Diffusion is certainly present in the experiments. It influences T_c and $H_c(T)$ in as much as it changes the electronic mean free path.

The critical fields of multiple films agree fairly well with the following model: Near H_c the gap function $\Delta(\mathbf{r})$ as well as the magnetization are negligibly small in the normal side of the film.

The multiple film then behaves like a single film, of thickness d_s, of the superconducting part alone with the value of the electronic specific heat approximately that of the pure metal, the critical temperature T_c of the multiple film, and an electronic mean free path l of the multiple film.

The fact that d_s and not d_n+d_s is important is in agreement with measurements of the microwave absorption in gold films plated onto bulk tin.[27,28]

[27] H. Meissner and R. V. Fanelli, Rev. Mod. Phys. 36, 194 (1964).

[28] R. Fanelli and H. Meissner, Phys. Rev. 147, 227 (1966).

ACKNOWLEDGMENTS

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Nonlinear Self-Coupling of Josephson Radiation in Superconducting Tunnel Junctions

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A study is made of the nonlinear feedback of the ac tunnelling current (first pointed out by Josephson) in a superconducting tunnel junction via the electromagnetic fields radiated by the current into the resonant-mode structure of the junction. The junction operating point is investigated, and a calculation is made of the dc current drawn by a strongly self-coupled fundamental mode together with weakly self-coupled harmonic modes. Attention is directed to the frequency dependence of the tunneling-current amplitudes reported briefly by Riedel, as a tentative explanation of the bumps in current-voltage characteristics which have been observed at $eV=2\Delta/$(integer), where 2Δ is the energy gap.

I. INTRODUCTION

SINCE the original prediction by Josephson[1] of an ac supercurrent generated in a superconducting tunnel junction by a dc bias voltage, considerable interest has developed in the electromagnetic radiation associated with the ac current. The study of the characteristics of the Josephson junction and detection of the radiation it produces are complicated by the high degree of nonlinearity of the device together with the large amount of feedback intrinsic to its configuration.

Several excellent reviews of the Josephson effect already have appeared,[2–4] describing the foundations of the theory and pointing out many of the widely varying features expected from a device of considerable intricacy and with a number of variable parameters. One purpose of the present article is to re-examine in detail the derivation and solutions of the equations representing the Josephson junction in order to clarify and improve the simplifying assumptions made in earlier work. This study should be relevant to possible applications of the Josephson junction as a source of radiation. In addition, attention is given to the new feature in tunneling with radiation present first pointed out briefly by Riedel.[5] He showed that the pair-tunneling current has a singularity when the applied dc voltage plus a harmonic of the radiation frequency equals the gap in the energy spectrum of the superconductors. We develop Riedel's result by showing that the single-particle tunneling current also has a similar behavior,

[1] B. D. Josephson, Phys. Letters 1, 251 (1962).

[2] P. W. Anderson, in *Lectures on the Many Body Problem*, edited by E. R. Caianiello (Academic Press Inc., New York, 1964), Vol. 2.

[3] B. D. Josephson, Rev. Mod. Phys. 36, 216 (1964); Advan. Phys. 14, 419 (1965).

[4] D. N. Langenberg, D. J. Scalapino, and B. N. Taylor, Proc. IEEE 54, 5690 (1966).

[5] E. Riedel, Z. Naturforsch. 19A, 1634 (1964).

and we extend the evaluation to the case of nonidentical superconductors on the two sides of the junction. The effect can then be shown to disappear when one of the metals is not superconducting. We explore whether these features can account for observations previously reported, but unexplained.

In the following Sec. II, we develop in a careful manner the general expression for the total tunneling current in the presence of arbitrary electromagnetic fields. Certain integrals dependent on the superconductors forming the junction are evaluated explicitly at zero temperature, and their singularities are discussed. In Sec. III, we review how the ac tunneling currents radiate into the electromagnetic mode structure of the junction. This forms the basis for the study in Sec. IV of the self-coupling feedback of the junction, regarded as an oscillator. As simplified examples we determine the operating point in the absence of harmonics, and then the harmonic generation when assumed to be a weak perturbation. Section V examines some consequences of the singularity in the current amplitude.

II. THE CURRENT AMPLITUDES

The first half of the theoretical description of the self-coupled junction is the constitutive relation, expressing the tunneling current in terms of the total time-dependent voltage across the junction. We adopt as our starting point the linear-response calculation of Ambegaokar and Baratoff,[6] which in turn is based on the effective tunneling Hamiltonian. This means that we regard the junction as composed of two bulk superconductors connected only weakly by an insulating layer. The insulating layer clearly must be thin enough so that an observable tunneling current can flow, but is assumed to be sufficiently thick that the tunneling process can be treated by lowest order perturbation theory. For very thin oxide layers, the two superconductors become more strongly coupled, so that lifetime effects and vertex corrections become significant.[7] Although they are especially important for junctions where a metallic short through the insulator exists, and may prove to be a highly significant theoretical approach to weak but completely metallic bridges between two superconductors, we will not consider such developments here. As a corollary, we assume that the only shielding currents which oppose the uniform entry of a magnetic field into the insulating layer are the weak tunneling currents themselves. In the case of metallic bridges, the shielding currents through the bridge can approach in magnitude the Meissner currents of the bulk superconductors, and the net field distribution will be highly nonuniform.

We begin with the equations of Ambegaokar and Baratoff[6] for the tunneling current density $\mathcal{J}(t)$ as a function of time:

$$\mathcal{J}(t)=-2e\,\mathrm{Re}\sum_{\mathbf{k}\mathbf{q}\sigma}\int_{-\infty}^{t}d\bar{t}\,\exp\eta\bar{t}\times\{|T_{\mathbf{k},\mathbf{q}}|^2[G_{\mathbf{k}}^{<}(\bar{t},t)G_{\mathbf{q}}^{>}(t,\bar{t})-G_{\mathbf{k}}^{>}(\bar{t},t)G_{\mathbf{q}}^{<}(t,\bar{t})]+T_{\mathbf{k},\mathbf{q}}T_{-\mathbf{k},-\mathbf{q}}[\bar{F}_{\mathbf{k}}^{>}(t,\bar{t})F_{\mathbf{q}}^{<}(\bar{t},t)-\bar{F}_{\mathbf{k}}^{<}(t,\bar{t})F_{\mathbf{q}}^{>}(\bar{t},t)]\}\,. \quad (1)$$

