Stability of a Lattice of Superfluid Vortices

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Landau's superfluid hydrodynamics is applied to the vibration spectrum of a lattice of rectilinear vortices in both charged and neutral superfluid systems. The resulting vortex dynamics is identical with that of classical hydrodynamics: each vortex moves with the local superfluid velocity at its core. The only mode considered is one in which the vortices move without bending. This mode is unstable for all lattice structures in a neutral system (liquid helium II); in a charged system (type-II superconductors) the mode is unstable for a square lattice but stable for a triangular lattice. The corresponding long-wavelength dispersion relation is $\omega=(eB/mc)q^2\lambda d(\sqrt{3}/32\pi)^{1/2}$, where B is the magnetic induction, λ is the London penetration depth, q is the wave number, and d is the lattice spacing. An elasticity theory of the lattice vibrations in the charged system is shown to predict identical results. These calculations agree qualitatively with those of de Gennes and Matricon but disagree with those of Abrikosov, Kemoklidze, and Khalatnikov; the discrepancies are discussed in detail.

I. INTRODUCTION

THE vibrational modes of an array of classical vortices have been studied for nearly a century as a criterion for hydrodynamic stability.[1,2] In recent years, similar vortex structures have been found in superfluid He II [3] and in type-II superconductors.[4] In his original paper,[4] Abrikosov compared the free energy of square and triangular vortex lattices. His numerical calculations were repeated and partially corrected by Matricon[5] and by Kleiner, Roth, and Autler,[6] who found that the triangular array always had lower free energy than the square one. The calculations were valid either very near H_{c2} for arbitrary values of the Ginzburg-Landau parameter[4,6] κ or for intermediate fields[5] and very low fields[4] ($H\gtrsim H_{c1}$) in the limit $\kappa\to\infty$. In order to test whether either structure was microscopically stable against small deviations from equilibrium, de Gennes and Matricon[7,5] presented a dynamical theory of vibrations of the lattice. This theory relied on the hydrodynamic concept of the Magnus force[8] and predicted several modes, which have not been observed. They verified that the triangular array was in fact microscopically stable, whereas the square was not. Additional confirmation of the stability of the triangular array was provided by the beautiful neutron-diffraction experiments of Cribier *et al.*,[9] who were able to distinguish between the two lattice structures. The hydrodynamic approach has been criticized for neglecting the positive background of the ionic lattice,[10] for predicting too large a Hall angle,[11,12] and for omitting current oscillations.[13] In particular, Abrikosov, Kemoklidze, and Khalatnikov[13] find imaginary frequencies for the vibration modes of both lattices, which implies an instability in the absence of dissipative forces.

The hydrodynamic theory has also been applied to a lattice of quantized vortices in superfluid helium II.[14]

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[1] J. J. Thomson, *A Treatise on the Motion of Vortex Rings* (Macmillan and Company, London, 1883), pp. 94–108.

[2] T. Von Karman, Nachr. Kgl. Ges. Wiss. Goettingen Math. Physik Kl., 509 (1911); 547 (1912).

[3] L. Onsager, Nuovo Cimento **6**, Suppl. II, 249 (1949); R. P. Feynman, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North Holland Publishing Company, Amsterdam, 1955), Vol. I, p. 17.

[4] A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)].

[5] J. Matricon, Phys. Letters **9**, 289 (1964).

[6] W. H. Kleiner, L. M. Roth, and S. H. Autler, Phys. Rev. **133**, A1226 (1964). See also G. Eilenberger, Z. Physik **180**, 32 (1964); and D. Saint James (to be published).

[7] P. G. de Gennes and J. Matricon, Rev. Mod. Phys. **36**, 45 (1964).

[8] J. Friedel, P. G. de Gennes, and J. Matricon, Appl. Phys. Letters **2**, 119 (1963).

[9] D. Cribier, B. Jacrot, L. M. Rao, and B. Farnoux, Phys. Letters **9**, 106 (1964), and private communications.

[10] J. Bardeen, Phys. Rev. Letters **13**, 747 (1964).

[11] P. H. Borcherds, C. E. Gough, W. F. Vinen, and A. C. Warren, Phil. Mag. **10**, 349 (1964).

[12] M. J. Stephen and J. Bardeen, Phys. Rev. Letters **14**, 112 (1965).

[13] A. A. Abrikosov, M. P. Kemoklidze, and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. **48**, 765 (1965) [English transl.: Soviet Phys.—JETP **21**, 506 (1965)].

[14] P. Pincus and K. A. Shapiro, Phys. Rev. Letters **15**, 103 (1965); **15**, 597 (E) (1965).

Due to a numerical error, Pincus and Shapiro concluded that a lattice would be stable. A more correct calculation in this case shows that the vibration frequencies are imaginary for any infinite lattice, and in fact there is no experimental evidence for any regular lattice. The concept of the Magnus force was not used explicitly by Pincus and Shapiro,[14] who formulated the problem directly in terms of the velocity field. This approach, which is also used here, starts from the observation that the vorticity is a convective quantity that follows the motion of the fluid particles.[15] Thus the vortex core, which contains the concentrated vorticity, moves with the local fluid velocity at its position. If a lattice of vortices is disturbed from its equilibrium configuration, the translational velocity of each vortex is merely the net velocity induced at its core by all the other vortices.

In this paper, we develop the above approach for a simple model of quantized vortices in superfluids that can be analyzed in complete mathematical detail. In this model the structure of the vortex cores is neglected so that it is permissible to apply Landau's dynamical equation for the superfluid velocity,[16] generalized to include electromagnetic forces. We make the plausible assumption that the vortices move without distortion [Eq. (7)] and then prove that for our model each vortex moves with the local velocity at its core [Eq. (13)]. In the limit of zero electric charge the model applies to helium II at very low temperatures. For finite charge it describes a hypothetical type-II superconductor of truly infinite κ parameter (or infinite H_{c2}), namely, a system in which the core diameter ξ is vanishingly small. As shown below, our dynamical theory predicts instability in liquid helium II for all lattice structures. The question of stability in type-II superconductors can be answered only by evaluating lattice sums, which is here done in analytic form. The square lattice is found to be unstable while the triangular lattice is stable, in agreement with earlier work.[4–6] The markedly different behavior in helium and in superconductors is due to the form of the interaction between vortices: the neutral system has a long-range interaction, while the charged system has a short-range interaction because the London penetration depth λ acts as a natural cutoff. In the present simplified model, the lattice arrangement can be stable only if the size of the sample is large compared with the range of interaction between vortices.

In the work of Matricon[5] the stability criterion for the lattice of flux lines is based on classical elasticity theory. For comparison, we have developed an elasticity theory from first principles. The elastic energy contains terms linear in the strains, as well as the usual quadratic ones, which indicates that the lattice cannot be an equilibrium configuration without an external applied (magnetic) pressure. In addition, certain symmetry conditions in the elastic theory of crystals are not satisfied for the vortex lattice. The generalized elasticity theory given here precisely reproduces the long-wavelength limit of the theory based on lattice dynamics.

The main conclusion of the present work is that the triangular lattice in our simple model of a type-II superconductor is stable, in agreement with experiments[9] and with the free-energy calculations based on the Ginzburg-Landau theory.[4–6] In contrast, Abrikosov, Kemoklidze, and Khalatnikov[13] treat the same model ($\kappa\to\infty$, $T=0$) and use the same dynamical equation, but find imaginary frequencies for both lattices in the absence of viscous damping forces. They employ an averaging procedure over areas containing many vortices, whereas we treat each vortex separately throughout the calculation. The process of averaging introduces additional assumptions, which appear to be the source of the discrepancy.

