The correlation, shown in Fig. 2, between the data for $(RE)^{3+}$ ions in CaF_2 and the calculated free ion data, decreased by 18 000 cm^{-1}, suggests that the variation of the location of the lowest $4f \rightarrow 5d$ transition of $(RE)^{3+}$ ions in crystals[11] containing anions of large electronegativity is simply like that of free ions. This correlation between the free ion and ion-in-crystal data may be useful in predicting the location of the lowest $4f^{n-1}\,5d$ level of free $(RE)^{3+}$ ions not yet observed experimentally.

[11] Our preliminary data on $(RE)^{3+}$ in SrF_2 and BaF_2 crystals are similar to those in CaF_2 with corresponding reduction in energy ~17 000 cm^{-1} for host SrF_2 and ~16 000 cm^{-1} for host BaF_2. For the host LaF_3 crystals, the present progress in purifying the crystal and hence in extending its transparent region toward vacuum ultraviolet will allow a similar measurement in the near future [Hugh M. Muir and Wm. Stein in a paper presented at Fifth Rare-Earth Research Conference, Ames, Iowa, 1965 (unpublished); M. Robinson and D. M. Cripe (to be published)].

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Simple Model for Collision Effects in Photoemission

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We treat the problem of energy loss to phonons in photoemission by making a one-dimensional approximation. We include pair creation and finite electron reflectivity at the surface. It is assumed that after pair creation neither primary nor secondary electrons can escape. Phonon scattering, pair creation, and reflectivity are assumed independent of electron energy, although this will often be a poor approximation. With the above approximations the problem is exactly soluble. For large phonon energy losses the results are similar to the predictions of age theory.

I. INTRODUCTION

PHOTOELECTRIC emission is potentially a very valuable tool for the study of the band structure of solids. The potential value of photoemission results from the richness of the measurements which are possible. Listed in order of progressive refinement these are total yield versus photon energy[1]; energy distribution of the yield at fixed photon energy[2]; and energy distribution of the yield at fixed photon energy and angle of emission.[3]

The above information is most readily interpretable in cases where the emitted electrons have suffered very little or no energy loss in the process of escaping from the solid.

In order to obtain the most reliable interpretation of the experimental data, however, a detailed knowledge of the energy-loss processes is necessary.

In the present paper we study a simple one-dimensional model in which we can allow for the effects of pair-creation scattering, phonon scattering, and finite reflection at the surface in a way which leads to an exactly soluble result for the total probability of escape and for the energy distribution of the emitted electrons assuming a delta-function initial energy.

We assume throughout that if an electron undergoes pair creation, neither the scattered primary nor the secondaries can escape. Clearly this assumption is valid only for electrons within a few volts of the vacuum potential.

The three-dimensional case has been studied in considerable detail by Baraff.[4] He has shown that simple results for the energy-distribution function of degraded electrons may be obtained using the approximations of neutron age theory. The age-theory approximation is good when phonon energy losses are large so that the energy-distribution function can be assumed to be a continuous function of the number of phonon-scattering events. Age theory also makes simplifying assumptions about the angular dependence of the distribution function. Naturally, our one-dimensional approximation amounts to an even worse approximation for the angular

[1] F. G. Allen and G. W. Gobeli, Phys. Rev. **144**, 558 (1966); M. L. Cohen and J. C. Phillips, *ibid.* **139**, A912 (1965); D. Brust, M. L. Cohen, and J. C. Phillips, Phys. Rev. Letters **9**, 389 (1962); W. E. Spicer and R. E. Simon, *ibid.* **9**, 389 (1962).

[2] C. N. Berglund and W. E. Spicer, Phys. Rev. **136**, A1044 (1964); D. Brust, *ibid.* **139**, A489 (1965).

[3] G. W. Gobeli, F. G. Allen, and E. O. Kane, Phys. Rev. Letters **12**, 94 (1964); E. O. Kane, *ibid.* **12**, 97 (1964).

[4] G. A. Baraff, Phys. Rev. **135**, A528 (1964).

distribution function. The energy-loss problem is treated exactly, however. Also, we are able to include the effects of reflection at the surface very easily and completely by our method.