Here $T_{\mathbf{k},\mathbf{q}}$ is a tunneling matrix element connecting plane-wave state $\mathbf{k}$ on one side of the insulator with plane-wave state $\mathbf{q}$ on the other side, and $\eta=0^+$. In the absence of electric and magnetic fields, the Green's functions G and F can be expressed via the spectral representations,[6]

$$G_{\mathbf{k}}^{>,<}(t,t')=e^{-i\mu(t-t')}\int_{-\infty}^{\infty}d\omega e^{-i\omega(t-t')}(\mp i)A_{\mathbf{k}}(\omega)f^{\pm}(\omega)\,,$$
$$F_{\mathbf{k}}^{>,<}(t,t')=e^{-i\mu(t+t')}\int_{-\infty}^{\infty}d\omega e^{-i\omega(t-t')}(\mp i)B_{\mathbf{k}}(\omega)f^{\pm}(\omega)\,,$$
$$\bar{F}_{\mathbf{k}}^{>,<}(t,t')=e^{i\mu(t+t')}\int_{-\infty}^{\infty}d\omega e^{-i\omega(t-t')}(\mp i)\bar{B}_{\mathbf{k}}(\omega)f^{\pm}(\omega)\,, \quad (2)$$

where $>$ goes with the upper sign and $<$ with the lower, and μ is the chemical potential. The spectral weight functions are

$$A_{\mathbf{k}}(\omega)=\tfrac{1}{2}\{[1+(\epsilon_{\mathbf{k}}/E_{\mathbf{k}})]\delta(\omega-E_{\mathbf{k}})+[1-(\epsilon_{\mathbf{k}}/E_{\mathbf{k}})]\delta(\omega+E_{\mathbf{k}})\}\,,$$
$$B_{\mathbf{k}}(\omega)=-i\tfrac{1}{2}(\Delta_{\mathbf{k}}/E_{\mathbf{k}})[\delta(\omega-E_{\mathbf{k}})-\delta(\omega+E_{\mathbf{k}})]\,, \quad (3)$$
$$\bar{B}_{\mathbf{k}}(\omega)=(\Delta_k^*/\Delta_k)B_{\mathbf{k}}(\omega)\,,$$

and the Fermi distribution factors are

$$f^{\pm}(\omega)=[e^{\mp\beta\omega}+1]^{-1}. \quad (4)$$

When electric and magnetic fields are present, the Green's functions become multiplied by additional phase factors,

$$G(t,t')\rightarrow G(t,t')\exp\left\{\tfrac{1}{2}i\int_{\pm\infty}^{\pm0}dx(\partial/\partial x)[\Phi(\mathbf{r}t)-\Phi(\mathbf{r}t')]\right\}\,,$$
$$F(t,t')\rightarrow F(t,t')\exp\left\{\tfrac{1}{2}i\int_{\pm\infty}^{\pm0}dx(\partial/\partial x)[\Phi(\mathbf{r}t)+\Phi(\mathbf{r}t')]\right\}\,, \quad (5)$$

provided the electric and magnetic fields vanish at $\pm\infty$. The x integration runs perpendicular to the plane of the junction, from the back face of the superconductor to the insulator. The geometry is illustrated in Fig. 1. The phase $\Phi(\mathbf{r}t)$ satisfies the gauge-invariant equations

$$\partial^2\Phi/\partial x\partial t-\partial^2\Phi/\partial t\partial x=(2e/\hbar)E_x\,,$$
$$\partial^2\Phi/\partial z\partial x-\partial^2\Phi/\partial x\partial z=(2e/\hbar c)H_y\,. \quad (6)$$

[6] V. Ambegaokar and A. Baratoff, Phys. Rev. Letters **10**, 486 (1963); Phys. Rev. Letters **11**, 104E (1963).

[7] W. L. McMillan (private communication and to be published).

Substituting Eqs. (2), (4), and (5) into Eq. (1), we arrive at

$$\mathcal{J}(t)=-2e\,\mathrm{Re}\sum_{\mathbf{k}\mathbf{q}\sigma}\int_{-\infty}^{t}d\bar{t}\,\exp\eta\bar{t}\int_{-\infty}^{\infty}d\omega_1 d\omega_2\exp[-i(\omega_1-\omega_2)(\bar{t}-t)][f^-(\omega_1)-f^-(\omega_2)]$$
$$\times\left\{|T_{\mathbf{k},\mathbf{q}}|^2A_{\mathbf{k}}(\omega_1)A_{\mathbf{q}}(\omega_2)\exp\left(\tfrac{1}{2}i\int_{-\infty}^{\infty}dx(\partial/\partial x)[\Phi(\mathbf{r}t)-\Phi(\mathbf{r}t')]\right)\right.$$
$$\left.+T_{\mathbf{k},\mathbf{q}}T_{-\mathbf{k},-\mathbf{q}}\bar{B}_{\mathbf{k}}(\omega_2)B_{\mathbf{q}}(\omega_1)\exp\left(\tfrac{1}{2}i\int_{-\infty}^{\infty}dx(\partial/\partial x)[\Phi(\mathbf{r}t)+\Phi(\mathbf{r}t')]\right)\right\}. \quad (7)$$

If we break the total electric and magnetic fields into dc parts applied externally, plus ac parts either generated by the currents in the junction or also applied externally and which satisfy Maxwell's equations, then Eqs. (6) are compatible with the solution

$$\int_{-\infty}^{\infty}dx\,\partial\Phi(\mathbf{r}t)/\partial x=\varphi-(2e/\hbar)\int_{-\infty}^{t}dt'\int_{-\infty}^{\infty}dxE_x(\mathbf{r}t')\,. \quad (8)$$

We define

$$\begin{aligned}\varphi&\equiv(2e/\hbar c)H_y^{\mathrm{dc}}(\lambda_1+\lambda_2+d)z-(2e/\hbar)V^{\mathrm{dc}}t\,,\\&\equiv k_0z-\omega_0t\,,\end{aligned} \quad (9)$$

where $\lambda_{1,2}$ are the weak-field penetration depths of the two superconductors, d is the thickness of the insulator, and V^{dc} is the dc voltage applied across the junction. Contrary to the statement by Eck, Scalapino, and Taylor,[8] there are no additional ac magnetic-field contributions to expression (8), since $E_x(\mathbf{r}t)$ and $H_y(\mathbf{r}t)$ are related by Maxwell's equations. It is now convenient to introduce the Fourier transform

$$\exp\left\{-i(e/\hbar)\int_{-\infty}^{t}dt'\int_{-\infty}^{\infty}dxE_x(\mathbf{r}t')\right\}=\int_{-\infty}^{\infty}d\omega W(\omega)e^{-i\omega t}. \quad (10)$$