In Sec. II, the basic hydrodynamic equation governing the motion of vortices is derived from Landau's superfluid dynamics. The resulting stability criterion is used in Sec. III to explain the qualitatively different behavior of vortices in liquid helium II (neutral) and in type-II superconductors (charged). Section IV contains the detailed theory of the square and triangular lattices in superconductors, where the lattice sums are evaluated explicitly. The elasticity theory of the lattice of quantized flux lines in type-II superconductors is developed in Sec. V. The validity of our model is considered in Sec. VI, along with a discussion of the averaging procedure of Ref. 13. Mathematical details are given in the Appendices.

II. THE STABILITY CRITERION

As a fundamental assumption, we describe the motion of the superfluid by Landau's dynamical equation, generalized to include electromagnetic forces

$$\partial \mathbf{v}/\partial t+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+\boldsymbol{\nabla}\mu=(e/m)(\mathbf{E}_0+c^{-1}\mathbf{v}\times\mathbf{H}), \quad (1)$$

which is just Eq. (3) of Ref. 13.[17] Here $\mathbf{H}$ is the magnetic field, μ is the chemical potential, $\mathbf{E}_0$ is the *applied* electric field (assumed small), and $\mathbf{v}$ is the superfluid velocity. All normal-fluid effects and frictional forces[13] are omitted; the validity of this important restriction is discussed in Sec. VI. In the presence of vorticity, the usual London equation must be generalized to[18]

$$\mathbf{Q}\equiv\operatorname{curl}\mathbf{v}+(e\mathbf{H}/mc)=\bar{\kappa}\textstyle\sum_n \boldsymbol{\nu}_n\delta(\mathbf{r}-\mathbf{r}_n), \quad (2)$$

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[15] A. Sommerfeld, *Mechanics of Deformable Bodies* (Academic Press Inc., New York, 1952), Chap. IV.

[16] L. Landau, J. Phys. (USSR) 5, 71 (1941); reprinted in I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (W. A. Benjamin, Inc., New York, 1965), p. 185.

[17] For superconductors, the linearized version of Eq. (1) is just the London acceleration equation, which can also be derived from the microscopic theory under certain assumptions. This is precisely the approach used in the treatment of the Josephson effect, as in B. D. Josephson, Rev. Mod. Phys. **36**, 216 (1964) and P. W. Anderson and A. H. Dayem, Phys. Rev. Letters **13**, 195 (1964).

[18] Our quantity Q differs from that of Ref. 13 by a factor *ne*.

where the summation runs over all vortices in the sample. As stated earlier, the present treatment is limited to systems in which the Ginzburg-Landau parameter κ is very large,[4] so that the core of each vortex (pointing along a unit vector $\boldsymbol{\nu}_n$) may be replaced by a two-dimensional delta function placed at the point $\mathbf{r}_n$. The quantity $\bar{\kappa}$ (which should be distinguished from the Ginzburg-Landau parameter κ) is the circulation about the vortex: $\bar{\kappa}=h/m_{\rm He}$ for liquid helium II, and $\bar{\kappa}=h/2m$ for superconductors. Throughout this section, the equations will be written in such a way that the correct result for He is obtained formally as the limit of vanishing electric charge ($e=0$).

Equation (1) may be rewritten as

$$\partial\mathbf{v}/\partial t-\mathbf{v}\times\operatorname{curl}\mathbf{v}=(e/m)[\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{H}], \tag{3}$$

where $\mathbf{E}$ is the total electric field

$$e\mathbf{E}/m=e\mathbf{E}_0/m-\nabla(\mu+\tfrac{1}{2}v^2), \tag{4}$$

in agreement with the definition of Abrikosov *et al.*[13] The curl of Eq. (3) may be combined with Maxwell's equation $\operatorname{curl}\mathbf{E}=-c^{-1}\partial\mathbf{H}/\partial t$ to yield

$$\partial\mathbf{Q}/\partial t-\operatorname{curl}(\mathbf{v}\times\mathbf{Q})=0. \tag{5}$$

This equation is valid both for superconductors and helium, since it does not contain the charge e explicitly, but only implicity in the definition of $\mathbf{Q}$ [Eq. (2)]. It can be shown[15] from Eq. (5) that $\mathbf{Q}$ is convective if $\operatorname{div}\mathbf{Q}=0$.

The stability of a lattice of vortices may be discussed directly in terms of Eq. (5). The equilibrium configuration is assumed to be a periodic array with positions given by $\{\mathbf{r}_L{}^0\}$ and with axes parallel to the z axis ($\boldsymbol{\nu}_L{}^0=\hat{z}$). The precise two-dimensional lattice structure need not be specified at this point, except that each lattice site must be an inversion center to ensure that the equilibrium configuration is stationary. The stability of the lattice depends on the behavior when each vortex is displaced a small distance $\mathbf{u}_L$ from its equilibrium position

$$\begin{aligned}\mathbf{r}_L&=\mathbf{r}_L{}^0+\mathbf{u}_L(t),\\ \mathbf{u}_L(t)&=\mathbf{s}\exp[i(\mathbf{q}\cdot\mathbf{r}_L{}^0-\omega t)].\end{aligned} \tag{6}$$

Only a special perturbation is considered here, in which $\boldsymbol{\nu}_L$ is unchanged (the vortices do not bend) and $\mathbf{q}$ lies in the xy plane. The crucial assumption is that the velocity pattern moves rigidly with the vortex core, so that the total velocity at any point $\mathbf{r}$ is given by

$$\mathbf{v}(\mathbf{r},t)=\sum_L \mathbf{v}_0[\mathbf{r}-\mathbf{r}_L{}^0-\mathbf{u}_L(t)], \tag{7}$$

where $\mathbf{v}_0(\mathbf{r})$ is the velocity field of a single vortex situated at the origin. Since for an individual vortex at rest we must have $\operatorname{div}\mathbf{v}_0=0$, Eq. (7) implies that $\operatorname{div}\mathbf{v}(\mathbf{r},t)=0$. Furthermore, the vector $\mathbf{v}$ lies in the xy plane and is independent of z, so that $\mathbf{Q}$ is parallel to the z axis, as is the applied magnetic field $\mathbf{H}$, and $\operatorname{div}\mathbf{Q}=0$. Thus Eq. (5) may be rewritten as

$$\partial\mathbf{Q}/\partial t+(\mathbf{v}\cdot\nabla)\mathbf{Q}=0, \tag{8}$$

or

$$-\sum_L(\dot{\mathbf{u}}_L\cdot\nabla)\delta[\mathbf{r}-\mathbf{r}_L{}^0-\mathbf{u}_L(t)] +[\mathbf{v}(\mathbf{r})\cdot\nabla]\sum_L\delta[\mathbf{r}-\mathbf{r}_L{}^0-\mathbf{u}_L(t)]=0. \tag{9}$$

Equation (9) is of the form

$$\begin{aligned}&\sum_L[\mathbf{f}(\mathbf{r},\mathbf{r}_L)\cdot\nabla]\delta(\mathbf{r}-\mathbf{r}_L)=0,\\ &\mathbf{f}(\mathbf{r},\mathbf{r}_L)=-\dot{\mathbf{u}}_L+\mathbf{v}(\mathbf{r}),\end{aligned} \tag{10}$$

which is an equation for a distribution centered at the points $\{\mathbf{r}_L\}$. This equation may be solved by multiplying by an arbitrary differentiable function and integrating over all space. We assume that $\mathbf{u}_L$ is much less than the lattice spacing d, so that the points $\mathbf{r}_L$ are well separated. The arbitrary function may be chosen to vanish except near a single $\mathbf{r}_L$; each term in the sum (10) must vanish identically, which implies

$$\mathbf{f}(\mathbf{r}_L,\mathbf{r}_L)=0, \tag{11}$$

$$(\nabla\cdot\mathbf{f})|_{\mathbf{r}=\mathbf{r}_L}=0. \tag{12}$$

The second condition (12) is obtained by a partial integration and is here satisfied identically. Equation (11) yields the relation

$$\dot{\mathbf{u}}_L=\mathbf{v}(\mathbf{r}_L), \tag{13}$$

which is just the classical hydrodynamic result[14,15] that each vortex moves with the local superfluid velocity at the position of its core. An alternative derivation of Eq. (13) starts from the observation that the vortices have negligible inertial mass,[19] so that the total force on each vortex must vanish. In the absence of external forces or frictional damping, the only force is the Magnus force

$$\mathbf{F}_M=\rho\bar{\kappa}\boldsymbol{\nu}_L\times[\dot{\mathbf{u}}_L-\mathbf{v}(\mathbf{r}_L)]=0, \tag{14}$$

which immediately leads to the same result. Equation (14) may also be written as

$$\rho\bar{\kappa}\boldsymbol{\nu}_L\times\dot{\mathbf{u}}_L=\rho\bar{\kappa}\boldsymbol{\nu}_L\times\mathbf{v}(\mathbf{r}_L). \tag{15}$$

