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Exact Nonlinear Electromagnetic Whistler Modes

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We solve exactly the nonlinear, relativistic equations of motion for an electron moving in a right circularly polarized wave which propagates along a static uniform magnetic field $B_0\mathbf{e}_z$. In the wave frame where the induction electric field disappears, we find two constants of the motion. With their aid, we examine the particle trajectories and obtain the periods and amplitudes of oscillation in the z direction. In the absence of collisions, the exact solution of the relativistic Vlasov equation in the wave frame is an arbitrary function of these constants. The requirement of self-consistency imposed by Maxwell's equations is examined and, in particular, we show that sufficient arbitrariness remains that no dispersion relation exists for these waves. However, for less general distribution functions, one may still have a dispersion relation independent of wave amplitude. When we require that the moments of the distribution be correct to first order in the amplitude of the wave, in analogy with the electrostatic case, we recapture the linearized distribution function together with a principal-value prescription for treating the usual singularity, and we also obtain the transverse Van Kampen modes.

I. INTRODUCTION

IN most conventional treatments of the problem of electromagnetic waves propagating in a fully ionized plasma,[1,2] the wave amplitudes are taken to be small quantities to permit linearization of the relevant equations. For the special case of propagation in the whistler mode parallel to an external magnetic field $B_0\mathbf{e}_z$, Stix has investigated the linearized equations of motion for electrons with no perpendicular energy, but with a zero-order velocity in the z direction.[3] The solutions have the following properties:

(a) The component of electron velocity parallel to the zero-order (external) magnetic field is constant through first order in the wave amplitude.

(b) The perpendicular velocity is oscillatory except at cyclotron resonance, where it increases linearly with time.

These solutions are restricted to small times over which the phase-space trajectories of the particles do not differ appreciably from the zero-order trajectories.

In this paper,[4] we begin by transforming to the wave frame and finding the two single-particle constants of the motion. In terms of these constants we solve the nonlinear relativistic equations of motion for an electron moving in the prescribed whistler field. In the nonlinear regime we find, as predicted by Stix,[5] an oscillatory nature to the motion in the z direction analogous to the motion in a longitudinal wave,[6,7] although the situation is considerably more complicated. We find that a number of oscillatory modes are possible depending upon the values of the constants of the motion. For one set of values, the nonlinear particle trajectories reduce to the usual helices of zero-order theory as the wave amplitude tends to zero. However, for other values of these constants we find trajectories that have no linear counterparts, and, in the limit of zero-wave amplitude, motion prescribed by such constants is not possible. Special attention is given to those electrons which have a z component of velocity near that particular velocity which makes the electrons feel the wave frequency Doppler-shifted to their own cyclotron frequency, i.e., the cyclotron-resonance velocity. These particles are analogous to the electrons trapped in an electrostatic wave. However, unlike the trapping of electrons in the potential troughs of an electrostatic wave, where the average velocity of all of the trapped electrons is the wave velocity, the average z velocity of the resonant electrons is not equal to the cyclotron-resonance velocity. Hence, if the motion is viewed from the cyclotron-resonance frame, the resonant electrons will be observed to have a slow drift in the z direction. This drift velocity tends to zero faster than the first power of the wave amplitude. A precise definition of particle trapping in the context of an electromagnetic wave is given in Appendix A, where the phase-space trajectories of the particles are examined in detail. The periods and

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[1] T. Pradhan, Phys. Rev. **107**, 1222 (1957).
[2] I. B. Bernstein, Phys. Rev. **109**, 10 (1958).
[3] T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill Book Company, Inc., New York, 1962), p. 160.
[4] The main results of this analysis have been announced in the form of an abstract; R. F. Lutomirski and R. N. Sudan, Bull. Am. Phys. Soc. **10**, 205 (1965).
[5] Reference 3, p. 163.
[6] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851 (1949).
[7] I. B. Bernstein, J. M. Green, and M. D. Kruskal, Phys. Rev. **108**, 507 (1957).

amplitudes of velocity oscillation are obtained in Sec. II, and in the limit of small wave amplitude the motion is compared with that predicted by linear theory (see Secs. II and IV).

Stix[8] has also hinted at the possible existence of solutions to the Vlasov and Maxwell equations for electromagnetic waves of arbitrary amplitude in analogy with the electrostatic waves considered by Bernstein *et al.*[7] For the case of parallel propagation we treat these equations relativistically in Sec. III and show that nonlinear solutions do indeed exist. The exact distribution function is not completely determined by the above equations, and we find these waves can propagate without satisfying any dispersion relation (neglecting collisions and questions pertaining to the stability of these solutions). However, it is possible to choose a distribution function such that these waves still satisfy a dispersion relation even in this nonlinear regime.

The electrostatic analogy is completed when we pass to the limit of vanishing wave amplitude in Sec. IV and find that, for the trapped electrons, the distribution has an expansion in half-integral powers of this amplitude. Requiring only that the moments of the distribution function be correct to first order in the wave amplitude, the usual linear results are obtained with a Dirac delta function appearing in the "first-order" distribution to account for the effects of the resonant or trapped particles. A Cauchy principal value prescription for treating the analytic portion of the first-order distribution function is also obtained explicitly.

II. EXACT PARTICLE TRAJECTORIES

We consider an infinite, collisionless plasma placed in a uniform magnetostatic field $B_0\mathbf{e}_z$. The ions are immobile and constitute a uniform background of positive charge. A right-circularly-polarized wave of arbitrary amplitude, frequency ω, and wave vector $\mathbf{k}$, is assumed to propagate parallel to $\mathbf{B}_0$. We restrict ourselves to whistler waves having phase velocity $\omega/k\equiv cv$ less than c, the speed of light, and transform to the wave frame where most of our calculations will be performed. In this frame the induction electric field vanishes and, because we only consider transverse waves with no density perturbations, the only fields encountered are the external and wave magnetic fields

$$\mathbf{B}=B_0\mathbf{e}_z+B_1(\mathbf{e}_x\cos kz-\mathbf{e}_y\sin kz)\,. \tag{1}$$

All quantities are measured in the wave frame unless otherwise indicated. The equation of motion can be written in the wave frame as

$$d\mathbf{u}/dt=-(e\beta/mc)(\mathbf{u}\times\mathbf{B})\,, \tag{2}$$

where e and m are the electronic charge and rest mass, $u=v/c\beta$ is the reduced velocity, and $\beta=(1-v^2/c^2)^{1/2}=(1+u^2)^{-1/2}$, with $u^2=u_x^2+u_y^2+u_z^2$.

We introduce a cylindrical coordinate system in $\mathbf{u}$ space with $u_\perp=(u_x^2+u_y^2)^{1/2}$, $\tan\phi=u_y/u_x$, and define $\zeta=kz+\phi$, the angle between the wave magnetic field and the transverse velocity of the electron. Then, with $dz/dt=\beta cu_z$, the components of Eq. (2) yield

$$du_z/dt=\beta\Omega_1u_\perp\sin\zeta\,, \tag{3a}$$

$$du_\perp/dt=-\beta\Omega_1u_z\sin\zeta\,, \tag{3b}$$

$$d\zeta/dt=\beta kcu_z+\beta\Omega_0-\beta\Omega_1(u_z/u_\perp)\cos\zeta\,, \tag{3c}$$

where $\Omega_0=eB_0/mc$ and $\Omega_1=eB_1/mc$. The system of Eqs. (3) is closed in the three variables $(u_\perp,u_z,\zeta)$ and there exists two constants of the motion