Then by substituting Eq. (10) into Eq. (7), we can do the time integral to yield

$$\mathcal{J}(t)=\mathrm{Im}\int_{-\infty}^{\infty}d\omega d\omega'\{W(\omega)W^*(\omega')e^{-i(\omega-\omega')t}j_1(\omega'+\tfrac{1}{2}\omega_0)+W(\omega)W(\omega')e^{-i(\omega+\omega')t+i\varphi+i\alpha}j_2(\omega'+\tfrac{1}{2}\omega_0)\}\,, \quad (11)$$

where we define the current amplitudes

$$\begin{aligned}j_1(\omega)&\equiv2e\sum_{\mathbf{k}\mathbf{q}\sigma}\int_{-\infty}^{\infty}d\omega_1d\omega_2[f^-(\omega_1)-f^-(\omega_2)]|T_{\mathbf{k}\mathbf{q}}|^2A_{\mathbf{k}}(\omega_1)A_{\mathbf{q}}(\omega_2)[\omega_1-\omega_2-\omega+i\eta]^{-1}\\e^{i\alpha}j_2(\omega)&\equiv2e\sum_{\mathbf{k}\mathbf{q}\sigma}\int_{-\infty}^{\infty}d\omega_1d\omega_2[f^-(\omega_1)-f^-(\omega_2)]T_{\mathbf{k},\mathbf{q}}T_{-\mathbf{k},-\mathbf{q}}\bar{B}_{\mathbf{k}}(\omega_1)B_{\mathbf{q}}(\omega_2)[\omega_1-\omega_2-\omega-i\eta]^{-1}.\end{aligned} \quad (12)$$

The argument of the complex quantity $\bar{B}_{\mathbf{k}}B_{\mathbf{q}}$ is made explicit in the phase α.

Before evaluating the current amplitudes in detail, it should be pointed out that $\mathrm{Re}j_1(\omega)$ is divergent. The divergence is connected through the Kramers-Kronig relations with the fact that the usual single-particle tunneling, involving $\mathrm{Im}j_1(\omega)$, increases ohmically ($\propto\omega$) for large ω. The divergence may be circumvented by making one subtraction in the spectral representations, a procedure validated by confirming in Eq. (11) that j_1 can be shifted by an arbitrary additive constant without affecting the observed current $\mathcal{J}(t)$. Hence, expressions (12) exist only after $j_1(\omega)$ is replaced by $j_1(\omega)-j_1(0)$. Upon substituting Eqs. (3) into (12), the ω_1 and ω_2 integrals can be done trivially, and the sums over wave vectors may be replaced by integrals.

The remaining expressions for j_1 and j_2 require numerical integration for evaluation at finite temperatures. However, at low temperatures ($T\ll\Delta(T)\equiv$ energy gap) the integrals can be evaluated analytically in closed form using

[8] R. E. Eck, D. J. Scalapino, and B. N. Taylor, in *Low Temperature Physics LT 9*, edited by J. G. Daunt, D. O. Edwards, F. J. Milford, and M. Yaqub (Plenum Press, Inc., New York, 1965), p. 415.

the changes of variable suggested by Anderson.[2] Since this regime is the most frequently encountered experimentally, no attempt is made here to give numerical results for all temperatures. The analytical expressions are

$$j_1(\omega)=\frac{1}{R}\frac{2\Delta_1\Delta_2}{\Delta_1+\Delta_2}\left\{\frac{1}{(1-x^2)^{1/2}}K\left(\left(\frac{\delta^2-x^2}{1-x^2}\right)^{1/2}\right)-2\frac{(1-x^2)^{1/2}}{1-\delta^2}E\left(\left(\frac{\delta^2-x^2}{1-x^2}\right)^{1/2}\right)-K(\delta)+\frac{2}{1-\delta^2}E(\delta)\right\},\quad 0\leq x\leq\delta;$$

$$=\frac{1}{R}\frac{2\Delta_1\Delta_2}{\Delta_1+\Delta_2}\left\{\frac{1}{(1-\delta^2)^{1/2}}\left[K\left(\left(\frac{x^2-\delta^2}{1-\delta^2}\right)^{1/2}\right)-2E\left(\left(\frac{x^2-\delta^2}{1-\delta^2}\right)^{1/2}\right)\right]-K(\delta)+\frac{2}{1-\delta^2}E(\delta)\right\},\quad \delta\leq x\leq 1;$$

$$=\frac{1}{R}\frac{2\Delta_1\Delta_2}{\Delta_1+\Delta_2}\left\{\left\{2\frac{(x^2-\delta^2)^{1/2}}{1-\delta^2}\left[K\left(\left(\frac{1-\delta^2}{x^2-\delta^2}\right)^{1/2}\right)-E\left(\left(\frac{1-\delta^2}{x^2-\delta^2}\right)^{1/2}\right)\right]-\frac{1}{(x^2-\delta^2)^{1/2}}K\left(\left(\frac{1-\delta^2}{x^2-\delta^2}\right)^{1/2}\right)\right.\right.$$

$$\left.\left.-K(\delta)+\frac{2}{1-\delta^2}E(\delta)\right\}+i\,\mathrm{sgn}\omega\left\{2\frac{(x^2-\delta^2)^{1/2}}{1-\delta^2}E\left(\left(\frac{x^2-1}{x^2-\delta^2}\right)^{1/2}\right)-\frac{1}{(x^2-\delta^2)^{1/2}}K\left(\left(\frac{x^2-1}{x^2-\delta^2}\right)^{1/2}\right)\right\}\right\},\quad x\geq 1. \tag{13}$$

$$j_2(\omega)=\frac{1}{R}\frac{2\Delta_1\Delta_2}{\Delta_1+\Delta_2}\frac{1}{(1-x^2)^{1/2}}K\left(\left(\frac{\delta^2-x^2}{1-x^2}\right)^{1/2}\right),\quad 0\leq x\leq\delta;$$

$$=\frac{1}{R}\frac{2\Delta_1\Delta_2}{\Delta_1+\Delta_2}\frac{1}{(1-\delta^2)^{1/2}}K\left(\left(\frac{x^2-\delta^2}{1-\delta^2}\right)^{1/2}\right),\quad \delta\leq x\leq 1;$$

$$=\frac{1}{R}\frac{2\Delta_1\Delta_2}{\Delta_1+\Delta_2}\frac{1}{(x^2-\delta^2)^{1/2}}\left[K\left(\left(\frac{1-\delta^2}{x^2-\delta^2}\right)^{1/2}\right)+i\,\mathrm{sgn}\omega K\left(\left(\frac{x^2-1}{x^2-\delta^2}\right)^{1/2}\right)\right],\quad x\leq 1.$$