It can be shown (Sec. V) that the right side of Eq. (15) represents the negative of the force due to the interaction with all the other vortices; this last point of view is that used by de Gennes and Matricon.[5,7]

It is important to prove that the equilibrium configuration is in fact stationary, which means that

$$\begin{aligned}\mathbf{v}^{\rm eq}(\mathbf{r}_L{}^0)&=\sum_{L'}\mathbf{v}_0(\mathbf{r}_L{}^0-\mathbf{r}_{L'}{}^0)\\ &=\sum_{L'}{}'\,\mathbf{v}_0(\mathbf{r}_L{}^0-\mathbf{r}_{L'}{}^0)+\mathbf{v}_0(0)\\ &=0.\end{aligned} \tag{16}$$

[19] G. W. Rayfield and F. Reif, Phys. Rev. **136**, A1194 (1964).

The primed sum (which means omit the single term $L'=L$) vanishes because of the inversion symmetry of the lattice, while $\mathbf{v}_0(0)$ is zero by definition. This last requirement follows from the equilibrium equation

$$\begin{aligned}\partial\mathbf{Q}/\partial t|_{\mathbf{r}=\mathbf{r}_L{}^0}&=(\mathbf{v}_0(\mathbf{r})\cdot\nabla)\mathbf{Q}|_{\mathbf{r}=\mathbf{r}_L{}^0}\\&=(\mathbf{v}_0(0)\cdot\nabla)\mathbf{Q}=0.\end{aligned}\tag{17}$$

The right side of Eq. (13) may thus be expanded as

$$\begin{aligned}\mathbf{v}(\mathbf{r}_L)&=\sum_{L'}\mathbf{v}_0(\mathbf{r}_L-\mathbf{r}_{L'})\\&=\sum_{L'}\mathbf{v}_0(\mathbf{r}_L{}^0-\mathbf{r}_{L'}{}^0+\mathbf{u}_L-\mathbf{u}_{L'})\\&=\sum_{L'}\mathbf{v}_0(\mathbf{r}_L{}^0-\mathbf{r}_{L'}{}^0)+\sum_{L'}{}'(\delta\mathbf{u}_{L'}\cdot\nabla)\mathbf{v}_0(\mathbf{r}_L{}^0-\mathbf{r}_{L'}{}^0),\end{aligned}\tag{18}$$

where $\delta\mathbf{u}_{L'}$, is defined as $\delta\mathbf{u}_{L'}=\mathbf{u}_L-\mathbf{u}_{L'}$. The first term vanishes according to Eq. (16) and the sum in the second term excludes the term $L'=L$. Equation (13) then reduces to a set of coupled linear equations,

$$\dot{\mathbf{u}}_L=\sum_{L'}{}'(\delta\mathbf{u}_{L'}\cdot\nabla)\mathbf{v}_0(\mathbf{r}_L{}^0-\mathbf{r}_{L'}{}^0),\tag{19}$$

which forms the basis of our stability criterion.[1,2,14]

III. DISTINCTION BETWEEN NEUTRAL AND CHARGED SUPERFLUIDS

In the present model, the distinction between charged and neutral superfluids arises only in the form of $\mathbf{v}_0(\mathbf{r})$. The velocity field of a single rectilinear vortex lying along the z axis is given by[4]

$$\mathbf{v}_0(\mathbf{r})=(\bar{\kappa}/2\pi\lambda)(\hat{z}\times\hat{r})K_1(r/\lambda),\tag{20}$$

where K_1 is the Bessel function of imaginary argument that vanishes exponentially for large values of its argument.[20] Here λ is the London penetration depth $\lambda=(mc^2/4\pi ne^2)^{1/2}$. For a neutral system, λ becomes infinite, and Eq. (20) reduces to the usual hydrodynamic result[15] that $\mathbf{v}_0(\mathbf{r})=(\bar{\kappa}/2\pi r)(\hat{z}\times\hat{r})$, since $K_1(x)\approx x^{-1}$ for $x\ll1$. Equation (20) may be written as

$$\mathbf{v}_0(\mathbf{r})=(\bar{\kappa}/2\pi)(\hat{z}\times\mathbf{r})f(r),\tag{21}$$

where

$$f(r)=(\lambda r)^{-1}K_1(r/\lambda).\tag{22}$$

If the lattice point L is taken as the origin of coordinates, the stability criterion (19) reduces to

$$\dot{\mathbf{u}}_0=\sum_L{}'(\delta\mathbf{u}_L\cdot\nabla)\mathbf{v}_0(-\mathbf{r}_L{}^0),\tag{23}$$

where the prime has been omitted on the dummy index L and $\delta\mathbf{u}_L=\mathbf{u}_0-\mathbf{u}_L$. A combination of Eqs. (21)–(23) yields

$$\dot{\mathbf{u}}_0=(\bar{\kappa}/2\pi)\hat{z}\times\sum_L{}'\{\delta\mathbf{u}_Lf(r_L)+(\mathbf{r}_L/r_L)(\delta\mathbf{u}_L\cdot\mathbf{r}_L)f'(r_L)\},\tag{24}$$

where $r_L=|\mathbf{r}_L|$ and the superscript 0 on $\mathbf{r}_L$ has been dropped. Equation (24) may be simplified by introducing the explicit form of the small disturbance [Eq. (6)]:

$$\begin{aligned}-i\omega s_x&=\alpha s_x+\beta s_y,\\-i\omega s_y&=\gamma s_x-\alpha s_y,\end{aligned}\tag{25}$$

where

$$\begin{aligned}\alpha&=(\bar{\kappa}/2\pi)\sum_L{}'(1-e^{i\mathbf{q}\cdot\mathbf{r}_L})[-(x_Ly_L/r_L)f'(r_L)],\\\beta&=(\bar{\kappa}/2\pi)\sum_L{}'(1-e^{i\mathbf{q}\cdot\mathbf{r}_L})\times[-f(r_L)-(y_L{}^2/r_L)f'(r_L)],\\\gamma&=(\bar{\kappa}/2\pi)\sum_L{}'(1-e^{i\mathbf{q}\cdot\mathbf{r}_L})[f(r_L)+(x_L{}^2/r_L)f'(r_L)].\end{aligned}\tag{26}$$

The eigenvalue equation for the vibration frequency is easily found to be

$$\omega^2=-\alpha^2-\beta\gamma=\eta^2-\alpha^2-\xi^2,\tag{27}$$

where

$$\begin{aligned}\beta&=\xi+\eta,\\\gamma&=\xi-\eta.\end{aligned}\tag{28}$$

It is not difficult to see that any lattice structure of vortices in a neutral superfluid (where λ is infinite) is unstable.[21] Since $f(r)\propto r^{-2}$ and $f'(r)=-2r^{-1}f(r)$, the quantities β and γ are identical, and ω^2 is an essentially negative quantity. Thus an arbitrary small disturbance of the equilibrium lattice structure grows exponentially with time. We interpret this instability to mean that the vortices in rotating liquid helium II are randomly distributed in the xy plane with essentially uniform density, like the molecules in a liquid.

