

$$\frac{\Delta\langle \sigma \rangle}{\Delta t} = \epsilon_m^2 \text{tr} \sigma K_m F_N = \frac{\langle \sigma \rangle}{T_2}, \quad (4.4d)$$

where the symbol $\Delta\langle \dots \rangle / \Delta t$ is the contribution to the time derivative of $\langle \dots \rangle$ from the relaxation operators K . The factor 2 in Eq. (4.4a) is a consequence of the fact that if $\langle a \rangle$ has a relaxation time T_r , then $\langle a^\dagger a \rangle$ has a relaxation time $(T_r/2)$. Higher order moments can have different relaxation times depending on the detailed structure of the K 's.

The most difficult problem introduced by dissipation is the noncommutivity of V_f and V_m with H_i . Although Eq. (4.3) is formally correct, we must know the eigenstates of the Hamiltonian $H(N)$ for the full system which includes H_i in order to explicitly carry out the evaluation of the matrix elements of K_f and K_m . If ϵ_f^2 and ϵ_m^2 were very much smaller than β^2 , we could carry out the Bogoliubov procedure of the previous two sections on the left-hand side of Eq. (4.3) and use the unperturbed-system Hamiltonian to evaluate the K 's. In a He-Ne laser this is not possible because near threshold $\beta^2 \sim 5 \times 10^{-4}$ and $\epsilon_m^2 \sim 6 \times 10^{-3}$. Consequently, we must return to the full Hamiltonian, Eq. (4.1), and carry out the expansions in powers of β^2 , ϵ_f^2 , and ϵ_m^2 at the same time. In the lowest order of the parameters β , ϵ_f , and ϵ_m the derivation is straightforward and the effects of the different expansions do not interfere with each other. The equations for the radiation density

matrix R and the matter density matrix ρ are

$$\begin{aligned} \frac{\partial R}{\partial t} + i \left[\left\{ \Omega + 2C_I \frac{\omega_L^2}{\omega_D} (n_+ - n_-) \right\} a^\dagger a + \epsilon_f^2 U_f, R \right] \\ = \frac{\delta R}{\delta t} + \epsilon_f^2 K_f R, \quad (4.5a) \end{aligned}$$

$$\begin{aligned} \frac{\partial \rho}{\partial t} + i \left[\left\{ \omega_0 + C_I \frac{\omega_L^2}{N\omega_D} \mathcal{E} \right\} \hat{\sigma} + \epsilon_m^2 U_m, \rho \right] \\ = \frac{\delta \rho}{\delta t} + \epsilon_m^2 K_m \rho, \quad (4.5b) \end{aligned}$$

where $\delta R / \delta t$ and $\delta \rho / \delta t$ are given in Eqs. (3.2) and (3.3). Now the operators K_f and K_m can be explicitly calculated because they are functions of the unperturbed-system Hamiltonian only.

The next terms in the expansion of the density matrix after β^2 , ϵ_f^2 , ϵ_m^2 are ϵ_m^4 , $\epsilon_m^2 \beta^2$, β^4 , $\epsilon_f^2 \beta^2$, and ϵ_f^4 in decreasing order of magnitude. The terms $\epsilon_m^2 \beta^2$ and $\epsilon_f^2 \beta^2$ arise because H_i does not commute with V_f and V_m . The structure of the terms $\epsilon_m^2 \beta^2$ and $\epsilon_f^2 \beta^2$ depends sensitively on the form of the interaction with the reservoirs. The problem of higher order terms is not academic because the experimentally observed Lamb dip requires the β^4 term.

After this lengthy discussion of dissipation, we consider here only the explicit effects of dissipation on the moment equations. However, problems of coherence require more knowledge of the solution of Eqs. (4.5a) and (4.5b) than the knowledge contained in the moment equations.

When we take the moments of Eqs. (4.5a) and (4.5b), we obtain with the aid of Eqs. (4.4a) and (4.4c) the following:

$$\mathcal{E}' = 2\beta^2 C_R \{ n_+ (\mathcal{E} + 1) - (1 - n_+) \mathcal{E} \} - 2\mathcal{E} / \bar{T}_r, \quad (4.6a)$$

$$\begin{aligned} n_+' = -\frac{2\beta^2}{N} C_R \{ n_+ (\mathcal{E} + 1) - (1 - n_+) \mathcal{E} \} \\ - (n_+ - 1) / \bar{T}_1, \quad (4.6b) \end{aligned}$$

where the prime indicates differentiation with respect to the dimensionless time $\omega_D t$. The bar over \bar{T}_r and \bar{T}_1 indicates they are multiplied by ω_D and are dimensionless. The one in $(n_+ - 1) / \bar{T}_1$ represents the effect of the pump which we need in order to supply the energy lost in dissipation. If we had N permanent two-level systems the value one could be replaced by any value between zero and one. In gas lasers the pump actually defines the number of systems through the one in $(n_+ - 1) / \bar{T}_1$.