Here, K and E are complete elliptic integrals of first and second kinds, respectively; and $x\equiv|\omega|/(\Delta_1+\Delta_2)$, $\delta\equiv|\Delta_1-\Delta_2|/(\Delta_1+\Delta_2)$. The junction resistance R per unit area in the normal state is given by

$$R^{-1}=4\pi e|N(0)T|^2.$$

The real and imaginary parts of $j_1(\omega)$ and $j_2(\omega)$ are plotted in Fig. 2 for the special case of equivalent superconductors, $\delta=0$. Alternatively, Fig. 3 plots the moduli and arguments of the current amplitudes. The most striking feature is the logarithmic singularity in both j_1 and j_2 at $|\omega|=\Delta_1+\Delta_2$, arising from the behavior of $K(x)$ near $x=1$. The singularity in j_2 has already been pointed out by Riedel.[5] Inspection of Eq. (13) shows that the singularity disappears if the two metals are not both superconducting, i.e., $\delta=1$. This behavior is due to the singularity in the BCS density of states just at the gap edge, which here is not compensated by coherence factors. A similar situation is well-known to occur in the calculation of the nuclear-magnetic-resonance relaxation time, where the singularity is rounded off by what is believed to be a combination of finite-pair lifetime due to strong phonon coupling, together with gap anisotropy, factors not included in our conventional BCS treatment. By analogy with NMR experiments on superconductors, however, we may expect the tunneling current amplitudes to rise by perhaps a factor of three to five at $|\omega|\sim\Delta_1+\Delta_2$. Further discussion and exploitation of these results will be postponed until Sec. V.

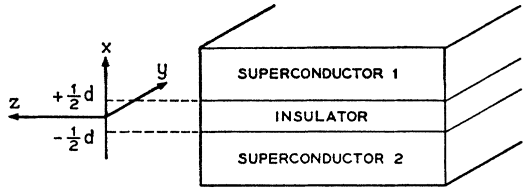

FIG. 1. Junction configuration and coordinate axes. The dc magnetic field is along the y axis.

III. ELECTROMAGNETIC RADIATION BY THE CURRENT

The ac tunneling currents for which we have just given a formal expression will radiate into the insulator and the surrounding superconductors, and the resulting electromagnetic fields will feed back to influence significantly the currents which generate them. The junction configuration is that of a section of a superconducting strip transmission line, and the impedance presented by the junction to the tunneling currents depends sensitively on the extent to which the current pattern matches the resonant modes of the strip line. Traveling wave modes of such a superconducting strip have been analyzed in some detail by Swihart,[9] while standing modes have been considered by Eck *et al.*,[8,4] by Coon and Fiske and by Kulik.[10] The problem of which of these two pictures is more relevant is related to the magnitude of the reflection coefficient at the

[9] J. C. Swihart, J. Appl. Phys. 32, 461 (1961).
[10] D. D. Coon and M. D. Fiske, Phys. Rev. 138, A744 (1965); I. O. Kulik, JETP Letters 1, 84 (1965).

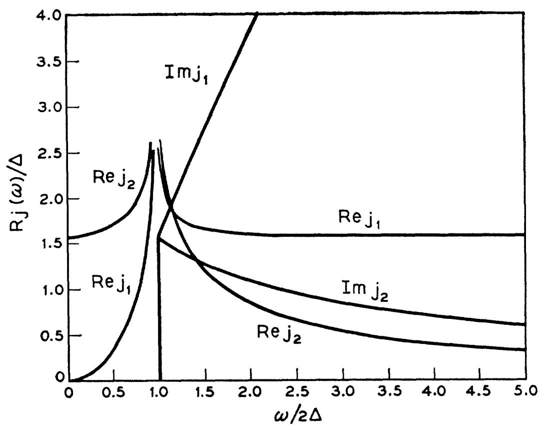

FIG. 2. Real and imaginary parts of the tunneling current amplitudes, $j_1(\omega)$ and $j_2(\omega)$, at zero temperature and for equivalent superconductors.

edges of the junction for power traveling down the line. This in turn could be expressed in terms of the fringing fields near the edges of the junction as influenced by the enclosure surrounding it, or alternatively in circuit language by the impedance mismatch between the strip line and the surrounding enclosure. Although the typical experimental situation is one of reflection coefficient very nearly unity, so that the standing wave picture is most appropriate, it is conceivable that enough power could be coupled out of the junction so that a picture intermediate between traveling and standing would be reasonable. Also, if the junction is sufficiently wide that the standing modes are spaced in frequency more closely than their width due to absorption, or stated in other words, if power is absorbed in the modes before it can return from reflection at the edges, then again a traveling-mode picture is applicable. In the following, we shall develop the analysis only for the simpler traveling wave case.

We begin with Maxwell's equations for the electric fields radiated from an oscillating charge distribution:

$$[\nabla^2-(\epsilon/c^2)\partial^2/\partial t^2]\mathbf{E}=(4\pi/c^2)\partial\mathcal{J}/\partial t, \quad \text{in the insulator}$$
$$[\nabla^2-(1/c^2)\partial^2/\partial t^2-\Lambda^{-2}]\mathbf{E}=0, \quad \text{in the superconductors.} \tag{14}$$

Here $\mathcal{J}$ is the tunneling current, Eq. (11); ϵ is the dielectric constant of the insulator; and Λ is a retarded operator, i.e., Λ is a function of frequency determined by the response of the superconductors. For present purposes, it is sufficiently accurate to use the local London approximation for the electromagnetic response, so that

$$\Lambda^{-2}\cong\lambda^{-2}+2i\delta^{-2}, \tag{15}$$

where λ is the London penetration depth and δ is the classical skin depth.

We next make the important simplifying assumption that $\mathcal{J}$ depends on t only through its explicit appearance in the variable φ, Eq. (9). This means that when a dc magnetic field is present we are assuming the only radiation influencing the current to be that self-generated by the current, with applied microwave power or feedback from enclosure circuitry to be negligible in comparison. When the dc magnetic field is negligible, the current is not spatially modulated and we need no such restrictive assumptions. The net result is that $\mathcal{J}=\mathcal{J}(\varphi)$, and the electric field satisfying Eqs. (14) is of the form $\mathbf{E}(\mathbf{r}t)=\mathbf{E}(x,\varphi)$. The full solution to Eqs. (14), including the boundary conditions across the insulator-superconductor interfaces, now becomes a straightforward exercise. If we Fourier-transform the current density,

$$\mathcal{J}(\varphi)=\int_{-\infty}^{\infty}d\mu\,\mathcal{J}_\mu e^{i\mu\varphi} \tag{16}$$

then we find

$$(-2e/\hbar)\int_{-\infty}^{t}dt'\int_{-\infty}^{\infty}dx E_x(\mathbf{r}t')=\int_{-\infty}^{\infty}d\mu\,\gamma_\mu\mathcal{J}_\mu e^{i\mu\varphi}. \tag{17}$$