This result is rigorous *only* for an infinite, inversion-symmetric lattice where each vortex is at rest in the equilibrium configuration, as in Eq. (16). Actual experiments on rotating He II are of course limited to finite systems, for instance a cylinder of radius R. In this case, the approximation of uniform vorticity implies[3,22] that the velocity field is just that of a solid body, rotating with angular velocity $\Omega=\bar{\kappa}n/2$ (n is the density of vortices). This velocity pattern is due to the circular geometry; the fluid is stationary only at the

[20] We follow the notation of G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambrigde University Press, Cambridge, 1962), 2nd ed., Chap. III.

[21] The lattice must be one that leads to convergent lattice sums. If the sums diverge, then the frequency is infinite, which also implies instability. Our calculation relies on the harmonic approximation, made in Eq. (23). It has been pointed out [L. H. Nosanow, (private communication)] that the harmonic approximation may lead to apparent instability when the lattice is, in fact, stable. This situation occurs in the treatment of solid helium [L. H. Nosanow and W. J. Mullin, Phys. Rev. Letters **14**, 133 (1965); **14**, 339(E) (1965)], where the effect arises from the zero-point motion. Experiments on rotating helium II are limited to low angular velocities, however, where the lattice constant is very large, so that zero-point motion could not affect our results. It is of course possible that the interaction potential might be affected when the vortices come close together, and the lattice might "freeze." Such an effect could occur only at prohibitively high angular velocities of the order of $\Omega\approx\frac{1}{2}nh/m\approx\frac{1}{2}h/md^2\approx\frac{1}{2}\times10^{13}$ rad/sec, since d would need to be ≈1 Å.

[22] A. L. Fetter, Phys. Rev. **138**, A429 (1965).

origin, which is the single point having inversion symmetry. The problem of stability in this geometry is very much more complicated than in the infinite system, since plane waves cannot be used to decouple the equations of motion of the different vortices; the normal modes must incorporate the symmetry of the particular system whose stability is being tested.[1] Moreover, since the force has infinite range, the image vortices due to the boundary have to be included in a proper calculation. We conjecture, however, that our calculation, which is valid only for an infinite lattice, may apply to a region near the axis of a large cylinder, since this region would be rotating very slowly. An analogous situation occurs in classical hydrodynamics in the discussion of the stability of a one-dimensional lattice.[23]

In the next section, detailed analysis of the stability in type-II superconductors (λ finite) shows that the square lattice is unstable for waves in certain directions, while the triangular lattice is stable. Before discussing the evaluation of the lattice sums, it is instructive to illustrate the distinction between finite and infinite penetration depth with the mathematical approximation of a continuum, in which the lattice sums are replaced by integrals. It must be emphasized that this has only pedagogic value; no rigorous conclusions can be drawn from the continuum model, which predicts stability for finite penetration depth, independent of lattice structure. Nevertheless, the continuum model demonstrates that stability depends on the size of the dimensionless parameter R/λ, where R is the size of the container. In Appendix A, it is shown that for $R/\lambda \ll 1$ (helium), the vibration frequency is given by

$$\omega^2 = -(\tfrac{1}{2}\bar{\kappa}n)^2[1-2(qR)^{-1}J_1(qR)]^2 + 0(R^2/\lambda^2), \quad (29)$$

where J_1 is the usual Bessel function,[20] while for $R/\lambda \gg 1$ (superconductors)

$$\omega^2 = (\tfrac{1}{2}\bar{\kappa}n)^2 \tfrac{1}{4} q^4 \lambda^2 d^2 (1+q^2\lambda^2)^{-1} + 0(\lambda^2/R^2). \quad (30)$$

Here n is the number of vortices per unit area, which, for superconductors, is equal to the ratio of the magnetic induction B to the quantum of magnetic flux $\varphi_0 = hc/2e$. Equation (30) may thus be written as

$$\omega = \tfrac{1}{4}(eB/mc) q^2 \lambda d (1+q^2\lambda^2)^{-1/2}, \quad (31)$$

which agrees qualitatively in the long-wavelength limit with the calculations of de Gennes and Matricon.[5,7]

It is interesting to calculate the path of the vortex core, which may be done most conveniently by evaluating the components $s_\perp$ and $s_{||}$ transverse and parallel to $\mathbf{q}$, respectively. We find

$$s_\perp = (Q+P)\cos\chi \cos(\mathbf{q}\cdot\mathbf{r}-\omega t) + (Q^2-P^2)^{1/2}\sin\chi \sin(\mathbf{q}\cdot\mathbf{r}-\omega t),$$

$$s_{||} = (Q-P)\sin\chi \cos(\mathbf{q}\cdot\mathbf{r}-\omega t) - (Q^2-P^2)^{1/2}\cos\chi \sin(\mathbf{q}\cdot\mathbf{r}-\omega t),$$

where Q and P are related to the quantities given in Eq. (A4) and χ is an arbitrary angle. Each vortex follows an elliptical orbit in which the ratio of the semiminor to semimajor axes is $d/4\lambda$ in the long-wavelength limit, and the semiminor axis is along $\mathbf{q}$. Qualitatively similar results are obtained for the triangular lattice in the long-wavelength limit.

IV. LATTICE STRUCTURE IN TYPE-II SUPERCONDUCTORS

The stability of a given lattice structure depends on the precise values of the quantities α, ξ, and η in Eqs. (26) and (28). Appendix A shows that the recursion relation for the Bessel functions may be used to write

$$\alpha = (\bar{\kappa}/2\pi\lambda^2)\sum_L{}' (1-e^{i\mathbf{q}\cdot\mathbf{r}_L})(x_L y_L/r_L^2)K_2(r_L/\lambda),$$

$$\xi = (\bar{\kappa}/4\pi\lambda^2)\sum_L{}' (1-e^{i\mathbf{q}\cdot\mathbf{r}_L})(y_L^2-x_L^2)r_L^{-2}K_2(r_L/\lambda), \quad (32)$$

$$\eta = (\bar{\kappa}/4\pi\lambda^2)\sum_L{}' (1-e^{i\mathbf{q}\cdot\mathbf{r}_L})K_0(r_L/\lambda).$$

The present discussion will be restricted to the long-wavelength limit, where $1-e^{i\mathbf{q}\cdot\mathbf{r}} \approx i\mathbf{q}\cdot\mathbf{r} + \frac{1}{2}(\mathbf{q}\cdot\mathbf{r})^2 + \cdots$. The linear contribution to the sums vanishes by symmetry, and the leading terms are proportional to q^2.

The difference between the square and triangular lattice may be seen by considering the quantity $-\xi+i\alpha$. Since the sums converge absolutely, the terms may be rearranged, summing first over concentric circles of fixed radius containing pth neighbors, and then over p. If $\mathbf{r}_L$ and $\mathbf{q}$ have polar coordinates (r_L,θ_L) and (q,χ), respectively, we have

$$-\xi+i\alpha \approx (\bar{\kappa}/8\pi\lambda^2)q^2 \sum_L{}' r_L^2 \times \cos^2(\theta_L-\chi)e^{2i\theta_L}K_2(r_L/\lambda). \quad (33)$$

The cosine may be rewritten as $\frac{1}{2}[1+\cos 2(\theta_L-\chi)]$, and Eq. (33) is equivalent to

$$-\xi+i\alpha = (\bar{\kappa}/16\pi\lambda^2)q^2 \sum_L{}' r_L^2 K_2(r_L/\lambda) \times [e^{2i\theta_L} + \tfrac{1}{2}e^{-2i\chi}e^{4i\theta_L} + \tfrac{1}{2}e^{2i\chi}]. \quad (34)$$

For the square lattice, the angular coordinate is of the form $\theta_L = \theta_0 + \frac{1}{2}\pi n$ ($n=0$, 1, 2, 3), where θ_0 is a constant. The sums over a fixed set of neighbors yield

$$\sum_{n=0}^{3} e^{i\pi n} = 0, \quad \sum_{n=0}^{3} e^{2\pi i n} \neq 0.$$

For the triangular lattice, the corresponding sums are

$$\sum_{n=0}^{5} \exp(i\tfrac{2}{3}n\pi) = 0, \quad \sum_{n=0}^{5} \exp(i\tfrac{4}{3}n\pi) = 0.$$

[23] H. Lamb, *Hydrodynamics* (Dover Publications, Inc., New York, 1945), p. 226.