$$\eta_1=u_\perp^2+u_z^2\,, \tag{4a}$$

$$\eta_2=(u_z+\nu_0)^2+2\nu_1u_\perp\cos\zeta\,, \tag{4b}$$

where $\nu_0=\Omega_0/kc$ is the cyclotron-resonance reduced velocity as measured in the wave frame, and $\nu_1=\Omega_1/kc$. Noting that $\beta=(1+\eta_1)^{-1/2}$ is constant (in the wave frame), we substitute the expressions (4a) and (4b) into Eq. (3a) and obtain

$$\frac{2}{k\beta c}\frac{du_z}{dt}=2u_z\frac{du_z}{d(kz)}=\pm(F(u_z))^{1/2}\,, \tag{5}$$

where

$$F(u_z)=(2\nu_1)^2(\eta_1-u_z^2)-((u_z+\nu_0)^2-\eta_2)^2\,. \tag{6}$$

We observe that Eq. (5) is mathematically just the differential equation describing one-dimensional motion in a "potential well" $F(u_z)$.[9] Because $F(u_z)\to-\infty$ as $u_z\to\pm\infty$, u_z must oscillate in this pseudo-potential well between two of the real zeros of $F(u_z)$, which we denote by (1,2). By calculating the zeros of the fourth-order polynomial $F(u_z)$, we can formally express t as a function of u_z in terms of elliptic integrals. The solution is formidable and cannot be inverted in terms of elementary functions to find u_z as a function of time, and for numerical computations, it is simpler to deal with Eq. (5) directly. However, much information can be obtained by considering the first and second terms of Eq. (6) separately and plotting the curves

$$y_1(u_z)=2\nu_1(\eta_1-u_z^2)^{1/2} \quad\text{and}\quad y_2(u_z)=|(u_z+\nu_0)^2-\eta_2|$$

(the positive square roots of the respective terms), and considering the regions where $y_1>y_2$. As an example, in Fig. 1 we sketch the two curves for the case $(\eta_1^{1/2}-\nu_0)^2>\eta_2>2\nu_1(\eta_1-\nu_0^2)^{1/2}$. Oscillations of u_z will take place between the values 1 and 2 given by the points of intersection of the curve y_1 with y_2. In Fig. 1, there are four real zeros and the motion is such that $u_z+\nu_0$ has a constant sign with electrons oscillating

[8] Reference 3, p. 166.

[9] C. S. Roberts and S. J. Buchsbaum, Phys. Rev. **135**, A381 (1964), have shown that the particle energy in the laboratory frame also obeys this type of equation.

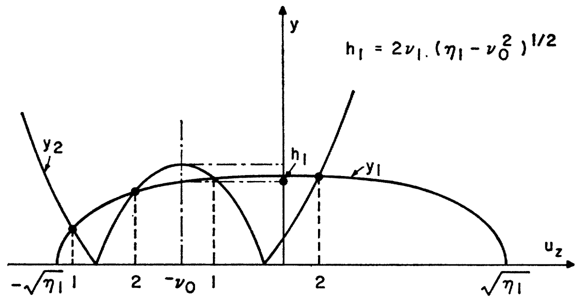

FIG. 1. The curves $y_1 = 2\nu_1(\eta_1 - u_z^2)^{1/2}$ and $y_2 = |(u_z+\nu_0)^2 - \eta_2|$ plotted for $(\eta_1^{1/2} - \nu_0)^2 > \eta_2 > 2\nu_1(\eta_1 - \nu_0^2)^{1/2}$.

about one of the two velocities $u_z = -\nu_0 \pm \sqrt{\eta_2}$. By sketching curves similar to those in Fig. 1, one can see that a number of modes of oscillation are possible depending upon the values of η_1, η_2, ν_0, and ν_1. The investigation of these nonlinear modes is of general interest, and a discussion of these oscillations with their limiting behavior as ν_1 tends to zero is given in Appendix A.

From Eq. (5), the period of oscillation in the wave frame is

$$T = \frac{4}{k\beta c}\int_1^2 (F(u_z))^{-1/2} du_z, \tag{7}$$

and, in the lab (primed) frame

$$T' = T/(1-\nu^2)^{1/2}. \tag{8}$$

Throughout the remainder of this section we concern ourselves with obtaining approximate expressions for some quantities of interest and neglect relativistic effects to simplify the calculations ($\nu, \nu_0, \nu_1 \ll 1$; $\beta \to 1$; $u_z \to v_z/c$). We consider the special case $\eta_1 \gg \nu_1^2$ and $\eta_1 \gg (\nu_0 + \sqrt{\eta_2})^2$, and note from Fig. 1 that the oscillations are then approximately symmetric about $u_z = -\nu_0$. Choosing to observe the motion from the cyclotron-resonance frame,

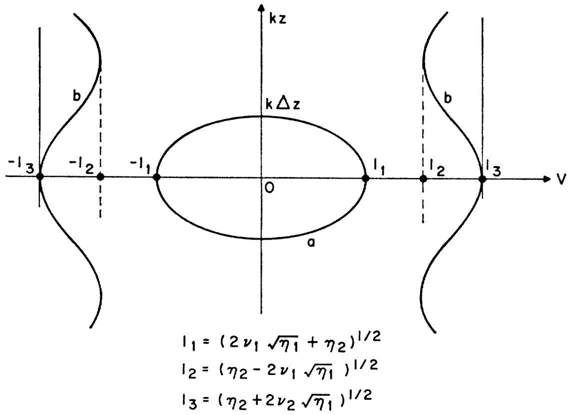

FIG. 2. The phase-plane trajectory $V^2 = \eta_2 + 2\nu_1(\sqrt{\eta_1})\cos(kZ)$ for (a) $|\eta_2| < 2\nu_1\sqrt{\eta_1}$, (b) $\eta_2 > 2\nu_1\sqrt{\eta_1}$.

a Galilean transformation of Eq. (5) with $Z = z + c\nu_0 t$, $V = v_z + c\nu_0$ yields

$$\frac{2}{kc^2}\frac{dV}{dt} = 2\frac{V}{c^2}\frac{dV}{dkZ} = \pm[F(V/c - \nu_0)]^{1/2}, \tag{9}$$

where with the indicated approximation

$$F(V/c - \nu_0) \approx (2\nu_1)^2\eta_1 - (V^2/c^2 - \eta_2)^2.$$

$F(V/c - \nu_0)$ is then approximately an even function of V with the two real roots $\pm(\eta_2 + 2\nu_1\sqrt{\eta_1})^{1/2}$. The trajectory in the (V,Z) plane is found from Eq. (9) to be

$$V^2 = \eta_2 + 2\nu_1\eta_1^{1/2}\cos(kZ + \theta_0), \tag{10}$$

where θ_0 depends trivially on the initial conditions and we set it equal to zero. A curve of Eq. (10) is given in Fig. 2. Physically our approximations imply that in the wave frame, the particle energy is very large, but that, over one oscillation, u_z does not vary greatly from the value $-\nu_0$. The particles considered thus have a large transverse energy. For $\eta_2 > 2\nu_1\sqrt{\eta_1}$, the electron velocity oscillates about one of the two values $V = \pm\sqrt{\eta_2}$. When $|\eta_2| < 2\nu_1\sqrt{\eta_1}$, in our approximation the phase plane trajectory is a pure libration indicating zero average velocity in the cyclotron-resonance frame. We will refer to these electrons as being "trapped" in this frame. A precise definition of particle trapping is given in Appendix A.

As $\nu_1 \to 0$, we observe that the amplitude of velocity oscillation approaches zero as $\nu_1(\eta_1/\eta_2)^{1/2}$, and for the trapped electrons tends to zero as $\sqrt{\nu_1}$. In the limit, only those electrons with $\eta_2 = 0$ stay trapped, and the trajectories of the remaining particles ($\eta_2 > 0$) approach the straight-line paths of linear theory.