The factor γ_μ relating the radiated electric fields to the current sources at frequency $\mu\omega_0$ is given by

$$\gamma_\mu=(4\pi/\mu^2)(2e/\hbar)\tilde{d}[\omega_0^2-(ck_0)^2\tilde{d}/(\Lambda_{1\mu}+\Lambda_{2\mu})]^{-1}, \tag{18}$$

and

$$\tilde{d}\equiv d/\epsilon.$$

We assume that the insulator thickness is small compared to the penetration depths of the superconductors, which in turn are small compared to the wavelength of the current pattern: $\mu kd\ll\mu k\lambda_\mu\ll 1$. We also suppose that the superconducting strips are thick compared with their penetration depths. Situations where these inequalities do not hold have been studied by Swihart[9] and by Ferrell.[11]

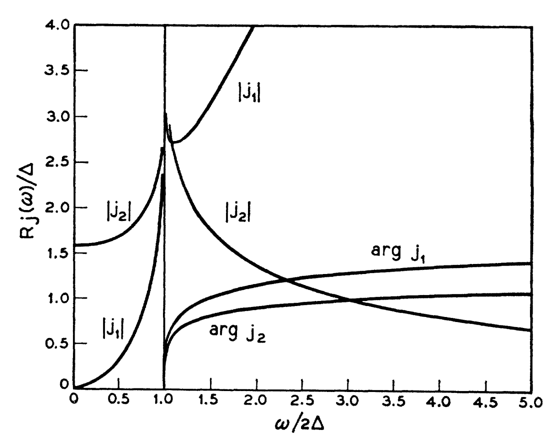

FIG. 3. Modulus and argument of the tunneling current amplitudes from Fig. 2.

[11] R. A. Ferrell, Phys. Rev. Letters **15**, 527 (1965).

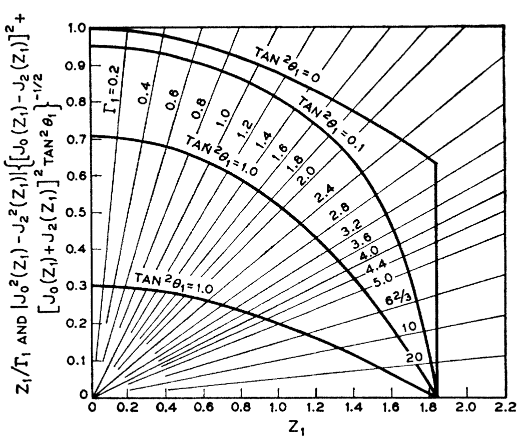

FIG. 4. Right- and left-hand sides of Eq. (25) as functions of Z_1, for various values of Γ_1 and θ_1. [The lowest curve is mislabeled. It should read $\tan^2\theta_1=10.0$.]

IV. SELF-COUPLING AND HARMONIC GENERATION

In the previous two sections, we have formulated the theoretical description of the self-coupled Josephson junction: the tunneling current and the total electromagnetic fields as functionals of each other. We now consider solutions for the operating point of the junction in certain special cases.

We first suppose that the only ac electric field which significantly influences the tunneling current is that self-generated at the fundamental frequency ω_0. In that case, we set

$$(-2e/\hbar)\int_{-\infty}^{t}dt'\int_{-\infty}^{\infty}dxE_x(\mathbf{r}t')\equiv Z_1\cos(\varphi+\phi_1)\,, \quad (19)$$

with Z_1 real. Hence the right-hand side of Eq. (10) becomes

$$\sum_n J_n(\tfrac{1}{2}Z_1)\exp[in(\varphi+\phi_1+\tfrac{1}{2}\pi)]\,, \quad (20)$$

where J_n is a Bessel function. Substituting into Eq. (11), we obtain

$$\mathcal{J}(\varphi)=\mathrm{Im}\sum_{n,n'}J_n(\tfrac{1}{2}Z_1)J_{n'}(\tfrac{1}{2}Z_1)$$
$$\times[e^{i(n-n')(\varphi+\phi_1+\frac{1}{2}\pi)}j_1((n'+\tfrac{1}{2})\omega_0)$$
$$+e^{i(n+n')(\varphi+\phi_1+\frac{1}{2}\pi)+i\varphi}j_2((n'+\tfrac{1}{2})\omega_0)]\,. \quad (21)$$

The operating point is determined by the simultaneous solution of Eq. (21) with the combination of Eqs. (17) and (19):

$$Z_1e^{i\phi_1}=2\gamma_1\int_0^{2\pi}d(\varphi/2\pi)e^{-i\varphi}\mathcal{J}(\varphi)\,. \quad (22)$$

At the operating point, the junction is drawing a dc current

$$\mathcal{J}^{\mathrm{dc}}=\int_0^{2\pi}d(\varphi/2\pi)\mathcal{J}(\varphi)\,. \quad (23)$$

Further analytical progress can be made in the special case of low dc voltages, where $\hbar\omega_0\ll\Delta_1+\Delta_2$, so that it is a good approximation to replace the current amplitudes by their zero-frequency values. This approximation[1,2] will be used throughout the remainder of this section. The summation in Eqs. (21) and (22) can then be evaluated straightforwardly. Introducing the notation

$$\gamma_1j_2(0)\equiv i\Gamma_1\cos\theta_1e^{-i\theta_1}\,, \quad (24)$$

with Γ_1 real, we obtain the transcendental equation for Z_1,

$$Z_1/\Gamma_1=|J_0^2(Z_1)-J_2^2(Z_1)|\{[J_0(Z_1)-J_2(Z_1)]^2+[J_0(Z_1)+J_2(Z_1)]^2\tan^2\theta_1\}^{-1/2}, \quad (25)$$

and also the relation for ϕ_1,

$$\tan\phi_1=[J_0(Z_1)+J_2(Z_1)][J_0(Z_1)-J_2(Z_1)]^{-1}\tan\theta_1\,. \quad (26)$$

When Eqs. (25) and (26) are satisfied, the expression (23) for the dc current can be manipulated into the simple form,

$$\mathcal{J}^{\mathrm{dc}}/j_2(0)=Z_1^2/2\Gamma_1\,. \quad (27)$$

The right- and left-hand sides of Eq. (25) are plotted in Fig. 4 as a function of Z_1 for various values of Γ_1