It is easy to see that the quantity $\xi^2+\alpha^2$, which enters directly in the expression for the lattice-vibration frequencies, is of the following form:

$$(\xi^2+\alpha^2)_4=A^2+B^2+2AB\cos 4\chi\,,\qquad(\xi^2+\alpha^2)_6=D^2\,, \tag{35}$$

where A, B, and D are independent of χ. Here and subsequently, the subscript 4 and 6 is used to distinguish between the square and triangular lattice (for the fourfold or sixfold rotation axis of symmetry). Similar arguments show that η is independent of χ. The evaluation of the coefficients is discussed in Appendix B, where analytic expressions are derived with the Poisson sum formula. The numerical values of the vibration frequencies for the square and triangular lattice, respectively, are given by

$$\omega^2=(eB/mc)^2q^4\lambda^2d^2[0.01989-0.03176\cos 4\chi]\,, \tag{36}$$

$$\omega^2=(eB/mc)^2q^4\lambda^2d^2(\sqrt{3}/32\pi)\,. \tag{37}$$

Equation (36) shows that the square lattice is unstable for waves propagating in directions such that $\omega^2<0$, while the triangular lattice is stable with a vibration frequency

$$\omega=0.1314(eB/mc)q^2\lambda d\,. \tag{38}$$

This result differs qualitatively from that of Abrikosov *et al.*,[13] and also quantitatively from that of Matricon[5] who finds a frequency approximately 10 times larger.

V. ELASTICITY THEORY OF LATTICE VIBRATIONS

Microscopic calculations of elastic constants are generally based on the long-wavelength limit of lattice dynamics.[24] It is therefore not surprising that the predictions of our theory are similar to those of Matricon,[5] who computed the elastic constants for the vortex lattice in type-II superconductors. Since there is a numerical discrepancy between his result and our Eq. (38), the application of elasticity theory to the vortex lattice will be derived from first principles. In contrast to the usual discussions of crystals in equilibrium, the elastic energy here contains terms linear in the strains because of the repulsive magnetic pressure between the vortex lines.

It is therefore necessary to perform a Legendre transformation to a new free energy which incorporates the external magnetic field; the magnetic flux density B is determined by the condition that this free energy be quadratic in the strains. A further complication arises because the elastic constants do not satisfy the standard symmetry relations.[25] An additional elastic constant must be introduced, which is just that necessary to reproduce the result derived in the theory of lattice dynamics [Eqs. (36) and (37)]. It must be noted that the approach of lattice dynamics is more general than that of elasticity, because it is not always permissible to expand the exponential in Eq. (26) under the summation sign. In a neutral system, such an expansion leads to divergent sums because of the long-range interaction This failure of elasticity theory when applied to a vortex lattice in liquid helium II is wholly distinct from the question of stability, which can be investigated only in terms of Eq. (26).

The following treatment will thus be limited to a lattice of flux lines in type-II superconductors, where the interaction has a finite range λ. Consider two vortices at the positions $\mathbf{r}(L)+\mathbf{u}(L)$ and $\mathbf{r}(L')+\mathbf{u}(L')$, where $\mathbf{u}(L)$ is the small displacement from the equilibrium position. In contrast to Sec. II, the lattice index is here written as an argument. It is convenient to consider the interaction energy per unit length of two vortices as a function of the square of the separation,[24] written as $\psi(|\mathbf{r}(LL')+\mathbf{u}(L)-\mathbf{u}(L')|^2)$, where $\mathbf{r}(LL')\equiv\mathbf{r}(L)-\mathbf{r}(L')$. By definition, ψ and its derivatives are taken to vanish for zero value of their argument; this is done to allow sums over L and L' to be unrestricted. The total interaction energy per unit length of lattice is

$$V=\tfrac{1}{2}\sum_{LL'}\psi(|\mathbf{r}(LL')+\mathbf{u}(L)-\mathbf{u}(L')|^2)\,. \tag{39}$$

It is sufficient in calculating elastic moduli to treat only homogeneous deformation,[24] defined by

$$u_i(L)=\sum_j u_{ij}r_j(L)\,, \tag{40}$$

where the subscripts refer to Cartesian components in the xy plane, and $u_{ij}=\partial u_i/\partial r_j$. The separation between vortices may be rewritten in terms of the strain tensor as

$$|\mathbf{r}(LL')+\mathbf{u}(L)-\mathbf{u}(L')|^2=|\mathbf{r}(LL')|^2+2\sum_{ij}r_i(LL')\times u_{ij}r_j(LL')+\sum_{ijk}u_{ij}u_{ik}r_j(LL')r_k(LL')\,. \tag{41}$$

The elastic energy density of the vortex lattice is obtained by expanding Eq. (39) to second order in the strains:

$$V/A=V_0/A+A^{-1}\sum_{ij}\{[u_{ij}+\tfrac{1}{2}\sum_k u_{ki}u_{kj}]\times[\sum_{LL'}\psi_{LL'}'r_i(LL')r_j(LL')]\}+A^{-1}\sum_{ijkl}u_{ij}u_{kl}\times[\sum_{LL'}\psi_{LL'}''r_i(LL')r_j(LL')r_k(LL')r_l(LL')]\,, \tag{42}$$

where A is the area of the lattice in the xy plane. Here $\psi_{LL'}'$ and $\psi_{LL'}''$ are the first and second derivatives of ψ, evaluated for the lattice separation corresponding to equilibrium in an applied magnetic field. The lattice

[24] See, for example, M. Born and K. Huang, *Dynamical Theory of Crystal Lattices* (Oxford University Press, London, 1954), Secs. 11 and 24–29.

[25] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), Sec. 10.

sums may be computed with negligible error by summing over $L-L'$, since the correction terms are exponentially small. The remaining sum over L' is just the total number of vortices N in the lattice. We shall introduce the abbreviation[24]

$$\{ij\}=n\sum_L \psi_L' r_i(L)r_j(L)\,,$$
$$\{ijkl\}=n\sum_L \psi_L'' r_i(L)r_j(L)r_k(L)r_l(L)\,, \tag{43}$$

where n is the density of vortex lines and the extra index $L'=0$ has been omitted. It is clear that these quantities are symmetric under interchange of any pair of arguments. The elastic energy density $\mathcal{E}$ may now be written as

$$\mathcal{E}=\sum_{ij} u_{ij}\{ij\}+\tfrac{1}{2}\sum_{ijk} u_{ki}u_{kj}\{ij\}+\sum_{ijkl} u_{ij}u_{kl}\{ijkl\}\,, \tag{44}$$

neglecting terms cubic in the strain.