The amplitude of oscillation in the Z direction may also be calculated from Eq. (9),

$$\Delta Z = \tfrac{1}{2}\lambda(1 + (2/\pi)\sin^{-1}(\eta_2/2\nu_1\sqrt{\eta_1})), \tag{11}$$

where λ is the wavelength. The total distance covered in the Z direction by a trapped electron (as viewed from the cyclotron-resonance frame) is less than one wavelength.

The period of oscillation is given by

$$T = \frac{8}{kc^2}\int [F(V/c - \nu_0)]^{-1/2} dV = \frac{8}{ck(\nu_1\sqrt{\eta_1})^{1/2}}K(p), \tag{12}$$

where the limits of integration are 0 and $(\eta_2 + 2\nu_1\sqrt{\eta_1})^{1/2}$, $K(p)$ is the complete elliptic integral of the first kind[10] and $p = \frac{1}{2}(1 + \eta_2/2\nu_1\sqrt{\eta_1})$. We note that for the trapped particles, p varies between 0 and 1, and for p not too near 1, $K(p)$ is a slowly varying function of p. [$K(0) = 1.57$, $K(0.7) = 1.80$, $K(0.98) = 3.10$, $K(1) = \infty$].[11] Then for

[10] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists* (Springer-Verlag, Berlin, 1954), Eq. (259.00), p. 133.
[11] Reference 10, p. 322.

$|\eta_2|<2\nu_1\sqrt{\eta_1}$, we have within a factor of 2

$$T/T_0\approx\nu_0/(\nu_1\sqrt{\eta_1})^{1/2}, \tag{13}$$

where T_0 is the cyclotron period. We observe that here, as in the electrostatic case, the frequency of oscillation varies as the square root of the wave field.

In Figs. 3, 4, and 5, we show the exact phase-plane trajectories in the cyclotron-resonance frame as obtained from numerical integration of Eq. (9). The normalized distance D shown in Figs. 3 and 4 displays the effect of the odd terms in V appearing in $F(V/c-\nu_0)$ which we have neglected in our approximations. The quantity D/k divided by the period T in Eq. (7) is the drift or average velocity which the trapped particles possess in this frame. By Taylor expansion of the ellipse y_1 in Fig. 1 about $u_z=-\nu_0$, one can show that this average velocity of the trapped particles tends to zero to higher order than the first power of ν_1. Two trajectories for untrapped particles are displayed in Fig. 5.

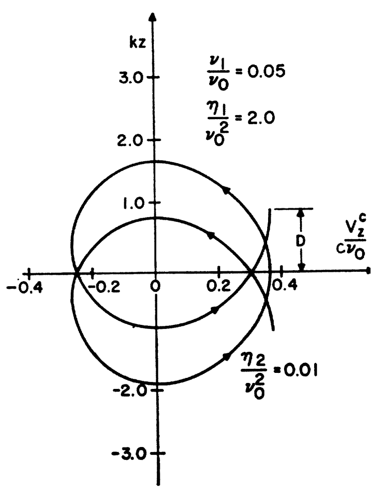

FIG. 3. Phase-plane plot of the solution of Eq. (9) obtained by numerical integration for $\eta_2/\nu_0^2=0.01$, $\nu_1/\nu_0=0.05$, $\eta_1/\nu_0^2=2.0$.

III. SELF-CONSISTENT DISTRIBUTION FUNCTIONS

For the assumptions given in the beginning of Sec. II, we find exact self-consistent stationary solutions to the Vlasov and Maxwell equations. In the relativistic case the distribution function is regarded as a function of (z,u_j,t), and an invariant form of the Vlasov equation[12] in cylindrical coordinates in $\mathbf{u}$ space for the magnetostatic field of Eq. (1) is

$$kcu_z\frac{\partial f}{\partial kz}+\left[\Omega_0-\Omega_1\frac{u_z}{u_\perp}\cos(kz+\phi)\right]\frac{\partial f}{\partial\phi}+\Omega_1\sin(kz+\phi)\left(u_\perp\frac{\partial f}{\partial u_z}-u_z\frac{\partial f}{\partial u_\perp}\right)=0. \tag{14}$$

[12] P. C. Clemmow and A. J. Willson, Proc. Cambridge Phil. Soc. 53, 222 (1957), Part I.

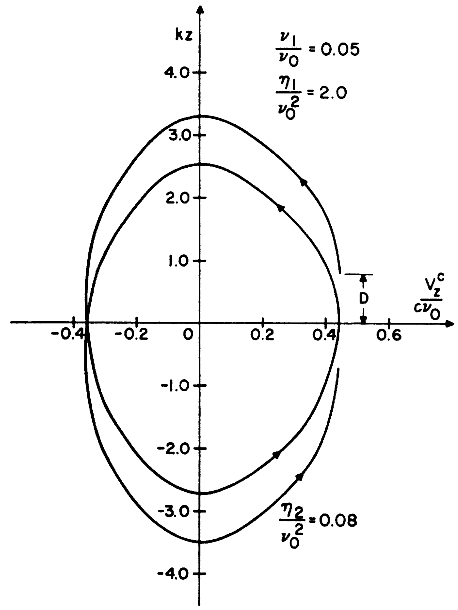

FIG. 4. Phase-plane plot of the solution of Eq. (9) for $\eta_2/\nu_0^2=0.08$, $\nu_1/\nu_0=0.05$, $\eta_1/\nu_0^2=2.0$.

From Eq. (14), $f(u_z,u_\perp,kz,\phi)=f(u_z,u_\perp,\zeta)$, where $\zeta=kz+\phi$. In terms of these new variables the Vlasov equation becomes

$$\left(u_z+\nu_0-\nu_1\frac{u_z}{u_\perp}\cos\zeta\right)\frac{\partial f}{\partial\zeta}+\nu_1\sin\zeta\left(u_\perp\frac{\partial f}{\partial u_z}-u_z\frac{\partial f}{\partial u_\perp}\right)=0. \tag{15}$$

The characteristic equations are easily integrated to give the general solution of Eq. (15):

$$f=n_0f(\eta_1,\eta_2), \tag{16}$$

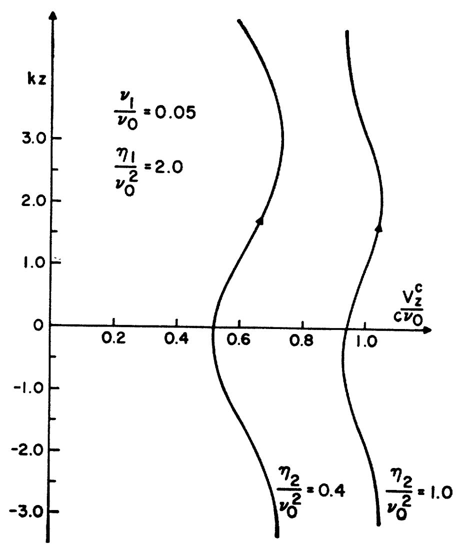

FIG. 5. Phase-plane plot of the solution of Eq. (9) for $\eta_2/\nu_0^2=0.4$, $\nu_1/\nu_0=0.05$, $\eta_1/\nu_0^2=2.0$, and $V>0$. Similar curves appear for $V<0$.

where η_1 and η_2 are the constants of the motion given by Eqs. 4(a) and 4(b), n_0 is the electron (and ion) density in the wave frame, and $f(\eta_1,\eta_2)$ is normalized to unity. (Note that n_0 is different in the lab and wave frames.)