We observe that our treatment of the Doppler motion introduces irreversibility into Eqs. (2.14) and (2.16). No pump is needed to compensate for the center-of-mass dephasing; however, when we introduce relaxation times for the energy of the radiation and matter, we

need to introduce a pump to compensate for the loss. Consequently, we have two types of irreversibility in the laser problem.

The steady-state solutions to Eqs. (4.6a) and (4.6b) are

$$(n_+)_s = \frac{\mathcal{E}_s + N(2\bar{T}_1 C_R \beta^2)^{-1}}{2\mathcal{E}_s + 1 + N(2\bar{T}_1 C_R \beta^2)^{-1}}, \quad (4.7a)$$

$$\begin{aligned} \mathcal{E}_s = & \frac{N}{4} \left(\frac{T_r}{T_1} + \frac{1}{N} - \frac{1}{2\bar{T}_1 C_R \beta^2} \right) \\ & \times \left[1 + \left\{ 1 + \frac{8T_r}{NT_1} \left(\frac{T_r}{T_1} + \frac{1}{N} - \frac{1}{2\bar{T}_1 C_R \beta^2} \right)^{-2} \right\}^{1/2} \right] \\ & \approx (T_r/2T_1)[N - N_T] + \frac{1}{2}, \quad (4.7b) \end{aligned}$$

where $N_T \equiv (\beta^2 C_R \bar{T}_1)^{-1}$ is the threshold inversion number of atoms needed to start laser action. The approximate sign in Eq. (4.7b) holds when $N \gg 1$. The steady state energy density in Eq. (4.7b) differs from the SCFA of I by only the zero-point energy.

When we substitute Eq. (4.7b) in Eq. (4.7a) and use the definition of the line shift, we obtain

$$\Delta\Omega \equiv 2C_I \langle \delta \rangle_s \omega_L^2 \omega_D^{-1} = C_I (C_R T_r)^{-1}. \quad (4.8)$$

The steady-state results, Eqs. (4.7a), (4.7b), and (4.8), are exact and require no approximations. When we consider the problem of the approach to the steady state in the presence of dissipation we must solve the time-dependent Eqs. (4.6a) and (4.6b). These are difficult to solve because there is no longer a constant of the motion. The system of Eqs. (4.6a) and (4.6b) is equivalent to a second-order differential equation with time-dependent transcendental coefficients. We can find a fairly accurate qualitative description of the approach to the steady state for two ranges. First, if N greatly exceeds threshold, we can treat the dissipation as a small perturbation on the dissipationless solution in Sec. III. For the important region from threshold to about twice threshold we can solve the following equation which we obtain when we add Eqs. (4.6a) and (4.6b)

$$\frac{\partial}{\partial t} \{ \mathcal{E} + N n_+ \} = - \frac{2\mathcal{E}}{T_r} - \frac{N(n_+ - 1)}{T_1}. \quad (4.9)$$

An approximate solution of Eq. (4.9) is

$$\begin{aligned} n_+ = & 1 - (2\mathcal{E}T_1)(N_T T_r)^{-1} \\ & \times [1 - (2T_1)(N - N_T)(N_T T_r)^{-1}] \quad (4.10) \end{aligned}$$

which is valid as long as the second term in square brackets is small compared with 1. For a He-Ne laser the second term in square brackets is about 0.06 when N is twice threshold. When we substitute Eq. (4.10) in Eq. (4.6a) and solve the resultant equation, we obtain

$$\mathcal{E}(\tau) = \mathcal{E}_s \frac{1 - e^{-B\tau}}{1 + \mathcal{E}_s e^{-B\tau}} \rightarrow \mathcal{E}_s \quad \text{for } t > (\omega_D B)^{-1}, \quad (4.11)$$