The stress in the lattice is found by calculating the change in $\mathcal{E}$ when the strain is increased from u_{ij} to $u_{ij}+du_{ij}$. The increment in the energy density is

$$d\mathcal{E}=\sum_{ij} S_{ij}du_{ij}+\sum_{ijkl} S_{ij,kl}du_{ij}u_{kl}\,, \tag{45}$$

where the following definitions have been introduced:

$$S_{ij}=\{ij\}\,,$$
$$S_{ij,kl}=\delta_{ik}\{jl\}+2\{ijkl\}\,. \tag{46}$$

The stress σ_{ij} is defined as the coefficient of du_{ij} in Eq. (45),

$$\sigma_{ij}=S_{ij}+\sum_{kl} S_{ij,kl}u_{kl}\,. \tag{47}$$

Equation (47) is a generalization of Hooke's law, since the stresses do not vanish with the strains; the constant term S_{ij}, which is shown to be diagonal in Appendix C, corresponds to an additional magnetic pressure. The vanishing of S_{ij} is the condition for equilibrium in the absence of external pressure,[24] and the vortex lattice is an equilibrium configuration only in the presence of an applied magnetic field. In the usual theory,[25] the elastic constants are symmetric under the interchange $(i\leftrightarrow j)$ or $(k\leftrightarrow l)$. This relation is not valid for $S_{ij,kl}$ precisely because S_{ij} is not zero. This peculiar aspect of the elastic theory of a vortex lattice may be traced to the lack of equilibrium in zero field, and does not seem to have been taken into account correctly by Matricon.[5]

The force $\mathbf{F}$ on a unit volume of the medium is given by the divergence of the stress tensor[26]

$$F_i=\sum_j \partial\sigma_{ij}/\partial r_j$$
$$=\sum_{jkl} S_{ij,kl}(\partial^2 u_k/\partial r_j\partial r_l)\,, \tag{48}$$

[26] This equation is correct even for the lattice of flux lines, since u_{ij} is not symmetrized. A discussion of this point may be found in Ref. 25, pp. 117–118.

where Eqs. (40) and (47) have been used. This force must balance the Magnus force per unit volume[7,8]; the resulting equation of motion is

$$n\rho\bar{\kappa}(\hat{z}\times\dot{\mathbf{u}})_i+F_i=0\,. \tag{49}$$

A detailed calculation shows that the elastic force F_i may be written as $-n\rho\bar{\kappa}[\hat{z}\times\mathbf{v}(\mathbf{r})]_i$, so that Eq. (49) is the same as Eq. (14). If the displacements correspond to a propagating wave [Eq. (6)], the equation of motion yields an eigenvalue condition for ω^2. The calculation is discussed in Appendix C; the final results are identical with Eqs. (36) and (37) for the square and triangular lattice, respectively. Thus a correctly formulated elasticity theory is completely equivalent to a theory based on lattice dynamics.

VI. DISCUSSION AND CONCLUSIONS

Section V has compared our work with Matricon's[5] elastic theory of a lattice of flux lines; both calculations predict an unstable square lattice and a stable triangular lattice, although the numerical values differ. A more serious discrepancy arises in comparison with the work of Abrikosov *et al.*,[13] who find imaginary frequencies for both lattices. We interpret this as an instability, since the growing exponential ($i\omega<0$) will be present for certain directions in the absence of dissipation. When friction forces are present ($\alpha\neq0$) their calculated frequency remains imaginary [see ω_1 in Eq. (6) of Ref. 13, which is the only mode considered here] but always must lead to a decaying exponential for sufficiently large α. It is difficult to see how randomly distributed friction forces could stabilize the vortices into a well-defined regular array, which would be unstable in the absence of these forces. Both our calculation and that of Abrikosov *et al.*[13] neglect the core structure and other normal fluid effects, and both use the same dynamical equation [Eq. (3)]. The different results arise from the following point: we make a definite assumption about the velocity field as a function of $\mathbf{r}$ and t, [Eq. (7)], which ensures that $\operatorname{div}\mathbf{v}=0$; Abrikosov *et al.*,[13] average the vorticity and velocity field over regions containing many vortex lines and then impose the condition $\operatorname{div}\mathbf{v}=0$. Such a procedure seems to us to be mathematically dangerous; in particular our Eq. (8) may be expanded to first order in the displacements from the equilibrium lattice sites as

$$\partial\mathbf{Q}_1/\partial t+(\mathbf{v}_1\cdot\nabla)\mathbf{Q}_0+(\mathbf{v}_0\cdot\nabla)\mathbf{Q}_1=0\,, \tag{50}$$

where the subscripts 0 and 1 mean the zeroth- and first-order terms. The last term vanishes at every lattice site, so that the equation of vortex dynamics [Eq. (13)] comes entirely from the first two terms of Eq. (50). The middle term of Eq. (50) appears to be neglected in the treatment of Ref. 13, since the *averaged* $\mathbf{Q}_0$ is a constant. We have not found a rigorous derivation of an equation for the smooth averaged quantities, but such an equation would presumably lead to Eq.

(13), which is just the result of classical hydrodynamics. In the long-wavelength limit, some suitable averaging procedure must exist, but the corresponding derivation of the forces from the microscopic viewpoint is probably very difficult. It is not clear that these problems were considered in previous treatments of vortex dynamics in liquid helium II, which were also based on an averaged vorticity.[27]

A more serious question arises with regard to the validity of our model in describing real type-II superconductors. The obvious corrections due to finite temperatures or crystal imperfections do not appear too serious, since these effects may be made arbitrarily small, at least in principle. The assumption (made also in Ref. 13) that the core diameter is negligibly small may in fact be more questionable. Indeed all known type-II superconductors have finite κ parameter and therefore finite core diameters (usually >50 Å). Thus there is a sizeable core region containing a large number of electronic states, in which the excitation spectrum is not of the BCS type, but more like the spectrum of the corresponding normal metal.[28] Even for pure superconductors, the superfluid electron density is not constant in space for stationary vortices and not constant in time for moving vortices. As emphasized by Stephen and Bardeen[12] the finite core size necessarily leads to normal current flow near the core of a moving vortex, in order to ensure that the total electronic density should not vary. For impure superconductors the finite core size also leads to an interaction of the vortex core with the ionic background, which may be strong enough to control the motion of the vortices.[12] Our model neglects these important effects and predicts a perfect Hall effect ($\theta_H=\frac{1}{2}\pi$),[29] so that it certainly cannot describe impure superconductors, which are known to have very small Hall angles[30] (of the same order as in the normal state). Whether our model is qualitatively correct for pure superconductors, and how pure the materials have to be for the model to apply, are still open questions.[30] No dynamical discussion of lattice stability has been given for a model which includes the core structure, so that it is impossible to tell whether the modes predicted here are in fact observable in real materials. The free-energy calculation of Ref. 6, which holds for finite κ near H_{c2}, does show that the triangular lattice is more stable than the square. It therefore seems plausible that the mode does exist in pure materials; the core forces will surely damp but perhaps not obliterate the mode. We have not been able to estimate the amount of this damping, nor can we suggest precisely how to excite this mode, if it exists. In order to answer these important questions the present model must be extended to include the core forces which occur for finite κ.

Our discussion of lattice stability has been limited to a single mode, and the lattice may exhibit unstable behavior for other modes. Indeed, Abrikosov *et al.*[13] find imaginary frequencies for a second transverse mode [ω_2 in their Eq. (6)] and real frequencies for the "helicon" mode [their Eq. (5)], which is independent of the existence of a stable lattice. As indicated above, their averaging procedure is open to question, and its predictions may not be reliable.

It is worth noticing the difference between a type-II superconductor and an ordinary electron gas in a magnetic field, which can sustain a low-frequency helicon mode ($\mathbf{q}\|\mathbf{H}$) for sufficiently pure systems ($\omega_c\tau\gg1$). The electron gas with only one type of carrier has no transverse modes ($\mathbf{q}\perp\mathbf{H}$) at low frequencies, independent of the value of τ. In a type-II superconductor, on the other hand, the magnetic field renders the order parameter doubly periodic in the xy plane; the corresponding loss of translational invariance [broken symmetry[31]], leads to an additional low-frequency transverse mode that is absent in the normal electron gas, even in the limit of infinite conductivity ($\tau\to\infty$).

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APPENDIX A

In this Appendix, we calculate the vibration modes of a vortex continuum, replacing the sums in Eq. (26) by integrals. This approximation smooths out the lattice and precludes comparison of the stability of one structure with another. Nevertheless, the simplicity of the method, which qualitatively reproduces the energy spectrum of the excitations in the type-II superconductor, is of pedagogic interest. In addition, the continuum theory shows that the stability of a vortex lattice depends on the range of the velocity distribution of an isolated vortex-instability in rotating helium II and stability in the mixed state of type-II superconductors.

[27] H. E. Hall and W. F. Vinen, Proc. Roy. Soc. (London) A238, 215 (1956).