The one-dimensional approximation amounts to the use of an average angle of projection of the electron's trajectory on the surface normal. The relation between the three-dimensional mean free path l_3 and the effective one-dimensional mean free path l_1 is then $l_1=\langle\cos\theta\rangle l_3$ where $\langle\cos\theta\rangle$ is the averaged angle of projection of the path direction on the surface normal. If we take a straight average over a half-sphere, $\langle\cos\theta\rangle=\frac{1}{2}$ and $l_3=2l_1$. Comparison with neutron age theory indicates that the appropriate average in this case yields $l_3=\sqrt{3}l_1$.

Actually, the most important practical limitation of our model is not the one-dimensional aspect but the necessity of using energy-independent parameters. We can, of course, allow the parameters to be a function of the initial energy but we cannot take into account the change in the parameters as the electron's energy is degraded by the emission of phonons.

When the pair-creation length is long compared to the phonon-scattering length, energy loss due to phonons is important and the energy independence of the parameters of our theory may lead to considerable error in cases of practical interest. In the opposite limit of pair-creation length short compared to the phonon-scattering length, phonon energy losses are negligible. In this case the escape-probability calculation simply corresponds to the calculation of the probability that the electron reach the surface from its point of origin and escape without undergoing any scattering at all. This problem is easily solved in the three-dimensional case and is discussed in Sec. III.

For the purpose of calculating escape probabilities in cases of practical interest, the Monte Carlo method of Stuart, Wooten, and Spicer[5] is easily adapted to include both three-dimensional effects and the energy dependence of the scattering parameters. The simple analytic results of the one-dimensional model are valuable mainly to guide and to interpret the machine calculations.

II. ONE-DIMENSIONAL MODEL

A. Escape Probability

In our model we introduce a pair-creation length l_{pair}; a phonon-scattering length l_{phonon}; a probability R of being reflected from the surface; and a light-absorption length l_α for the initial spatial distribution of the photoelectrons.

We assume that an electron which has created a pair cannot escape (nor can the secondary). We further assume that the parameters of our theory are independent of the phonon energy loss.

The probability $p(x)$ that an electron should travel a distance x without phonon or pair scattering and then phonon scatter in the interval dx is

$$\begin{aligned} p(x)&=ae^{-cx}dx,\\ a&\equiv l_{\text{phonon}}^{-1},\\ b&\equiv l_{\text{pair}}^{-1},\\ c&\equiv a+b, \end{aligned} \tag{1}$$

where c is the reciprocal scattering length.

We take the surface as our origin with positive x being in the solid.

The probability, $p_0(x)$, that an electron created at x should travel to the surface and escape without scattering is given by

$$p_0(x)=(1-R)\tfrac{1}{2}e^{-cx} \tag{2}$$

R is the probability that the electron is reflected back into the solid upon reaching the surface. The factor $\frac{1}{2}$ reflects the fact that the electron can be going either towards the surface or away from it when it is created.

To obtain the probability, p_n, that an electron escape from the surface after suffering exactly n phonon scatterings, we derive a recursive integral equation by following the electron from the moment of creation to its first collision. After the first collision, the probability of escape is given by p_{n-1} which is assumed known. We find

$$p_n(x)=\frac{a}{2}\left\{\int_0^x p_{n-1}(y)e^{-c(x-y)}dy+R\int_0^\infty p_{n-1}(y)e^{-c(x+y)}dy+\int_x^\infty p_{n-1}(y)e^{-c(y-x)}dy\right\}. \tag{3}$$

In the first term on the right-hand side of Eq. (3) the electron created at x has moved a distance $x-y$ toward the surface before scattering at the point y. In the second term the electron has reached the surface and has been reflected with probability R before scattering at y, having moved a distance $x+y$ after being created. In the third term the electron was initially headed away from the surface and traveled a distance $y-x$ before scattering at the point, y.

We now introduce $q(x)$, the total probability of escaping from the surface:

$$q(x)=\sum_{n=0}^{\infty} p_n(x). \tag{4}$$

By summing Eq. (3) from $n=1$ to ∞ we obtain an integral equation for $q(x)$:

$$q(x)=\frac{a}{2}\left\{\int_0^x q(y)e^{-c(x-y)}dy+R\int_0^\infty q(y)e^{-c(x+y)}dy+\int_x^\infty q(y)e^{-c(y-x)}dy\right\}+p_0(x). \tag{5}$$

[5] R. Stuart, F. Wooten, and W. E. Spicer, Phys. Rev. 135, A495 (1964).