where

$$\begin{aligned} B = & \left[(2\beta^2 C_R)^2 \left(\left\{ 1 - \frac{N_T}{N} - \frac{2}{N_T} \left(\frac{T_1}{T_r} \right) \right. \right. \right. \\ & \times \left. \left. \left. \left[1 + 2 \left(\frac{T_1}{T_r} \right) \left(\frac{N_T}{N} - 1 \right) \right] \right\}^2 \right. \right. \\ & \left. \left. + \frac{8}{N_T} \left(\frac{2T_1}{T_r} \right) \left[1 + 2 \left(\frac{T_1}{T_r} \right) \left(\frac{N_T}{N} - 1 \right) \right] \right) \right]^{1/2} \\ & \approx (2/T_r)(1/N_T)(N - N_T). \end{aligned}$$

In the SCFA of I we needed an initial field to start the approach to the steady state. Once a small field appeared the rate of approach was the same as the approximate value of B in Eq. (4.11). Here, however, spontaneous emission provides the trigger and the only condition is that N be greater than N_T . When we compare B with A , the rate of approach to the steady state in the absence of dissipation, we find $B = (N - N_T)A$. Thus, for $N \sim 0.1N_T$, the rate of approach to the steady state is about 10^6 times faster in the presence of dissipation. When we substitute Eq. (4.11) into Eqs. (4.5a) and (4.5b), the coupled density-matrix equations uncouple and R and ρ satisfy linear differential equations with time-dependent coefficients. For $t > (\omega_D B)^{-1}$ the time-dependent coefficients achieve their stationary values and R and ρ satisfy uncoupled linear equations with constant coefficients. When N is about equal to $0.1N_T$ the quantity $(\omega_D B)^{-1}$ is about 10^{-5} sec.

V. HIGHER ORDER TERMS

We now show one of the advantages of the density-matrix method of this paper over the operator equations of motion approach^{1,3,4} by discussing qualitatively the contribution of higher order terms to $(\partial R/\partial t)$. Since β^2 is so small for gas lasers, it would seem that there would be little need to calculate the β^4 terms of the equations of motion for R . However, the Lamb dip which Lamb predicted and which has been observed experimentally requires knowledge of higher order dynamically induced correlations.

The fourth-order contribution to $(\partial R/\partial t)$ consists of many terms. Each term consists of a product of R with four radiation creation and annihilation operators multiplied by the sums of average values of four polarization operators which in turn are multiplied by a threefold time integral over the center-of-mass coordinates. A typical contribution to $(\partial R/\partial t)$ in fourth order is

$$(\gamma^4 D \omega_D^{-4}) a^\dagger a R a^\dagger a \sum_\alpha \sum_\beta \langle \sigma_\alpha^\dagger \sigma_\alpha \sigma_\beta^\dagger \sigma_\beta \rangle, \quad (5.1)$$

where

$$\begin{aligned} D \equiv & \int_0^\infty d\tau_3 \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 e^{i\bar{\Delta}\tau_3} e^{i\bar{\Delta}(\tau_2 - \tau_1)} \\ & \times \{ \Gamma[X_\alpha(\tau_3)] \Gamma[X_\beta(\tau_2)] \Gamma[X_\beta(\tau_1)] \Gamma[X_\alpha(0)] \}_{\text{av}} \end{aligned}$$

and in D the average is over the center-of-mass distribution function $\mathfrak{F}_1(X, V)$ as in Eq. (2.10). The other terms to fourth order have the same structure and differ only in the order of operators.

When we separate the $\alpha=\beta$ term of Eq. (5.1) from the rest of the sum, we obtain

$$\begin{aligned} & (\tilde{\gamma}^4 \omega_D^{-4} D) a^\dagger a R a^\dagger a [N(N-1) \langle \sigma_\alpha^\dagger \sigma_\alpha \sigma_\beta^\dagger \sigma_\beta \rangle \\ & \quad + N \langle \sigma_\alpha^\dagger \sigma_\alpha \sigma_\alpha^\dagger \sigma_\alpha \rangle] \\ & = (\tilde{\gamma}^4 \omega_D^{-4} D) a^\dagger a R a^\dagger a [N^2 \langle \sigma_\alpha^\dagger \sigma_\alpha \rangle \langle \sigma_\beta^\dagger \sigma_\beta \rangle + N \langle \sigma_\alpha^\dagger \sigma_\alpha \rangle] \\ & = (\tilde{\gamma}^4 \omega_D^{-4} D) a^\dagger a R a^\dagger a [N^2 n_+^2 + N n_+], \end{aligned} \quad (5.2)$$