[28] C. Caroli, P. G. de Gennes, and J. Matricon, Phys. Letters 9, 307 (1964); K. Maki, Phys. Rev. 139, A702 (1965).

[29] Consider a single vortex moving with velocity $\mathbf{v}_L$ in a uniform velocity field $\mathbf{v}_T$, and set $\mathbf{v}=\mathbf{v}_T+\mathbf{v}_0(\mathbf{r}-\mathbf{v}_Lt)$. From Eqs. (5) or (8), we find

$$-(\mathbf{v}_L\cdot\nabla)Q+(\mathbf{v}_T\cdot\nabla)Q=0,$$

so that $\mathbf{v}_L=\mathbf{v}_T$. This leads to a voltage *perpendicular* to the transport current $\mathbf{v}_T$ and a Hall angle $\theta_H=\frac{1}{2}\pi$. [See, B. D. Josephson, Phys. Letters 16, 242 (1965)].

[30] Proceedings of Sussex Symposium on Quantum Fluids (to be published); P. G. de Gennes, P. Nozières, and W. F. Vinen (to be published).

[31] J. Goldstone, Nuovo Cimento 19, 154 (1961); P. W. Anderson, Phys. Rev. 112, 1900 (1958); R. V. Lange, Phys. Rev. Letters 14, 3 (1965), and Phys. Rev. 146, 301 (1966).

We begin with the coefficients α, β, and γ defined in Eq. (26), which determine the spectrum of excitations with wave vector $\mathbf{q}$ ($\mathbf{q}\perp\hat{z}$):

$$\alpha=-(\bar{\kappa}/2\pi)\sum_L{}'(1-e^{i\mathbf{q}\cdot\mathbf{r}_L})(x_Ly_L/r_L)f'(r_L),$$

$$\beta=-(\bar{\kappa}/2\pi)\sum_L{}'(1-e^{i\mathbf{q}\cdot\mathbf{r}_L})[f(r_L)+(y_L^2/r_L)f'(r_L)],$$

$$\gamma=+(\bar{\kappa}/2\pi)\sum_L{}'(1-e^{i\mathbf{q}\cdot\mathbf{r}_L})[f(r_L)+(x_L^2/r_L)f'(r_L)],$$

where $f(r)=(\lambda r)^{-1}K_1(r/\lambda)$. The derivative of $f(r)$ is easily obtained as

$$f'(r)=(\lambda^2r)^{-1}[K_1'(r/\lambda)-(\lambda/r)K_1(r/\lambda)]=-(\lambda^2r)^{-1}K_2(r/\lambda),\quad\text{(A1)}$$

where the recursion formula for Bessel functions,[32]

$$K_1'(z)=z^{-1}K_1(z)-K_2(z),$$

has been used. A combination of Eqs. (A1), (26), and (28) with the recursion relation $(2/z)K_1(z)=K_2(z)-K_0(z)$ yields Eq. (32).

The quantities α, ξ, and η will now be evaluated in the continuum approximation. Since the lattice is isotropic in this limit, the polar axis may be taken along $\mathbf{q}$; only ξ and η need to be computed because α then vanishes by symmetry. When the sums are replaced by integrals, the resulting equations may be written as

$$\xi=-(\bar{\kappa}n/4\pi\lambda^2)\int_d^R rdr\int_0^{2\pi}d\theta(1-e^{iqr\cos\theta})\times\cos2\theta K_2(r/\lambda),\quad\text{(A2)}$$

$$\eta=(\bar{\kappa}n/4\pi\lambda^2)\int_d^R rdr\int_0^{2\pi}d\theta(1-e^{iqr\cos\theta})K_0(r/\lambda),$$

where n is the density of vortices. The upper cutoff on the radial integral R is to be interpreted as the macroscopic size of the sample, while the lower cutoff d is the lattice constant. The angular integrations give

$$\begin{aligned}\xi&=-(\bar{\kappa}n/2\lambda^2)\int_d^R rdrJ_2(qr)K_2(r/\lambda),\\ \eta&=(\bar{\kappa}n/2\lambda^2)\int_d^R rdr[1-J_0(qr)]K_0(r/\lambda).\end{aligned}\quad\text{(A3)}$$

Consider first the case of rotating liquid helium II, where $\lambda\to\infty$. Then η vanishes and ξ remains finite, because $K_2(x)\approx2x^{-2}$ and $K_0(x)\approx-\ln x$ for small x. Thus Eq. (27) yields $\omega^2=-\xi^2$, which shows the instability of a lattice structure in the limit $R/\lambda\ll1$.

When qd is small, Eq. (A3) becomes

$$\xi\approx-\tfrac{1}{2}\bar{\kappa}n[1-2(qR)^{-1}J_1(qR)],$$

which is equivalent to Eq. (29). In contrast, the mixed state of a type-II superconductor usually satisfies the following inequalities: $R/\lambda\gg1$ and $d/\lambda\ll1$. In this case, the upper limit in Eq. (A3) may be replaced by infinity with negligible error, and we find

$$\begin{aligned}\xi&\approx-\tfrac{1}{2}\bar{\kappa}n\{(q\lambda)^2[1+(q\lambda)^2]^{-1}-\tfrac{1}{8}(qd)^2\},\\ \eta&\approx\tfrac{1}{2}\bar{\kappa}n(q\lambda)^2[1+(q\lambda)^2]^{-1},\end{aligned}\quad\text{(A4)}$$

to lowest order in $(qd)^2\ll1$ but for arbitrary $q\lambda$. Equations (A4) and (27) lead directly to Eq. (31). Thus a finite-range velocity pattern produces a stable lattice in the continuum model, while an infinite-range velocity pattern produces an instability.

APPENDIX B

The calculation of the spectrum of lattice vibrations in type-II superconductors has been reduced to the evaluation of certain lattice sums given in Eq. (32). It is convenient to introduce a general notation

$$\begin{aligned}\Sigma_1&=d^{-2}\sum_L{}'r_L^2K_0(r_L/\lambda),\\ \Sigma_2&=d^{-2}\sum_L{}'r_L^2K_2(r_L/\lambda),\\ \Sigma_x&=d^{-2}\sum_L{}'(x_L^4/r_L^2)K_2(r_L/\lambda),\\ \Sigma_y&=d^{-2}\sum_L{}'(y_L^4/r_L^2)K_2(r_L/\lambda),\\ \Sigma_{xy}&=d^{-2}\sum_L{}'(x_L^2y_L^2/r_L^2)K_2(r_L/\lambda),\end{aligned}\quad\text{(B1)}$$

where d is the lattice spacing. When the distinction between square and triangular lattices becomes important, Eq. (B1) will be written with a subscript 4 or 6 (for the fourfold or sixfold rotation axis.). The last four sums are related by the obvious condition

$$\Sigma_x+2\Sigma_{xy}+\Sigma_y=\Sigma_2,\quad\text{(B2)}$$

which can provide a check on the accuracy of the separate calculations.