By differentiating Eq. (5) twice we easily obtain the differential equation

$$q''(x)=(c^2-ac)q. \tag{6}$$

Since $c>a$ according to Eq. (1), the physical solution of (6) for a semi-infinite medium is

$$\begin{aligned} q(x)&=Be^{-\mu x},\\ \mu&=(c^2-ac)^{1/2}; \end{aligned} \tag{7}$$

μ is evidently a reciprocal escape length. The coefficient B may be evaluated by substitution in Eq. (5). We find

$$B=\frac{(1-R)c}{(1-R)c+\mu(1+R)}. \tag{8}$$

According to Eq. (7), B is the probability that an electron created at the surface will escape. Equation (7) shows that $\mu\leq c$. The limit $\mu\approx c$ occurs when $a\ll b$ so that phonon scattering is unlikely compared to pair creation. In this limit $B\simeq(1-R)/2$. The electron gets out if it is headed in the right direction and is not reflected, otherwise it is lost. In the limit $b\ll a$ we have $\mu\ll c$ and $B\simeq 1$. Since pair creation is unlikely, the electron is almost certain to escape eventually.

We assume that the exciting light I is absorbed according to

$$I=I_0e^{-\alpha x}. \tag{9}$$

We can then use Eq. (7) to give, Q, the average probability of escape of an excited electron

$$Q=(B\alpha)/(\alpha+\mu), \tag{10}$$

with B given by Eq. (8) and μ by Eq. (7). An equation similar to (10) has been given by Spicer.[6] A more realistic expression for Q is obtained if we allow for the energy dependence of the parameters as described in footnote.[7] If α is large, the electrons are created very near the surface and the probability of escape is B. If α is small compared to μ, the probability of escape is proportional to the escape length, μ^{-1}.

It is extremely fortunate that one can include the effects of light absorption, phonon scattering, pair creation, and reflection at the surface in a formula as simple as Eq. (10).

B. Phonon Energy Loss

If we wish to know the phonon energy loss we must count the number of phonon scattering events.

Equation (3) shows that $p_n(x)$ is proportional to a^n providing we regard c as a parameter independent of a. If we then expand Q as a power series in a:

$$Q=\sum_{n=0}^{\infty} g_n a^n, \tag{11}$$

we have that g_na^n is the probability that the electron escape after suffering exactly n phonon collisions.

If we average over phonon absorption and emission events the average energy loss, $E_{\rm ph}$, per scattering event is

$$E_{\rm ph}=\left\langle\frac{\hbar\omega_{\rm ph}}{2n_{\rm ph}(\omega)+1}\right\rangle, \tag{12}$$

where $n_{\rm ph}(\omega)$ is the number of phonons excited and the average is over all phonon modes. The average total phonon energy loss is $\bar{n}E_{\rm ph}$, with $\bar{n}$ given by

$$\begin{aligned} \bar{n}&=\sum_{n=0}^{\infty} ng_na^n/\sum_{n=0}^{\infty} g_na^n,\\ \bar{n}&=\frac{ada/da}{Q}. \end{aligned} \tag{13}$$

Evaluating Eq. (13) we get

$$\bar{n}=\frac{ca}{2\mu}\left\{\frac{1+R}{(1-R)c+(1+R)\mu}+\frac{1}{\alpha+\mu}\right\}. \tag{14}$$

In the limit of phonon scattering weak compared to pair production ($a\ll b$) we have ($\mu\simeq c\simeq b$)

$$n\simeq\frac{a}{2}\left\{\frac{1+R}{2b}+\frac{1}{\alpha+b}\right\}; \tag{15}$$

hence $\bar{n}\ll 1$. Energy loss due to phonons is obviously negligible in this limit. In the opposite limit, $a\gg b$, where pair creation is weak compared to phonon scattering $c\simeq a$, $\mu\simeq(ab)^{1/2}$

$$\bar{n}\simeq\frac{1}{2}\left(\frac{a}{b}\right)^{1/2}\left\{\frac{1+R}{(1-R)+(1+R)(b/a)^{1/2}}+\frac{a}{\alpha+(ab)^{1/2}}\right\}. \tag{16}$$

In this limit phonon energy losses are large, $\bar{n}\sim\frac{1}{2}(a/b)^{1/2}$ at least. For $1-R<(b/a)^{1/2}$ or $\alpha<(ab)^{1/2}$ the losses are even greater, $\bar{n}\sim a/b$. A high reflectance or a low α (long photon-absorption length) always increases the loss as can be seen in general from Eq. (14). This result is obvious on physical grounds.