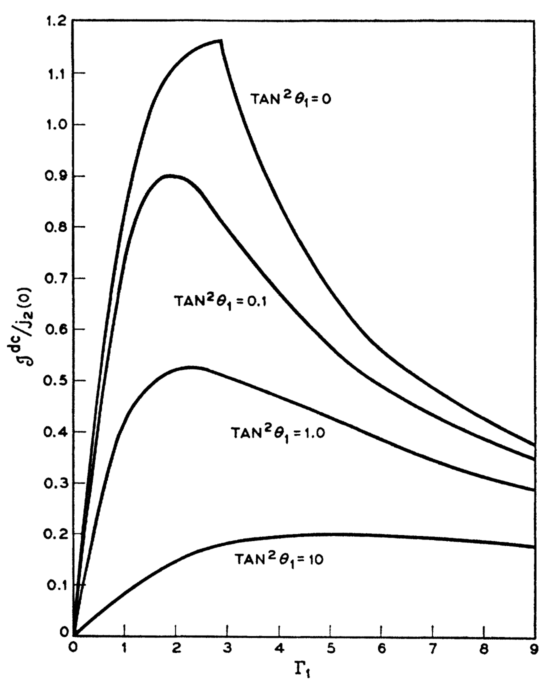

FIG. 5. The dc current drawn by the junction as a function of Γ_1 and θ_1 when the junction is strongly self-coupled only at the fundamental frequency.

and θ_1. The intersections of the two sets of curves give the family of solutions $Z_1=Z_1(\Gamma_1,\theta_1)$. Interpreting Eq. (24), the parameter Γ_1 is a dimensionless coupling constant measuring the strength of feedback of the junction radiation, and is proportional to the product of the tunneling current amplitude and the inverse width in frequency of the junction modes due to absorption in the superconductors:

$$\Gamma_1=(4\pi/c^2k_0^2)(2e/\hbar)j_2(0)/\mathrm{Im}(\Lambda_1+\Lambda_2)^{-1}. \quad (28)$$

The parameter θ_1 measures the degree to which the tunneling current pattern is tuned to the resonant modes,

$$\tan\theta_1=\frac{(\omega_0/ck_0)^2-\tilde{d}\,\mathrm{Re}(\Lambda_1+\Lambda_2)^{-1}}{\tilde{d}\,\mathrm{Im}(\Lambda_1+\Lambda_2)^{-1}}. \quad (29)$$

The total dc current, using Eq. (27) and Fig. 4, is plotted in Fig. 5 as a function of Γ_1. When the junction is only weakly self-coupled so that $\Gamma_1\cos\theta_1\ll 1$, then Eqs. (25) and (27) may be expanded to yield

$$Z_1\cong\Gamma_1\cos\theta_1,\quad \mathcal{J}^{\mathrm{dc}}/j_2(0)\cong\tfrac{1}{2}\Gamma_1\cos^2\theta_1, \quad (30)$$

results already obtained by Eck *et al.*[12] For a strongly self-coupled junction, on the other hand, Z_1 saturates at the first zero of $J_0(Z_1)-J_2(Z_1)$, namely $Z_1=1.84$, and the dc current tends to zero inversely with Γ_1. These latter results disagree with statements in Ref. 4, which claim that any zero of $J_0(Z_1)-J_2(Z_1)$ is a satisfactory solution. An immediate consequence of their conclusion, via Eq. (26), is that the junction could draw a very large dc current relative to the zero-voltage current. The only way to achieve such a large self-coupling, however, is to make the junction mode resonances extremely narrow by decreasing the losses into the superconducting metals. But the requirement of very low loss in the junction is inconsistent with the large amount of power being supplied by the external battery via the large dc current.

Equation (21) shows that the ac tunneling current contains components at all harmonics of the fundamental, in contradiction to the initial supposition that only the fundamental had appreciable feedback. However, since the coupling of the current into a higher mode decreases as the inverse square of the harmonic [Eq. (18)], it is not unreasonable to improve the approximation by assuming the self-coupling constants for all higher harmonics to be nonvanishing but nevertheless small. The total dc current will be modified to include contributions to lowest order in the Γ_n, $n>1$, where in analogy with Eq. (24), we define

$$\gamma_n j_2(0)\equiv i\Gamma_n\cos\theta_n e^{-i\theta_n}. \quad (31)$$

It is important to recognize, however, that the dc current contribution proportional to Γ_n results not just from losses into the nth harmonic mode, but also from

[12] R. E. Eck, D. J. Scalapino, and B. N. Taylor, Phys. Rev. Letters 13, 15 (1964).

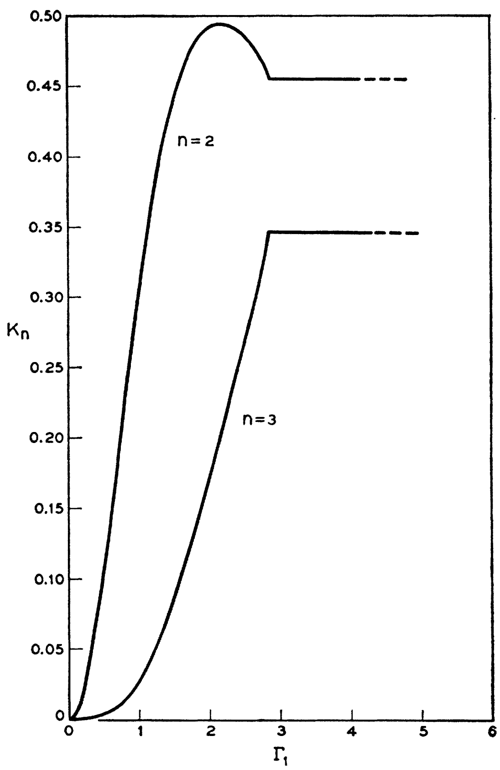

FIG. 6. The quantity K_n, defined by Eq. (33) together with Eq. (25), as a function of Γ_1, for $n=2$ and 3, when $\theta_1=0$.

changes in the loss into the fundamental mode, since the presence of self-coupled harmonics shifts the fundamental operating point.