Maxwell's equation in the wave frame is

$$\nabla\times\mathbf{B}=B_1k(\cos kz\mathbf{e}_x-\sin kz\mathbf{e}_y)$$
$$=-\frac{4\pi n_0e}{c}\left[c\nu\mathbf{e}_z+\int\beta c\mathbf{u}f(\eta_1,\eta_2)d^3u\right]. \quad (17)$$

Noticing that the curl of our assumed magnetic field has no component in the z direction, the total current along $\mathbf{B}_0$ must be zero. If the ions are stationary in the lab system, then the electron current in this frame must be zero. However, in the linearized theory, one obtains a zero-order distribution function of the form $f_0=f_0(u_\perp,u_z)$, which allows an arbitrary zero-order electron drift along the magnetic field. In other words, one usually assumes that the magnetic field from the electron current can be neglected in comparison to the wave magnetic field. This assumption cannot be made in an exact self-consistent analysis, and we restrict ourselves to distributions yielding zero net current along the magnetic field in the lab system. (A similar statement holds if the ions are not considered immobile.) Then in cylindrical coordinates, the components of Eq. (17) are

$$\int_0^\infty du_\perp\,u_\perp^2\int_{-\infty}^\infty du_z\,\beta\int_{-\pi}^{\pi}f(\eta_1,\eta_2)\cos\zeta d\zeta$$
$$=-\nu_1(kc/\omega_p)^2, \quad (18a)$$

$$\int_0^\infty du_\perp\,u_\perp\int_{-\infty}^\infty du_z\,\beta u_z\int_{-\pi}^{\pi}f(\eta_1,\eta_2)d\zeta=-\nu, \quad (18b)$$

$$\int_0^\infty du_\perp\,u_\perp\int_{-\infty}^\infty du_z\int_{-\pi}^{\pi}f(\eta_1,\eta_2)d\zeta=1, \quad (18c)$$

where $\omega_p=(4\pi n_0e^2/m)^{1/2}$. Equation (18a) follows from either the x or y component of Eq. (17), (18b) from the z component, and (18c) is the normalization integral. We wish to rewrite Eqs. (18) with η_1, u_z, and η_2 as variables of integration for reasons which will become apparent in the next section. The transformation is straightforward if we first consider a spherical coordinate system in $\mathbf{u}$ space. Then $\eta_1=u^2$ is the square of the radius vector, and one need only change the ϕ integration to one over η_2 which yields

$$\int_0^\infty d\eta_1\,\beta\int_{-\sqrt{\eta_1}}^{\sqrt{\eta_1}}du_z\int_{a-b}^{a+b}\frac{(\eta_2-a)f(\eta_1,\eta_2)}{[b^2-(\eta_2-a)^2]^{1/2}}d\eta_2$$
$$=-2\nu_1^2(kc/\omega_p)^2, \quad (19a)$$

$$\int_0^\infty d\eta_1\,\beta\int_{-\sqrt{\eta_1}}^{\sqrt{\eta_1}}du_z\,u_z$$
$$\times\int_{a-b}^{a+b}\frac{f(\eta_1,\eta_2)}{[b^2-(\eta_2-a)^2]^{1/2}}d\eta_2=-\nu, \quad (19b)$$

$$\int_0^\infty d\eta_1\int_{-\sqrt{\eta_1}}^{\sqrt{\eta_1}}du_z\int_{a-b}^{a+b}\frac{f(\eta_1,\eta_2)}{[b^2-(\eta_2-a)^2]^{1/2}}d\eta_2=1, \quad (19c)$$

where $\beta=(1+\eta_1)^{-1/2}$, $a=(u_z+\nu_0)^2$, and $b=2\nu_1(\eta_1-u_z^2)^{1/2}$. The specification of the zeroth and three first velocity moments (Maxwell's equations) of the distribution function $f(\eta_1,\eta_2)$ imply the above integrals over f must be satisfied. f is not uniquely determined from the above equations, and the degree of arbitrariness may be illustrated by expanding f in a Fourier cosine series

$$f(\eta_1,\,a+b\cos\zeta)=\tfrac{1}{2}C_0+C_1\cos\zeta+C_2\cos2\zeta+\cdots,$$

where $C_n=(1/\pi)\int_{-\pi}^{\pi}f\cos(n\zeta)d\zeta$. Substituting in Eqs. (18) yields

$$\pi\int_0^\infty du_\perp\,u_\perp^2\int_{-\infty}^\infty du_z\,\beta C_1(u_\perp,u_z)=-\nu_1(kc/\omega_p)^2, \quad (20a)$$

$$\pi\int_0^\infty du_\perp\,u_\perp\int_{-\infty}^\infty du_z\,\beta u_zC_0(u_\perp,u_z)=-\nu, \quad (20b)$$

$$\pi\int_0^\infty du_\perp\,u_\perp\int_{-\infty}^\infty du_z\,C_0(u_\perp,u_z)=1. \quad (20c)$$

C_0 and C_1 may be picked independently to satisfy Eqs. (20) for any values of k and ν. Therefore, there is no dispersion relation for these waves in the sense of a usual correspondence between frequency and wave velocity. This feature arises from our postulating a steady-state phenomena without regard to the manner in which the wave was established.[13]

IV. SMALL-AMPLITUDE WAVES

In linear theory it is assumed that one can separate the distribution into the sum of two terms in the form $f_{\text{lin}}=f_0+f_1$, where f_0 is the zero-order distribution for a plasma immersed in a uniform magnetic field $\mathbf{B}_0$, and f_1, the first-order term, is assumed to be proportional to the amplitude of the wave, or ν_1. This separation has

[13] If one considers a distribution of the form $f(\eta_1,\eta_2)=(1/\beta)F(A\eta_1+B\eta_2)$, where F is an arbitrary function of any linear combination of η_1 and η_2 with A and $B\neq0$, then it is interesting to note that sufficient arbitrariness on f is removed to yield a dispersion relation for these waves. In the nonrelativistic limit ($\nu\ll1$), this relation reduces to that for whistlers in a cold plasma [see, for example, J. A. Ratcliffe, *The Magnetoionic Theory and its Applications to the Ionosphere* (Cambridge University Press, Cambridge, England, 1959), p. 19]

$$(kc/\omega)^2=1+\omega_p^2/\omega(\Omega_0-\omega).$$

been justified for the electrostatic problem[7] by observing it is necessary and sufficient that f_{lin} reproduce the average value of any function ψ of physical interest correct to first order in ν_1. For the electromagnetic case the functions of interest are periodic functions of ζ, and it suffices to consider a single Fourier component of ψ, which implies $\psi\equiv\psi(\eta_1,u_z,\eta_2/2\nu_1)$. Then, from the development of Eqs. (19), this average is given by

$$\bar{\psi}=\int_0^\infty d\eta_1\int_{-\sqrt{\eta_1}}^{\sqrt{\eta_1}}du_z\int_{a-b}^{a+b}\frac{f(\eta_1,\eta_2)}{[b^2-(\eta_2-a)^2]^{1/2}}\psi d\eta_2. \quad (21)$$

To justify the separation of the distribution function for the electromagnetic problem, our approach is to first identify the zero-order distribution function of linear theory f_0 by considering the particle trajectories in phase space and determining those values of η_1 and η_2 for which $f(\eta_1,\eta_2)$ possesses a limit as ν_1 tends to zero. The distribution function and the particle trajectories are studied and the electromagnetically trapped particles are defined in Appendix A, where it is demonstrated that the (η_1,η_2) plane may be divided into two regions. Particles in region I defined by

$$\begin{aligned}&\text{(i)}\quad \eta_1<\nu_0^2, \quad\text{and}\quad (\nu_0-\sqrt{\eta_1})^2\leq\eta_2\leq(\nu_0+\sqrt{\eta_1})^2,\\&\text{(ii)}\quad \eta_1\geq\nu_0^2, \quad\text{and}\quad 0\leq\eta_2\leq(\nu_0+\sqrt{\eta_1})^2,\end{aligned} \quad (22a)$$

have well-defined trajectories as $\nu_1\to 0$. No trajectories exist at $\nu_1=0$ for particles in region II defined by

$$\begin{aligned}&\text{(i)}\quad \eta_1<\nu_0^2, \quad\text{and}\quad \eta_2<(\nu_0-\sqrt{\eta_1})^2\\&\qquad\qquad\qquad\qquad\text{or}\quad \eta_2>(\nu_0+\sqrt{\eta_1})^2, \quad (22b)\\&\text{(ii)}\quad \eta_1\geq\nu_0^2, \quad\text{and}\quad \eta_2<0 \quad\text{or}\quad \eta_2>(\nu_0+\sqrt{\eta_1})^2.\end{aligned}$$