C. Phonon Energy Loss Distribution

In cases where the phonon energy loss is large, we are interested in knowing the distribution function for the energy loss. We expect that the energy distribution of the emitted electrons will be very different from the initial distribution in which they were created. The

[6] W. E. Spicer, Phys. Rev. **112**, 114 (1958).

[7] If we wished to take account of the energy dependence of the collision parameters somewhat crudely we would integrate over the distribution of initial energies $\alpha(E)$ produced by the light. Eq. (12) would then become $Q=\int dEB(E)\alpha(E)/[\alpha_0+\mu(E)]$; $\alpha_0=\int dE\alpha(E)$. This equation is, of course, not consistent with the neglect of the energy dependence of the collision parameters during escape as the electron energy is degraded by phonon emission. In the case of small phonon energy losses it is approximately correct.

probability P_n of suffering n phonon collisions is given by

$$P_n=g_n a^n, \tag{17}$$

as shown in Eq. (11). We may also write

$$g_n=(n!)^{-1}(d^nQ/da^n)_{a=0}. \tag{18}$$

In order to calculate the derivatives we write Q in the form

$$\begin{aligned}&Q=(\rho\alpha')/(\alpha'-\rho)[((1-a')^{1/2}+\rho)^{-1}\\&\qquad-((1-a')^{1/2}+\alpha')^{-1}],\\&\alpha'\equiv\alpha/c,\\&a'\equiv a/c,\\&\rho\equiv(1-R)/(1+R).\end{aligned} \tag{19}$$

We consider

$$f_n\equiv(n!)^{-1}(d^n)/(d(a')^n)((1-a')^{1/2}+\rho)^{-1};\quad n=1,2,\cdots. \tag{20}$$

It is easily shown by induction that f_n is of the form

$$f_n=\sum_{j=2}^{n+1}\beta_j{}^n(1-a')^{-(2n-j+1)/2}\{(1-a')^{1/2}+\rho\}^{-j};\quad n\geq 1. \tag{21}$$

We can differentiate Eq. (21) to determine recursion relations for the coefficients

$$\beta_j{}^n=\frac{1}{2n}[(2n-j-1)\beta_j{}^{n-1}+(j-1)\beta_{j-1}{}^{n-1}],\quad n\geq 2;\ 2<j<n+1; \tag{22}$$

$$\beta_2{}^n=[(2n-3)/2n]\beta_2{}^{n-1} \tag{23}$$

$$\beta_{n+1}{}^n=\tfrac{1}{2}\beta_n{}^{n-1} \tag{24}$$

$$\beta_2{}^1=\tfrac{1}{2}. \tag{25}$$

We have determined by a detailed analysis of the recursion relations that they have the following solution:

$$\beta_j{}^n=\frac{(j-1)(2n-j)!}{n!(n-j+1)!2^{2n-j+1}},\quad n\geq 1,\quad 2\leq j\leq n+1. \tag{26}$$

The validity of this relation is easily verified by substitution in Eqs. (22), (23), and (24). The uniqueness of the solution follows by mathematical induction.

With the use of Eqs. (17) through (21) we can write P_n in the form

$$\begin{aligned}&P_0=\left(\frac{1-R}{2}\right)\frac{\alpha}{\alpha+c},\\&P_n=\left(\frac{a}{c}\right)^n\rho\alpha'\sum_{j=2}^{n+1}\beta_j{}^n h_j;\quad n\geq 1,\\&h_j\equiv(\alpha'-\rho)^{-1}\{((1+\rho)^j)^{-1}-((1+\alpha')^j)^{-1}\}.\end{aligned} \tag{27}$$

To show the positive definite character of h_j we can write it in the form

$$h_j=\sum_{t=0}^{j-1}\frac{(1+\rho)^t(1+\alpha')^{j-1-t}}{(1+\rho)^j(1+\alpha')^j}. \tag{28}$$

For $\rho=\alpha'$; h_j becomes

$$h_j=j(1+\rho)^{-j-1}. \tag{29}$$

When n is small, the summations we have given are a useful description of the energy-loss distribution.