A substantial amount of algebra is required to obtain the resulting total dc current, but since the manipulations are quite straightforward, we only quote the final expression. In the particular case where the junction is tuned precisely to the fundamental mode, $\theta_1=0$, the result is

$$\mathcal{J}^{\mathrm{dc}}j_2(0)=(Z_1^2/2\Gamma_1)+\sum_{n\geq 2}(-1)^n\Gamma_n\cos^2\theta_n K_n(Z_1), \quad (32)$$

where we introduce

$$K_n(Z_1)\equiv n[J_{n-1}(Z_1)-(-1)^n J_{n+1}(Z_1)]^2, \quad (33)$$

and where Z_1 still satisfies Eq. (25). The quantity $K_n(Z_1)$ is plotted in Fig. 6 as a function of Γ_1 for $n=2$ and 3. Except for values of $\Gamma_1\gg 1$, K_n is itself substantially smaller than $Z_1^2/2\Gamma_1$ (crudely and empirically, by a factor of $\sim 2^{-n}$ in the range $\Gamma_1\sim 1$). Since Γ_n itself decreases like n^{-2}, it would seem that treating the higher harmonics as small perturbations is a better approximation than might have been expected *a priori*, unless $\Gamma_1\gg 1$. It should be noted that harmonics make a nonvanishing contribution to the dc current for all $Z_1\neq 0$, in disagreement with the statement in Ref. 4 that Eq. (25) constitutes a "threshold" condition for the existence of harmonics.

In addition, we remark that the junction equations are also compatible with the presence of fractional subharmonics in the very strong coupling situation, as pointed out theoretically and observed experimentally by Langenberg, *et al.*[13] We are not able to enlarge on their discussion.

V. INFLUENCE OF THE SINGULARITY IN THE CURRENT AMPLITUDE

The analysis of the previous section was carried out assuming that the current amplitudes $j_{1,2}(\omega)$ were independent of frequency, an approximation shown in Sec. II to be valid only for $\omega \gg \Delta_1+\Delta_2$. We now explore possible consequences of the logarithmic singularity in the real part of the amplitudes. Before considering the self-coupled junction, however, we turn to a simpler situation where the investigation can be carried out more fully.

We take the case of a dc-biased junction, without any feedback or magnetic field, but driven by a uniform microwave voltage V^{rf}, at angular frequency ν. Experiments of this type have been reported by Shapiro.[14] In this situation, the right-hand side of Eq. (10) becomes $\sum_n J_n(eV^{\mathrm{rf}}/\hbar\nu)e^{in\nu t}$, and from Eq. (11) the current is

$$\mathcal{J}(t)=\mathrm{Im}\sum_{nn'} J_nJ_{n'}\big[e^{i(n-n')\nu t}j_1(n'\nu-\tfrac{1}{2}\omega_0) + e^{i(n+n')\nu t-i\omega_0 t+i\alpha}j_2(n'\nu-\tfrac{1}{2}\omega_0)\big]. \quad (34)$$

The second line contributes to the dc component of the current only if $\omega_0=N\nu$, where N is a non-negative integer. Then

$$\mathcal{J}^{\mathrm{dc}}=\mathrm{Im}\sum_n \big[J_n^2 j_1((n-\tfrac{1}{2}N)\nu) + J_{N-n}J_n e^{i\alpha}j_2((n-\tfrac{1}{2}N)\nu)\big]. \quad (35)$$

Inspection of Fig. 2 shows that $\mathrm{Re}j_2(\omega)$ is nearly constant except for $|\omega|\sim 2\Delta$. We thus single out from the summation in Eq. (35) that term for which $|n-\frac{1}{2}N|\nu\equiv\frac{1}{2}M\nu$ is most nearly equal to 2Δ, and in the remainder we approximate $\mathrm{Re}j_2(\omega)\cong\mathrm{Re}j_2(0)$. Also using the symmetry relation visible in Eq. (13), $j_2(\omega)=j_2^*(-\omega)$, we arrive at

$$\begin{aligned}\mathcal{J}^{\mathrm{dc}}\cong\sum_{m>0} &\big[J_{\frac{1}{2}(N+m)}{}^2(eV^{\mathrm{rf}}/\hbar\nu)-J_{\frac{1}{2}(N-m)}{}^2(eV^{\mathrm{rf}}/\hbar\nu)\big]\\ &\times\mathrm{Im}j_1(\tfrac{1}{2}m\nu)+\big[J_N(2eV^{\mathrm{rf}}/\hbar\nu)\\ &+2J_{\frac{1}{2}(N+M)}(eV^{\mathrm{rf}}/\hbar\nu)\\ &\times J_{\frac{1}{2}(N-M)}(eV^{\mathrm{rf}}/\hbar\nu)\hat{\jmath}(\tfrac{1}{2}M\nu)\big]\,\mathrm{Re}j_2(0)\sin\alpha, \end{aligned}\quad (36)$$

where

$$\hat{\jmath}(\omega)\equiv\big[\mathrm{Re}j_2(\omega)/\mathrm{Re}j_2(0)\big]-1. \quad (37)$$

Since $\mathrm{Im}j_1$ is just the usual single-particle tunneling characteristic, the first sum in Eq. (36) is identical to the dc characteristic calculated by Tien and Gordon[15] to explain the observations of Dayem and Martin.[16] The second term is the one used by Shapiro[14] to interpret his observations as a detection of the ac Josephson effect, since the term appears in the dc characteristic only when the microwave frequency satisfies $N\hbar\nu=2eV^{\mathrm{dc}}$. The third term is a consequence of the frequency dependence of the current amplitudes, and makes a significant contribution to the dc current when $\frac{1}{2}M\nu\sim 2\Delta$. This latter effect may be able to account for previously unexplained anomalies in Shapiro's data.[14] We remark parenthetically that neither $\mathrm{Re}j_1$ nor $\mathrm{Im}j_2$ contributes to $\mathcal{J}^{\mathrm{dc}}$ in this case. It should also be noted that the quantity V^{rf} is to be interpreted as the actual voltage drop across the junction, not the voltage drop in an equivalent section of the microwave cavity in the absence of the junction. These two quantities differ substantially because of the strong impedance mismatch between cavity and junction. As a result, the large discrepancy in the magnitude of V^{rf} estimated by Tien and Gordon[15] is probably spurious, although quantitative comparison cannot be made with precision.