It is further shown that, for certain η_1, η_2, $f(\eta_1,\eta_2)$ may have two distinct values, depending upon whether $u_z+\nu_0$ is less or greater than zero. Then, for (η_1,η_2) in region I, we define

$$\begin{aligned}f&=f^+(\eta_1,\eta_2) \quad\text{for}\quad u_z+\nu_0>0,\\&=f^-(\eta_1,\eta_2) \quad\text{for}\quad u_z+\nu_0<0,\end{aligned} \quad (23)$$

and hence, in region I, $f=f^++f^-$. For the trapped electrons in this region [see Appendix A, $\eta_1\geq\nu_0^2$, $0\leq\eta_2\leq\alpha\nu_1$, $\alpha\equiv 2(\eta_1-\nu_0^2)^{1/2}$], the distribution is $f^+=f^-=\frac{1}{2}f$. The zero-order distribution function of linear theory is then identified with[14]

$$\begin{aligned}f_0(\eta_1,u_z)&=f^+[\eta_1,(u_z+\nu_0)^2], \quad u_z+\nu_0>0,\\&=f^-[\eta_1,(u_z+\nu_0)^2], \quad u_z+\nu_0<0.\end{aligned} \quad (24)$$

One usually assumes that $f_0(\eta_1,u_z)$ is an analytic function of η_1 and u_z for all real values of these variables.[15] The continuity of f_0 at $u_z=-\nu_0$ implies that

$$f_0(\eta_1,-\nu_0)=f^+(\eta_1,0)=f^-(\eta_1,0)=\tfrac{1}{2}f(\eta_1,0).$$

We also assume that $\partial f_0/\partial u_z\neq 0$ near $u_z=-\nu_0$. Then

$$f_0(\eta_1,u_z)=f_0(\eta_1,-\nu_0)+(u_z+\nu_0)\frac{\partial f_0}{\partial u_z}\bigg|_{u_z=-\nu_0}+\cdots.$$

This latter assumption imples that f^+ and f^- have expansions in half-integral powers of η_2 near $\eta_2=0$, i.e.,

$$f^\pm=f_0(\eta_1,-\nu_0)\pm(\sqrt{\eta_2})\frac{\partial f_0}{\partial u_z}\bigg|_{-\nu_0}+\cdots, \quad (25)$$

and $f^\pm$ are not analytic functions of η_2 near $\eta_2=0$, however f^++f^- will have a normal Taylor series expansion about $\eta_2=0$.

In Appendix B, the total region of integration in Eq. (21) is divided into sums of integrals over appropriate subregions (by considering the allowed values of η_2 and u_z for fixed η_1), where f may be identified with f_0 in the limit $\nu_1=0$. Any integral which tends to zero faster than the first power of ν_1 is neglected, and we obtain

$$\begin{aligned}&\int_0^\infty d\eta_1\int_{-\sqrt{\eta_1}}^{\sqrt{\eta_1}}du_z\,(f_0,\psi)+\nu_1\int_0^\infty d\eta_1\bigg[\int_{-\sqrt{\eta_1}}^{-\nu_0-(2\alpha\nu_1)^{1/2}}du_z+\int_{-\nu_0+(2\alpha\nu_1)^{1/2}}^{\sqrt{\eta_1}}du_z\bigg]\\&\qquad\times\frac{u_\perp(\partial f_0/\partial u_z)-u_z(\partial f_0/\partial u_\perp)}{u_z+\nu_0}(\cos\xi,\psi)+\int_{\nu_0^2}^\infty d\eta_1\int_{-\nu_0-(2\alpha\nu_1)^{1/2}}^{-\nu_0+(2\alpha\nu_1)^{1/2}}du_z\,(f-f_0,\psi),\end{aligned} \quad (26)$$

where we have defined

$$\begin{aligned}(f,\psi)&\equiv\int_{a-b}^{a+b}\frac{f(\eta_1,\eta_2)\psi}{[b^2-(\eta_2-a)^2]^{1/2}}d\eta_2\\&=\frac{1}{2}\int_{-\pi}^{\pi}f(\eta_1,a+b\cos\zeta)\psi d\xi.\end{aligned}$$

Therefore as $\nu_1\to 0$ the prescription for treating the singularity at $u_z=-\nu_0$ in the u_z integration in the second integral is to take the Cauchy principal value. Then for the first two integrals in Eq. (26), we transform back to the more familiar variables $(u_\perp,u_z,\zeta)$,and write these

[14] In the course of writing this paper, our attention has been drawn to the work of C. Brossier, Nucl. Fusion 4, 137 (1964). Brossier has considered the limits of the zeroth and three first velocity moments of the distribution function as the wave amplitude is reduced to zero in a nonrelativistic analysis. However, his identification of the zero-order distribution function is not correct, and he does not obtain a Dirac delta function in the first-order distribution.

[15] I. B. Bernstein, in *Radiation and Waves in Plasmas*, edited by M. Mitchner (Stanford University Press, Stanford, California, 1961), p. 36.

terms as

$$\int_0^\infty du_\perp\, u_\perp \int_{-\infty}^{\infty} du_z \times \int_{-\pi}^{\pi}\left[f_0+\nu_1\mathcal{P}\left(\frac{u_\perp(\partial f_0/\partial u_z)-u_z(\partial f_0/\partial u_\perp)}{u_z+\nu_0}\cos\zeta\right)\right]\psi d\zeta, \quad (27)$$

where $\mathcal{P}$ denotes the Cauchy principal value.

In Appendix C, it is shown that, to lowest order in ν_1, the last integral in Eq. (26) can be written as Eq. (C1), an integration over the distribution of trapped electrons defined in Appendix A. Then, utilizing Maxwell's equations [Eqs. (19a)–(19c)], with $\psi=\beta(\eta_2-a)$, and $\psi=1$, it is shown that the trapped electron distribution possesses an expansion in half-integral powers of η_2 about $\eta_2=0$. Substituting this expansion in Eq. (C1), we find that the integration of ψ over the distribution of trapped electrons yields a term proportional to ν_1. Combining this expression with Eq. (27), we find that the distribution function

$$f_{\rm lin}=f_0(u_\perp,u_z)+\nu_1\mathcal{P}\frac{u_\perp(\partial f_0/\partial u_z)-u_z(\partial f_0/\partial u_\perp)}{u_z+\nu_0}\cos(kz+\phi)+\nu_1 J\delta(u_z+\nu_0) \quad (28)$$

will reproduce the average value of any function ψ of physical interest correct to first order in ν_1. J denotes the expression in the curly brackets in (C4), which can be evaluated if both f_0, and the coefficients in the expansion of $f(\eta_1,\eta_2)$ about $\eta_2=0$ are specified, and $\delta(u_z+\nu_0)$ is the Dirac delta function. From Eq. (C2b), this integral will vanish when ψ does not depend upon ζ, i.e., for any function of the form $\psi(\eta_1,u_z)\equiv\psi(u_\perp,u_z)$. A Lorentz transformation of the first two terms of Eq. (28) to the laboratory frame yields

$$f_0(u_\perp,u_z)+\nu_1\mathcal{P}\frac{u_\perp(\partial f_0/\partial u_z)-(\beta u_z-\nu)(\partial f_0/\partial u_\perp)}{u_z-(\nu-\beta\nu_0)}\times\cos(kz+\phi-\omega t), \quad (29)$$

which are just those terms which enter into the usual linear theory of electromagnetic oscillations. All quantities in (29) are now measured in the lab frame. The delta function has been introduced previously by Pradhan,[1] who applied the normal mode analysis of Van Kampen[16] to the electromagnetic problem. As in the electrostatic case, the nonlinear method yields the physical nature of the singularity, namely, the need to account for the resonant or trapped electrons within the structure of the linear theory.[17] While the true distribution was well behaved, the first-order solutions are singular reflecting the fact that these waves do not damp.