When n is large, the important contributions to P_n come from values of j for which $j\ll n$. Accordingly, we examine this case by expressing the factorials in Eq. (26) by Stirling's approximation and further by taking $j\ll n$. We obtain

$$\beta_j{}^n\simeq\frac{(j-1)e^{-j^2/4n}}{2\pi^{1/2}n^{3/2}},\quad n\gg 1,\ j. \tag{30}$$

Equation (30) shows that the important values of j are those for which $j\lesssim 2\sqrt{n}$, which demonstrates the consistency of assuming $j\ll n$ for large n.

Equation (27) shows that the large n limit is of interest only for $a/c\simeq 1$, otherwise P_n is negligible. Using Eq. (1) for c and assuming $b\ll a$, we may write

$$(a/c)^n\simeq e^{-bn/a}. \tag{31}$$

Collecting these results we may write P_n in the large n limit as

$$P_n\simeq\frac{e^{-bn/a}\rho\alpha'}{2\pi^{1/2}n^{3/2}}\sum_{j=2}^{\infty}(j-1)e^{-j^2/4n}h_j,\quad n\gg 1,\ j, \tag{32}$$

with h_j given by Eq. (27).

Let $\{\rho,\alpha'\}_{\min}$ be the smaller of the quantities ρ and α'. Equations (27) and (40) show that the j dependence of h_j is governed by $\{\rho,\alpha'\}_{\min}$. If $\{\rho,\alpha'\}_{\min}\geq 1$, only the very low values of j are important and the summation as given in (32) is useful. Equation (32) shows that P_n varies with n as $e^{-bn/a}/n^{3/2}$ in this case.

If $\{\rho,\alpha'\}_{\min}\ll 1$, a much wider range of j values contribute in Eq. (32). We may then approximate h_j in Eq. (27) by

$$h_j=(\alpha'-\rho)^{-1}[e^{-\rho j}-e^{-\alpha' j}],\quad \{\rho,\alpha'\}_{\min}\ll 1. \tag{33}$$

In the case $\{\rho,\alpha'\}_{\min}\ll 1$ we may replace the sum in Eq. (32) by an integral since the main contribution comes from large j values:

$$P_n=\frac{e^{-bn/a}}{2\pi^{1/2}n^{3/2}}\left(\frac{\rho\alpha'}{\alpha'-\rho}\right)\int_0^{\infty}xe^{-x^2/4n}(e^{-\rho x}-e^{-\alpha' x})dx,\quad \{\rho,\alpha'\}_{\min}\ll 1,\quad n\gg 1. \tag{34}$$

The integral in Eq. (34) may be expressed in terms of

the conjugate error function, Erfc, as follows:

$$P_n=\frac{e^{-bn/a}}{2\pi^{1/2}n^{3/2}}\frac{(\rho\alpha')}{(\alpha'-\rho)}\{F(\rho)-F(\alpha')\},$$
$$\{\rho,\alpha'\}_{\min}\ll 1,\quad n\gg 1; \tag{35}$$
$$F(\rho)=2n-4n^{3/2}\rho e^{n\rho^2}\,\mathrm{Erfc}(n^{1/2}\rho);$$
$$\mathrm{Erfc}(x)\equiv\int_x^\infty e^{-x^2}dx.$$

Equation (35) is not particularly transparent. We illustrate its behavior by considering limiting cases which are most easily obtained from Eq. (34):

$$p_n=e^{-bn/a}\rho\alpha', \qquad \rho\sqrt{n}\ll 1,\ \alpha'\sqrt{n}\ll 1; \tag{36}$$

$$P_n=(e^{-bn/a}\rho)/(2n^{1/2}), \qquad \rho\sqrt{n}\ll 1,\ \alpha'\sqrt{n}\gg 1, \tag{37}$$

$$P_n=(e^{-bn/a}\alpha')/(2n^{1/2}), \qquad \rho\sqrt{n}\gg 1,\ \alpha'\sqrt{n}\ll 1; \tag{38}$$

$$P_n=(e^{-bn/a})/(2\pi^{1/2}n^{3/2})(\rho^{-1}+\alpha'^{-1}), \qquad \rho\sqrt{n}\gg 1,\ \alpha'\sqrt{n}\gg 1,\ \{\rho,\alpha'\}_{\min}\ll 1. \tag{39}$$

For comparison we also give the leading term of Eq. (32):

$$P_n=\frac{e^{-bn/a}}{2\pi^{1/2}n^{3/2}}(\rho^{-1}+\alpha'^{-1}), \qquad \alpha'\gg 1,\ \rho\gg 1. \tag{40}$$

Equations (39) and (40) are identical, hence the restriction on Eq. (39), $\{\rho,\alpha'\}_{\min}\ll 1$, is probably not too important.