With this simpler example in mind, we return to the case of the self-coupled junction, without applied microwaves. Assuming once again that only the fundamental frequency feeds back significantly, we recover Eqs. (21)–(23). But now we isolate those terms in the n' sum for which $|n'+\frac{1}{2}|\omega_0\equiv\frac{1}{2}M\omega_0$ is closest to 2Δ, and only in the remainder approximate $j_{1,2}(\omega)\cong j_{1,2}(0)$. Combining the procedures of Sec. IV and the example just above, we find that Z_1 satisfies the equation

$$Z_1^2/2\Gamma_1=\{J_1(Z_1)+J_{\frac{1}{2}(1+M)}(\tfrac{1}{2}Z_1)\times J_{\frac{1}{2}(1-M)}(\tfrac{1}{2}Z_1)\hat{\jmath}(\tfrac{1}{2}M\omega_0)\}\cos\phi_1, \quad (38)$$

which generalizes Eq. (25). The equation for ϕ_1, however, is complicated by the presence of $\mathrm{Re}j_1$, which we take $\cong\mathrm{Re}j_2$ in the neighborhood of the singularity. In the special case where $\theta_1=0$, we find $\phi_1\neq 0$:

$$\sin\phi_1=\frac{\frac{1}{2}\big[J_{\frac{1}{2}(1+M)}{}^2(\frac{1}{2}Z_1)+J_{\frac{1}{2}(1-M)}{}^2(\frac{1}{2}Z_1)\big]'\,\hat{\jmath}(\frac{1}{2}M\omega_0)}{\big[J_1(Z_1)+J_{\frac{1}{2}(1-M)}(\frac{1}{2}Z_1)J_{\frac{1}{2}(1+M)}(\frac{1}{2}Z_1)\hat{\jmath}(\frac{1}{2}M\omega_0)\big]'}, \quad (39)$$

where the prime denotes differentiation of the Bessel functions with respect to their argument. On the other hand, the dc current still satisfies the simple relation (27).

The implicit equation for the operating point, Eqs. (38) and (39), has now become rather too formidable for numerical solution without resort to computers. Furthermore, a number of simplifications were made in its derivation which would render a detailed

[13] D. N. Langenberg, D. J. Scalapino, B. N. Taylor, and R. E. Eck, Phys. Rev. Letters **15**, 294 (1965).

[14] S. Shapiro, Phys. Rev. Letters **11**, 80 (1963); S. Shapiro, A. R. Janus, and S. Holly, Rev. Mod. Phys. **36**, 223 (1964).

[15] P. K. Tien and J. P. Gordon, Phys. Rev. **129**, 647 (1963).

[16] A. H. Dayem and R. J. Martin, Phys. Rev. Letters **8**, 246 (1962).

solution of qualitative significance only. We believe that any full scale numerical effort should be made on the more complete Eq. (21)–(23). Nevertheless, certain general features are visible in Eq. (38) which are of relevance to experiment. Because the function $\mathcal{J}(\frac{1}{2}M\omega_0)$ peaks sharply in the neighborhood of $\frac{1}{2}M\omega_0 \sim 2\Delta$, or $eV^{\mathrm{dc}} \sim 2\Delta/$(odd integer), we expect that the dc current characteristic will exhibit peaks at the corresponding discrete voltages. The amplitude of the peaks depends on the junction feedback at the fundamental, Γ_1, and on the odd integer M. The current peaks will be superimposed on a rising background due to single-particle tunneling, $\sim \mathrm{Im} j_1$. This latter contribution has been omitted for simplicity from Eqs. (38) and (27), but can be recognized as the first term in Eq. (36), describing the nonself-coupled, microwave driven junction. The single-particle tunneling will also contribute an additional shoulder to the peaks, due to the steep rise of $\mathrm{Im} j_1(\frac{1}{2}M\omega_0)$ at $eV^{\mathrm{dc}} = 2\Delta/M$; this is just the Dayem-Martin-Tien-Gordon effect,[15,16] as discussed above.

Such shouldered peaks in the current voltage characteristic have been observed by a number of workers,[17] but reported only sparingly.[18] However, in addition to the effect at $eV^{\mathrm{dc}} = 2\Delta/$(odd integer), similar phenomena also occur at $eV^{\mathrm{dc}} = 2\Delta/$(even integer). A qualitative explanation of the latter series of steps must be obtained from the feedback of higher harmonics, rather than from the fundamental. Referring to Eqs. (32) and (28), one finds that the dc current drawn by the nth higher mode is proportional to $\Gamma_n \cos^2\theta_n$, which in turn is proportional to the electromagnetic absorption (more precisely, the surface resistance) of the superconductors at frequency $n\omega_0$. At low temperatures, this loss rises sharply when $n\omega_0$ exceeds 2Δ, and thus the dc current should also show shoulders at $eV^{\mathrm{dc}} = 2\Delta/$(even integer).

The explanations put forward above for the two series of steps have been devised and discussed previously by many workers,[17] although only schematically. Even though we have not yet carried out a detailed numerical evaluation, we feel that the present work provides a formulation which is the appropriate theoretical framework for a quantitative description of the effects. In addition, several pertinent comments can be made from the present formulas. The first is the observation from Eq. (28) that the coupling parameters Γ_n decrease with increasing dc magnetic field like H^{-2}. This means that for magnetic fields sufficiently large that $\Gamma_n \ll 1$ and so that the current steps are nearly suppressed, their amplitude should vanish also as H^{-2}, although this functional form for $\mathcal{J}^{\mathrm{dc}}$ need not continue to hold at lower field values. It should be noted that the current steps are observed only for fields well above those necessary to match precisely to the junction resonant modes.

We also remark that in the case of junctions formed of identical superconductors, it is not obvious from the theoretical formulation that the two series of steps (even and odd) will be as closely similar as observed, since the mechanisms invoked for the two series are conceptually quite different. Nevertheless, as seen in Eq. (32) on the one hand, and Eq. (38) on the other, the current steps in the two cases each involve the product of Bessel functions times a superconductor linear response function in the vicinity of the gap edge. Only a detailed evaluation, however, can affirm that the step shapes of the two series are very nearly equivalent. The case of inequivalent superconductors should provide a more direct and severe test of the present ideas, since the two series are expected to split. The odd series is predicted to occur at $eV^{\mathrm{dc}} = (\Delta_1 + \Delta_2)/$(odd integer), while the even series should double into both $eV^{\mathrm{dc}} = 2\Delta_1/$(even integer) and $eV^{\mathrm{dc}} = 2\Delta_2/$(even integer). The experimental confirmation or refutation of these expectations would do much to clarify our understanding of the behavior of strongly self-coupled tunneling junctions.

[17] M. D. Fiske, I. Giaever, and S. Silverstein (private communication); D. N. Langenberg, D. J. Scalapino, and B. N. Taylor (private communication); J. M. Rowell (private communication) and discussion remark, Rev. Mod. Phys. **36**, 215 (1964); D. H. Douglass, Jr. (private communication), and Bull. Am. Phys. Soc. **11**, 87 (1966).

[18] I. K. Yanson, V. M. Svistunov, and I. M. Dmitrenko, Zh. Eksperim. i Teor. Fiz. **47**, 2091 (1964) [English transl.: Soviet Phys.—JETP **20**, 1404 (1965)]; S. M. Marcus, Phys. Letters **19**, 623 (1966); **20**, 236(1966). Effects reported by B. N. Taylor and E. Burstein, Phys. Rev. Letters **10**, 14 (1963), are rather similar. We disagree with the theoretical interpretation supplied by each of these authors.

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