where we use $\sigma_\alpha^\dagger \sigma_\alpha \sigma_\beta^\dagger \sigma_\beta = \sigma_\alpha^\dagger \sigma_\alpha$. The breakup of the term $\langle \sigma_\alpha^\dagger \sigma_\alpha \sigma_\beta^\dagger \sigma_\beta \rangle$ follows from the Bogoliubov² expansion procedure for the kinetic equation where $\rho_2(\alpha, \beta) = \rho_1(\alpha) \rho_1(\beta)$. In a laser N is large so the second term in square brackets is usually negligible compared with the first term. This suggests that β^2 is the correct expansion parameter.

The $\tilde{\gamma}^4 N^2$ terms arise when two different particles within a wavelength of light apart exchange a virtual photon in a time small compared with ω_D^{-1} which is the time an atom with the average thermal velocity takes

to move a distance equal to a wavelength of light. The $\tilde{\gamma}^4 N$ terms represent a second-Born-approximation scattering between a single particle and the radiation field. The second-Born-approximation terms are proportional to $\tilde{\gamma}^4 N$ instead of $\tilde{\gamma}^4 N^2$ because the fundamental process involves a single particle instead of a pair of particles.

In a second paper we explicitly evaluate and sum the $\tilde{\gamma}^4$ terms in $(\partial R / \partial t)$ and the coupled $(\partial \rho / \partial t)$. The explanation of the Lamb dip and the steady state to order $\tilde{\gamma}^4$ suggested by this section differs fundamentally from Lamb's³ explanation. Lamb retains only the $\tilde{\gamma}^4 N$ term because he follows the detailed behavior of a single particle. He does not obtain the $\tilde{\gamma}^4 N^2$ terms because his theory starts with average values that preclude the development of the dynamically induced particle-particle correlations which give rise to the $\tilde{\gamma}^4 N^2$ terms.

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Erratum

Theory of Superconductors Containing Magnetic Impurities, PETER FULDE AND KAZUMI MAKI [Phys. Rev. **141**, 275 (1966)]. The $\kappa_2(t)$ parameter used in this paper is based on the original theory of type-II superconductors given by one of the authors (K.M.).¹ This theory has recently been corrected by Caroli, Cyrot, and de Gennes.² The $\kappa_2(t)$ parameter in this revised formulation depends not only on temperature but also on the concentration of the magnetic impurities. The explicit form of $\kappa_2(t)$ is given by¹

$$\kappa_2(t) = \frac{\pi^2}{2[7\zeta(3)]^{1/2}} f(\rho)^{1/2} g(\rho)^{-1}, \quad (1)$$

where

$$f(\rho) = \sum_{n=0}^{\infty} \left\{ \frac{1}{(n+\frac{1}{2}+\rho)^3} - \rho_i \frac{1}{(n+\frac{1}{2}+\rho)^4} \right\}, \quad (2)$$

$$g(\rho) = \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2}+\rho)^2}, \quad (3)$$

and

¹ K. Maki, Physics **1**, 21 (1964).

² C. Caroli, M. Cyrot, and P. G. de Gennes (unpublished).

$$\rho = \rho_i + \rho_H, \quad \rho_i = \frac{1}{2\pi\tau_s T}, \quad \text{and} \quad \rho_H = \frac{\tau_{tr} v^2 e H_{c2}}{6\pi T}.$$

The temperature dependence of ρ is determined by the equation

$$\ln \frac{T}{T_{c0}} + \psi\left(\frac{1}{2} + \rho\right) - \psi\left(\frac{1}{2}\right) = 0, \quad (4)$$

where $\psi(z)$ is the digamma function.

From the above expressions, we see that $\kappa_2(t)$ depends on the concentration of magnetic impurities through parameter ρ_i . [The effect of magnetic impurity and the external magnetic field is not additive as is seen from Eq. (2).] Therefore, our conclusion that the jump in the specific heat at the transition from the mixed state to the normal state is a function of temperature only, is incorrect. The effects of the magnetic impurity and the magnetic field on the jump of specific heat is different, though this difference is appreciable only at low temperatures. (The magnetic impurity gives rise to a larger jump in the specific heat than the magnetic field.)