Equations (36) through (40) are all dominated by the exponential decay term which is governed by b/a, the probability of pair creation relative to phonon scattering.

The weaker power-law dependences on n can be seen to have their origin in the following effects.

In the process of emitting n phonons an electron will diffuse an average distance, $l_D=n^{1/2}/c$.

If the barrier reflection R is near 1, ($\rho\ll 1$), few electrons will be lost through the surface. If $\alpha^{-1}\gg l_D$ ($\alpha'\ll 1$), the initial distribution will not be perturbed by diffusion over the time in which n phonons are emitted. Hence P_n will have no additional dependence on n in agreement with Eq. (36).

If $\rho\ll 1$ and $\alpha^{-1}\gg l_D$, few electrons will be lost through the surface but the initial distribution will be broadened by diffusion so that the electron concentration near the surface will drop like $1/\sqrt{n}$. This explains the additional n dependence in Eq. (37).

If ρ is large ($R\sim 0$), electrons will be lost rapidly through the surface. These electrons are replaced by diffusion from the nearly uniform initial distribution ($\alpha'\ll 1$) to give the $n^{-1/2}$ dependence in Eq. (38).

In Eq. (39) the initial electron distribution is localized near the surface ($\alpha'\sqrt{n}\gg 1$) and the surface reflection is low. The loss of electrons from the surface is most rapid in this case, leading to an $n^{-3/2}$ additional n dependence in P_n.

III. THREE-DIMENSIONAL RESULT FOR THE LIMIT $a\ll b$

If the phonon mean free path a^{-1} is very long compared to the pair creation length, b^{-1} the probability of escape, Q, is the probability that the electron reach the surface without scattering and escape without being reflected. This simple limiting case can easily be treated in three dimensions. We have

$$Q=\frac{\alpha(1-R)}{2}\int_0^\infty dx\int_0^1 d\cos\theta\, e^{-\alpha x}e^{-c_3x/\cos\theta} \tag{41}$$

θ is the angle of the electron's direction measured from the surface normal. We write c_3 to indicate that we are here dealing with the true three-dimensional mean free paths. Equation (41) can be integrated to give

$$Q=\tfrac{1}{2}\alpha(1-R)\{1-(c_3/\alpha)\ln(1+(\alpha/c_3))\}. \tag{42}$$

Evidently no simple correspondence can be made between Eqs. (42) and (10) in general. In the limiting case $c_3\ll\alpha$, i.e., escape length long compared to the light absorption length, Eq. (42) becomes

$$Q\simeq\tfrac{1}{2}(1-R),\quad c_3\ll\alpha. \tag{43}$$

Since the electron has no chance of being "turned around" by phonon scattering, the escape probability is the probability $\frac{1}{2}$ that the electron is initially headed toward the surface times the probability $(1-R)$ that it will pass through the surface without scattering. This result agrees with Eq. (10) on setting $c=\mu\ll\alpha$.

In the opposite limit, $c_3\gg\alpha$, Eq. (42) becomes

$$Q=\tfrac{1}{2}(1-R)(\alpha/2c_3),\quad c_3\gg\alpha. \tag{44}$$

This result agrees with Eq. (10) on setting $c=\mu=2c_3$. This is just the case $l_3=2l_1$ which results using the uniform average of $\cos\theta$ to relate l_3 and l_1 through $l_1=\langle\cos\theta\rangle l_3$.

One can consistently introduce energy-dependent parameters into Eqs. (41) and (42), obtaining

$$Q=\frac{1}{2}\int dE\alpha(E)(1-R(E))\times\left\{1-\frac{c_3(E)}{\alpha_0}\ln\left(1+\frac{\alpha_0}{c_3(E)}\right)\right\}, \tag{45}$$
$$\alpha_0=\int\alpha(E)dE, \tag{46}$$

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