[16] N. G. Van Kampen, Physica **21**, 949 (1955).

[17] It is noteworthy that the delta function at $u_z=-\nu_0$ arises "physically" because the drift velocity of the trapped particles in the cyclotron-resonance frame tends to zero faster than the first power of ν_1.

ACKNOWLEDGMENTS

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APPENDIX A

We introduce a coordinate system in phase space with polar axis u_z, cylindrical radius $u_\perp$, and azimuthal angle ζ, where we consider the spherical surface $\eta_1=u_\perp^2+u_z^2$, and the right parabolic cylinder $\eta_2=(u_z+\nu_0)^2+2\nu_1u_\perp\cos\zeta$ for given η_1 and η_2. The circle and parabola shown in Fig. 6 are the intersections of the respective surfaces with the plane $u_\perp\sin\zeta=0$. The parabolic surface is symmetric about $u_z=-\nu_0$ and has its vertex at $P=\eta_2/2\nu_1$. The intersection of these surfaces determines the curve(s) along which f has a particular value. This curve has the general appearance of the seam of a tennis ball if P lies between Q and Q'.

Then, for any "radius" $\sqrt{\eta_1}$, if $\eta_2<0$, as $\nu_1\to 0$ the point P will move to the left until there is no intersection. When $\eta_2>0$, as $\nu_1\to 0$ the point P moves infinitely to the right in such a manner that the parabolic surface approaches the two planes $u_z=-\nu_0\pm\sqrt{\eta_2}$ in the vicinity of the sphere. When $\eta_2<(\nu_0-\sqrt{\eta_1})^2$, the axis of the parabola is sufficiently below the sphere that there again exists no intersection as $\nu_1\to 0$. The same holds true for $\eta_2>(\nu_0+\sqrt{\eta_1})^2$ because the upper and lower limiting planes will then be above and below the sphere, respectively. Thus, for the above values of η_1 and η_2, motion is possible only in the strictly nonlinear regime. In the limit as the wave amplitude tends to zero, these modes of oscillation cease to exist.

To further discuss the particle trajectories, for $\eta_1>\nu_0^2$, we define the quantity $\alpha\equiv 2(\eta_1-\nu_0^2)^{1/2}$ (see Fig. 6). Then, for $\eta_1>\nu_0^2$, if $-\alpha\nu_1<\eta_2<0$, the point P lies between $-\alpha/2$ and 0 (i.e., $-\alpha/2<u_\perp\cos\zeta<0$). The z component of velocity oscillates about $u_z=-\nu_0$, and the

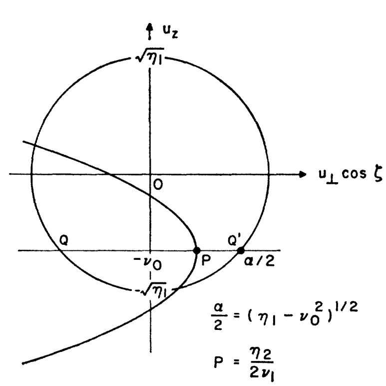

FIG. 6. The intersections of the sphere $\eta_1=u_\perp^2+u_z^2$ and the right parabolic cylinder $\eta_2=(u_z+\nu_0)^2+2\nu_1u_\perp\cos\zeta$ with the plane $u_\perp\sin\zeta=0$. The $u_\perp\sin\zeta$ axis points into the paper. The curves are drawn for $\eta_1>\nu_0^2$ and $0<\eta_2<\alpha\nu_1$.

allowed values of the angle ζ lie between $\pi-\cos^{-1}(\eta_2/\alpha\nu_1)$ and $\pi+\cos^{-1}(\eta_2/\alpha\nu_1)$. Thus, at any point z, the mean value of ϕ is $\bar{\phi}=\pi-kz$. With Eq. (1), we observe that this mean value of ϕ implies that the electrons are so organized that their mean transverse velocity is antiparallel to the magnetic field of the wave.[18] Essentially the same trajectories exist for $0<\eta_2<\alpha\nu_1$, except that the amplitudes of oscillation of ζ about $\zeta=\pi$ are greater, i.e., between $\pi/2$ and π. The basic difference between these modes is that the mode governed by $-\alpha\nu_1<\eta_2<0$ disappears completely as $\nu_1\to 0$, while the $0<\eta_2<\alpha\nu_1$ splits into two modes when $\nu_1<\eta_2/\alpha$. Thus it is precisely those electrons having $\eta_1>\nu_0^2$ and $-\alpha\nu_1<\eta_2<\alpha\nu_1$ which oscillate about $u_z=-\nu_0$ and feel the wave Doppler-shifted to their own cyclotron frequency. The distribution for these electrons is equally divided in u_z about $u_z=-\nu_0$, and in analogy with the electrostatic case, we define these resonant electrons as being "trapped."

Considering the remaining modes with $\eta_2>0$, for $(\nu_0-\sqrt{\eta_2})^2<\eta_1<\nu_0^2$, the motion is such that $u_z+\nu_0>0$, and as $\nu_1\to 0$, u_z approaches the limit $-\nu_0+\sqrt{\eta_2}$. For $\nu_0^2<\eta_1<(\nu_0+\sqrt{\eta_2})^2$, the lowest point on the sphere is below the axis of the parabola and above the plane $u_z=-\nu_0-\sqrt{\eta_2}$. When $\eta_2<\alpha\nu_1$ we have the trapped particle motion discussed previously, and when $\eta_2>\alpha\nu_1$ the motion is such that $u_z+\nu_0$ has a constant sign. As $\nu_1\to 0$ the upper parabolic surface flattens out to yield the limiting velocity $u_z=-\nu_0+\sqrt{\eta_2}$. The bottom surface might intersect the sphere depending upon the relative values of η_1, η_2, and α, but not in the limit at $\nu_1=0$. Finally, with $\eta_1>(\nu_0+\sqrt{\eta_2})^2$, $u_z+\nu_0$ again has a constant sign when $\eta_2>\alpha\nu_1$. In the limit these electrons have one of the two limiting velocities $u_z=-\nu_0\pm\sqrt{\eta_2}$.

From the preceding discussion we can divide the set of (η_1,η_2) into two regions, summarized by (22a) and (22b), according to whether or not the spherical and parabolic surfaces yield an intersection at $\nu_1=0$. For all $\eta_2\neq 0$ in region I, as $\nu_1\to 0$, the intersections approach one or two circles each of which is entirely above or below the plane $u_z=-\nu_0$. All values of $\zeta(-\pi\leq\zeta\leq\pi)$ are assumed by the electrons following their phase-space trajectories with angular velocity $d\zeta/dt\to\beta ck(u_z+\nu_0)$, the Doppler-shifted cyclotron frequency. For $\eta_2=0$ in region I, the intersection approaches the semicircle $u_\perp^2=\eta_1-\nu_0^2$ with $\pi/2\leq\zeta\leq 3\pi/2$ in the plane $u_z=-\nu_0$, and $d\zeta/dt\to 0$.

APPENDIX B

(i) $0<\eta_1<\nu_0^2$. Noting the conditions in (22a) and (22b) we plot the parabola $y_1(u_z)\equiv a=(u_z+\nu_0)^2$, and the ellipses $y_2=(\nu_0-\sqrt{\eta_1})^2\pm b$ and $y_3=(\nu_0+\sqrt{\eta_1})^2\pm b$, where $b=2\nu_1(\eta_1-u_z^2)^{1/2}$, in Fig. 7. The two points of intersection of y_1 and y_2 occur at $u_z=-\sqrt{\eta_1}$, P_1, and y_1 and y_3 intersect at $u_z=P_2$, $\sqrt{\eta_1}$, as indicated on the diagram. Then, for $-\sqrt{\eta_1}<u_z<P_1$: $a-b<(\nu_0-\sqrt{\eta_1})^2<a+b$; for $P_1<u_z<P_2$: $(\nu_0-\sqrt{\eta_1})^2<a-b<\eta_2<a+b<(\nu_0+\sqrt{\eta_1})^2$ (i.e., region I); and for $P_2<u_z<\sqrt{\eta_1}$: $a-b<(\nu_0+\sqrt{\eta_1})^2<a+b$. Hence the portion of the integration in Eq. (21) from $0<\eta_1<\nu_0^2$ may be written as

$$\int_0^{\nu_0^2} d\eta_1\left\{\int_{-\sqrt{\eta_1}}^{P_1} du_z\left[\int_{a-b}^{(\nu_0-\sqrt{\eta_1})^2}\frac{f\psi}{[b^2-(\eta_2-a)^2]^{1/2}}d\eta_2+\int_{(\nu_0-\sqrt{\eta_1})^2}^{a+b}\frac{f^+\psi}{[b^2-(\eta_2-a)^2]^{1/2}}d\eta_2\right]\right.$$
$$\left.+\int_{P_1}^{P_2} du_z\int_{a-b}^{a+b}\frac{f^+\psi}{[b^2-(\eta_2-a)^2]^{1/2}}d\eta_2+\int_{P_2}^{\sqrt{\eta_1}} du_z\left[\int_{a-b}^{(\nu_0+\sqrt{\eta_1})^2}\frac{f^+\psi}{[b^2-(\eta_2-a)^2]^{1/2}}d\eta_2+\int_{(\nu_0+\sqrt{\eta_1})^2}^{a+b}\frac{f\psi}{[b^2-(\eta_2-a)^2]^{1/2}}d\eta_2\right]\right\}. \tag{B1}$$

In the limit at $\nu_1=0$, (B1) becomes

$$\int_0^{\nu_0^2} d\eta_1\int_{-\sqrt{\eta_1}}^{\sqrt{\eta_1}} du_z\int_{a-b}^{a+b}\frac{f^+\psi}{[b^2-(\eta_2-a)^2]^{1/2}}d\eta_2. \tag{B2}$$

The integrals in (B1) differ from the integral in (B2) by integrations over the shaded areas in Fig. 7. These latter integrals tend to zero as $\nu_1^{3/2}$ because $P_1\to-\sqrt{\eta_1}$ and $P_2\to\sqrt{\eta_1}$ as ν_1 and $\int_0^{\nu_1}x^{-1/2}dx\to 0$ as $\nu_1^{1/2}$.

(ii) $\eta_1\geq\nu_0^2$. In order to bring out the effects of the trapped electrons, we consider the parabola $y_1=a=(u_z+\nu_0)^2$ and the ellipses $y_2=\alpha\nu_1\pm b$ and $y_3=(\nu_0+\sqrt{\eta_1})^2\pm b$ plotted in Fig. 8. One can show that for any $\eta_1>\nu_0^2$, if we pick $\nu_1<\nu_1^*$, where $2\nu_1^*=[(\sqrt{\eta_1})-\nu_0]^{3/2}/[(\sqrt{\eta_1})+\nu_0]^{1/2}$, then the point on the bottom ellipse $u_z=-\sqrt{\eta_1}$ is to the left of the parabola, and there will be two arcs of intersection of y_1 and y_2: (P_3,P_4) and $(-\nu_0, P_5)$. This value of ν_1 depends only upon η_1 and as $\nu_1\to 0$, this is the only case to consider. We observe that for $-\sqrt{\eta_1}<u_z<P_3$ and $P_5<u_z<P_6$: $\alpha\nu_1<a-b<\eta_2<a+b<(\nu_0+\sqrt{\eta_1})^2$; $P_3<u_z<P_5$: $a-b<\alpha\nu_1$; $P_6<u_z<\sqrt{\eta_1}$: $a-b<(\nu_0+\sqrt{\eta_1})^2<a+b$.

[18] A qualitative discussion of such phase organization has been given by N. Brice, J. Geophys. Res. 68, 4626 (1961).

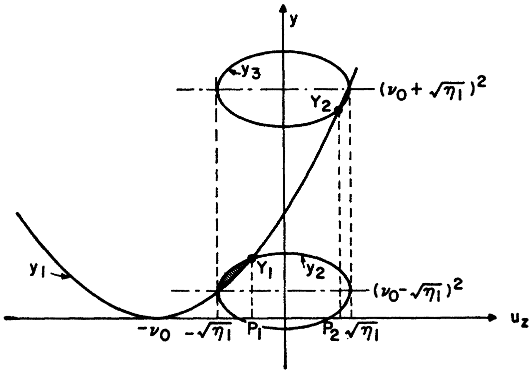

FIG. 7. The curves $y_1=(u_z+\nu_0)^2$, $y_2=(\nu_0-\sqrt{\eta_1})^2\pm b$, and $y_3=(\nu_0+\sqrt{\eta_1})^2\pm b$, where $b=2\nu_1(\eta_1-u_z^2)^{1/2}$.

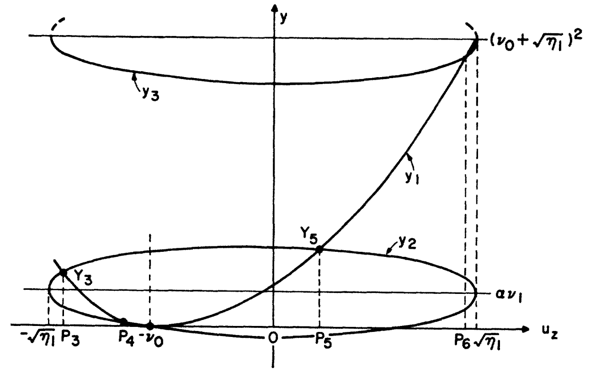

FIG. 8. The curves $y_1=(u_z+\nu_0)^2$, $y_2=\alpha\nu_1\pm b$, and $y_3=(\nu_0+\sqrt{\eta_1})^2\pm b$. For $-P_4<u_z<-\nu_0$, the parabola y_1 is below the ellipse y_2.

Then the integral from ν_0^2 to ∞ can be written as

$$\int_{\nu_0^2}^{\infty} d\eta_1 \left\{ \int_{-\sqrt{\eta_1}}^{P_3} du_z\,(f^-,\psi)+\int_{P_3}^{P_5} du_z\,(f,\psi)+\int_{P_5}^{P_6} du_z\,(f^+,\psi) + \int_{P_6}^{\sqrt{\eta_1}} du_z \left[\int_{a-b}^{(\nu_0+\sqrt{\eta_1})^2} \frac{f^+\psi}{[b^2-(\eta_2-a)^2]^{1/2}}d\eta_2 + \int_{(\nu_0+\sqrt{\eta_1})^2}^{a+b} \frac{f\psi}{[b^2-(\eta_2-a)^2]^{1/2}}d\eta_2 \right] \right\}. \quad \text{(B3)}$$

As $\nu_1\to 0$, the points in Fig. 8 Y_3, $Y_5\to 2\alpha\nu_1$, $P_3\to -\nu_0-(2\alpha\nu_1)^{1/2}$, $P_5\to -\nu_0+(2\alpha\nu_1)^{1/2}$. Then following the same argument used in establishing (B2) from (B1) we have for ν_1 small

$$\int_{\nu_0^2}^{\infty} d\eta_1 \left[\int_{-\sqrt{\eta_1}}^{-\nu_0-(2\alpha\nu_1)^{1/2}} du_z\,(f^-,\psi)+\int_{-\nu_0+(2\alpha\nu_1)^{1/2}}^{\sqrt{\eta_1}} du_z\,(f^+,\psi)+\int_{\nu_0^2}^{\infty} d\eta_1 \int_{-\nu_0-(2\alpha\nu_1)^{1/2}}^{-\nu_0+(2\alpha\nu_1)^{1/2}} du_z\,(f,\psi) \right]. \quad \text{(B4)}$$

Expanding the integrands of (B2), and the first two integrals in (B4), in a Taylor series about $\nu_1=0$, and using the relations

$$u_\perp \frac{\partial f^\pm}{\partial u_z} - u_z \frac{\partial f^\pm}{\partial u_z} = 2\frac{\partial f^\pm}{\partial \eta_2}[u_\perp(u_z+\nu_0)-\nu_1 u_z \cos\zeta]$$

and $\partial\eta_2/\partial\nu_1=2u_\perp \cos\zeta$ results in (26).

APPENDIX C

To lowest order in ν_1 over the range of integration indicated by the limits in the last inetgral in Eq. (26), $b=2\nu_1(\eta_1-u_z^2)^{1/2}$ may be replaced by $\alpha\nu_1$. The (η_2,u_z) integration is then over the area in the (η_2,u_z) plane between the parabolas $y_1=(u_z+\nu_0)^2-\alpha\nu_1$ and $y_2=(u_z+\nu_0)^2+\alpha\nu_1$ and bounded by the lines $u_z=-\nu_0\pm\alpha\nu_1$. Inverting the order of integration, f can be identified with either f^+ or f^- in two of the three resulting integrals. Changing variables from η_2 to $x=\eta_2/\alpha\nu_1$ and recalling that ψ depends on η_2 only as $\eta_2/2\nu_1$, these integrals may be written to lowest order in ν_1 as

$$\nu_1\int_{\nu_0^2}^{\infty} d\eta_1\,\alpha\int_1^3 dx \int_{-\nu_0-(2\alpha\nu_1)^{1/2}}^{-\nu_0-[\alpha\nu_1(x-1)]^{1/2}} du_z \frac{[f^-(\eta_1,\alpha\nu_1 x)-f_0]\psi(\eta_1,x,u_z)}{\{(\alpha\nu_1)^2-[(u_z+\nu_0)^2-\alpha\nu_1 x]^2\}^{1/2}}$$
$$+\nu_1\int_{\nu_0^2}^{\infty} d\eta_1\,\alpha\int_1^3 dx \int_{-\nu_0+[\alpha\nu_1(x-1)]^{1/2}}^{-\nu_0+(2\alpha\nu_1)^{1/2}} du_z \frac{[f^+(\eta_1,\alpha\nu_1 x)-f_0]\psi(\eta_1,x,u_z)}{\{(\alpha\nu_1)^2-[(u_z+\nu_0)^2-\alpha\nu_1 x]^2\}^{1/2}}$$
$$+\nu_1\int_{\nu_0^2}^{\infty} d\eta_1\,\alpha\int_{-1}^1 dx \int_{-\nu_0-[\alpha\nu_1(1+x)]^{1/2}}^{-\nu_0+[\alpha\nu_1(1+x)]^{1/2}} du_z \frac{\frac{1}{2}[f(\eta_1,\alpha\nu_1 x)-f(\eta_1,0)]\psi}{\{(\alpha\nu_1)^2-[(u_z+\nu_0)^2-\alpha\nu_1 x]^2\}^{1/2}}. \quad \text{(C1)}$$

But from Eq. (25) we have $f^{\pm}(\eta_1,\alpha\nu_1 x)-f_0=\pm(\alpha\nu_1 x)(\partial f_0/\partial u_z)|_{-\nu_0}+\cdots$. Substituting these expressions for f^- and f^+ into the first and second integrals in (C1), respectively, and expanding ψ in a Taylor series in u_z about $u_z=-\nu_0$, we find that to first order in ν_1 these two integrals exactly cancel. The last integral in (C1) is an integration of ψ over the distribution of trapped electrons defined in Appendix A. To determine the nature of the expansion of this distribution for small η_2 we consider Maxwell's equations (19) with $\psi=\beta(\eta_2-a)$ and $\psi=1$. With these substitutions we obtain

$$\int_0^\infty d\eta_1\,\alpha^{3/2}\int_{-1}^1 dx\,[xE_1(x)-E_2(x)][f(\eta_1,\alpha\nu_1 x)-f(\eta_1,0)]=\nu_1^{1/2}R(f_0)\,, \tag{C2a}$$

$$\int_0^\infty d\eta_1\,\alpha^{1/2}\int_{-1}^1 dx\,E_1(x)[f(\eta_1,\alpha\nu_1 x)-f(\eta_1,0)]=0\,, \tag{C2b}$$

respectively, where

$$R(f_0)\equiv-2\left(\frac{kc}{\omega_p}\right)^2-2\pi\int_0^\infty du_\perp\,u_\perp{}^2\mathcal{P}\int_{-\infty}^\infty \beta\frac{u_\perp(\partial f_0/\partial u_z)-u_z(\partial f_0/\partial u_\perp)}{u_z+\nu_0}du_z\,,$$

and E_1 and E_2 are complete elliptic integrals which we define by

$$E_1(x)=\int_0^{(1+x)^{1/2}}[1-(w^2-x)^2]^{-1/2}dw \quad\text{and}\quad E_2(x)=\int_0^{(1+x)^{1/2}}[1-(w^2-x)^2]^{-1/2}w^2dw\,.$$

[It may be noted that taking $\psi=u_z$ also yields Eq. (C2b).] Hence, assuming only that f is continuous at $\eta_2=0$ we have

$$f(\eta_1,\eta_2)-f(\eta_1,0)=Ag_1(\eta_1)|\eta_2|^{1/2}+\cdots\,,\quad\text{for}\quad \eta_2>0\,, \tag{C3a}$$

$$=Bg_2(\eta_1)|\eta_2|^{1/2}+\cdots\,,\quad\text{for}\quad \eta_2<0\,, \tag{C3b}$$

where A and B are constants and g_1 and g_2 are functions of η_1 alone. Substituting Eqs. (C3) into Eq. (C2) yields two linear equations which can in principle be solved for A and B provided f_0, and the expansion coefficients of f about $\eta_2=0$, g_1 and g_2, are known. Therefore while we have assumed that $f(\eta_1,\eta_2)$ is continuous about $\eta_2=0$, $\partial f/\partial\eta_2$ is not necessarily continuous.

Substituting Eqs. (C3) into the last integral in Eq. (C1) and expanding ψ in a Taylor series about $u_z=-\nu_0$, we find the contribution to first order in ν_1 to be

$$\nu_1\int_{\nu_0{}^2}^\infty d\eta_1\int_{-1}^1 dx\,\{|x|^{1/2}E_1(x)[Ag_1(\eta_1)H(x)+Bg_2(\eta_1)H(-x)]\}\psi(\eta_1,\,x,\,-\nu_0)\,, \tag{C4}$$

where $H(x)$ is the Heaviside function

$$H(x)=1\,,\quad x>0$$
$$=0\,,\quad x<0\,.$$