The sums defined in Eq. (B1) obey certain additional symmetry relations in the special case of a square or a triangular lattice. If θ_L is the angle between $\mathbf{r}_L$ and the x axis, we have

$$\begin{aligned}\Sigma_x&=\tfrac{1}{8}\sum_L{}'r_L^2(3+4\cos2\theta_L+\cos4\theta_L)K_2(r_L/\lambda),\\ \Sigma_y&=\tfrac{1}{8}\sum_L{}'r_L^2(3-4\cos2\theta_L+\cos4\theta_L)K_2(r_L/\lambda),\\ \Sigma_{xy}&=\tfrac{1}{8}\sum_L{}'r_L^2(1-\cos4\theta_L)K_2(r_L/\lambda).\end{aligned}\quad\text{(B3)}$$

The sums converge absolutely, and it is permissible to

[32] See, for example, H. B. Dwight, *Tables of Integrals and Other Mathematical Functions* (The Macmillan Company, New York, 1957), 3rd ed., p. 177.

perform the sums in concentric circles containing equivalent neighbors. The square lattice is invariant under fourfold rotations, while the triangular lattice is invariant under sixfold rotations. It follows that the term in Eqs. (B3) containing $\cos 2\theta_L$ vanishes for both lattices; the term containing $\cos 4\theta_L$ vanishes for the triangular lattice, but is finite for the square lattice [see the discussion following Eq. (34)]. This distinction is crucial in the theory of stability of different lattice structures. The conclusions drawn from these symmetry arguments are that

$$\Sigma_{x4}=\Sigma_{y4};\quad \Sigma_{x6}=\Sigma_{y6}=\tfrac{3}{8}\Sigma_{26};\quad \Sigma_{xy6}=\tfrac{1}{8}\Sigma_{26}. \qquad \text{(B4)}$$

For the triangular lattice, it is therefore sufficient to calculate Σ_{16} and Σ_{26}; for the square lattice, it is necessary to calculate Σ_{x4} and Σ_{xy4} in addition to Σ_{14} and Σ_{24}.

The vibration frequency of the vortex lattice may be written as [Eqs. (27) and (35)]

$$\omega^2=\eta^2-\alpha^2-\xi^2=\eta^2-|-\xi+i\alpha|^2, \qquad \text{(B5)}$$

where, according to Eqs. (32), (33), (B1), and (B4)

$$\eta=(\bar{\kappa}/16\pi\lambda^2)q^2d^2\Sigma_1$$
$$-\xi+i\alpha=(\bar{\kappa}/32\pi\lambda^2)q^2d^2[e^{2i\chi}\Sigma_2+e^{-2i\chi}(\Sigma_2-8\Sigma_{xy})]. \qquad \text{(B6)}$$

Here χ is the angle between $\mathbf{q}$ and the x axis. Comparison with Eqs. (34) and (35) shows that the constants A, B, and D may be expressed as

$$A=(\bar{\kappa}/32\pi\lambda^2)q^2d^2\Sigma_{24},$$
$$B=(\bar{\kappa}/32\pi\lambda^2)q^2d^2(\Sigma_{24}-8\Sigma_{xy4}), \qquad \text{(B7)}$$
$$D=(\bar{\kappa}/32\pi\lambda^2)q^2d^2\Sigma_{26}.$$

The corresponding vibration frequencies are

$$\omega^2=\eta^2-A^2-B^2-2AB\cos 4\chi,$$
$$\omega^2=\eta^2-D^2, \qquad \text{(B8)}$$

for the square and triangular lattices, respectively.

The lattice sums (B1) converge absolutely, but the terms become small only at distances large compared to the penetration depth ($r_L\gg\lambda$). In the intermediate-density range ($d/\lambda\lesssim 1$), a direct calculation is impractical, since a great many terms must be included. Furthermore, the evaluation of physically interesting quantities often leads to a cancellation between the leading terms of the series, so that the precise form of the first correction term is of great importance. The analytic expressions derived below avoid both of these difficulties: Our series converge rapidly for $d/\lambda\lesssim 1$, and the low-order corrections may be found with little effort.

Square Lattice

Consider first the square lattice. For the calculation of Σ_{x4} and Σ_{xy4}, it is necessary to treat a general rectangular lattice, with basis vectors

$$\mathbf{a}_1=d\hat{x},\quad \mathbf{a}_2=d'\hat{y}; \qquad \text{(B9)}$$

an arbitrary lattice vector $\mathbf{r}_L$ may be written as

$$\mathbf{r}_L=l\mathbf{a}_1+m\mathbf{a}_2, \qquad \text{(B10)}$$

where l and m are positive or negative integers. Let $\Xi(\mu,\mu')$ be defined as

$$\Xi(\mu,\mu')\equiv\sum_{lm}{}' K_0[(l^2\mu^2+m^2\mu'^2)^{1/2}], \qquad \text{(B11)}$$

where $\mu=d/\lambda$ and $\mu'=d'/\lambda$. Here the summation is over all values of l and m, omitting the single term $l=m=0$. It is not difficult to show that[32]

$$\Sigma_{24}=4\mu^2(\partial/\partial\mu^2)^2\Xi(\mu,\mu),$$
$$\Sigma_{x4}=4\mu^2[(\partial/\partial\mu^2)^2\Xi(\mu,\mu')]|_{\mu'=\mu}, \qquad \text{(B12)}$$
$$\Sigma_{xy4}=4\mu^2[(\partial/\partial\mu^2)(\partial/\partial\mu'^2)\Xi(\mu,\mu')]|_{\mu'=\mu},$$

so that Eq. (B11) contains all the relevant lattice sums except Σ_{14}, which requires a separate treatment.

Equation (B11) may be rewritten with a standard integral representation of the Bessel function[33]

$$K_0(z)=\frac{1}{2}\int_0^\infty d\tau\,\tau^{-1}e^{-\tau}\exp(-z^2/4\tau), \qquad \text{(B13)}$$

which yields

$$\Xi(\mu,\mu')=\frac{1}{2}\int_0^\infty d\tau\,\tau^{-1}e^{-\tau}\sum_{lm}{}'$$
$$\times\exp[-(l^2\mu^2+m^2\mu'^2)/4\tau]. \qquad \text{(B14)}$$

If the sum over l and m were unrestricted, Eq. (B14) could be transformed with the Poisson sum formula.[34,35] Unfortunately, the integral diverges logarithmically at the origin when the term $l=m=0$ is included, and a limiting procedure must be used:

$$\Xi(\mu,\mu')=\lim_{\epsilon\to 0}\Xi(\mu,\mu',\epsilon),$$
$$\Xi(\mu,\mu',\epsilon)=\frac{1}{2}\int_\epsilon^\infty d\tau\,\tau^{-1}e^{-\tau}$$
$$\times\{\sum_{lm}\exp[-(l^2\mu^2+m^2\mu'^2)/4\tau]-1\}. \qquad \text{(B15)}$$

Here the summation is over *all* l and m.

The Poisson sum formula can be written as[36]

$$\sum_{lm}\exp[-(l^2\mu^2+m^2\mu'^2)/4\tau]$$
$$=(4\pi\tau/\mu\mu')\sum_{lm}\exp[-4\pi^2\tau(l^2\mu^{-2}+m^2\mu'^{-2})]. \qquad \text{(B16)}$$

[33] See, for example, Ref. 20, p. 183, 6.22 (15).
[34] See, for example, M. J. Lighthill, *Introduction to Fourier Series and Generalized Functions* (Cambridge University Press, Cambridge, England, 1960), p. 70.
[35] This is related to the Ewald method, which can be found in J. M. Ziman, *Principles of the Theory of Solids* (Cambridge University Press, Cambridge, 1964), pp. 37–42.
[36] The summation of the left-hand side can be considered as the product of two sums, over l and m separately. The standard formula (Ref. 34) can be applied to each factor, which yields the right-hand side. A more general derivation, applicable to an arbitrary two-dimensional lattice, may be found in Ref. 35. The latter approach must be used in treating a triangular lattice.

When Eq. (B16) is substituted into Eq. (B15), the resulting integrals are straightforward, and we find

$$\Xi(\mu,\mu',\epsilon)=\frac{2\pi}{\mu\mu'}\sum_{lm}\times\frac{e^{-\epsilon}\exp[-4\pi^2\epsilon(l^2\mu^{-2}+m^2\mu'^{-2})]}{1+4\pi^2(l^2\mu^{-2}+m^2\mu'^{-2})}-\tfrac{1}{2}E_1(\epsilon)\,, \quad \text{(B17)}$$

where the exponential integral E_1 is defined as[37]

$$E_1(x)=\int_x^\infty dt\, t^{-1}e^{-t}\,. \quad \text{(B18)}$$

Equation (B17) is finite as ϵ vanishes, since the logarithmic divergence of the sum exactly cancels the divergence of $E_1(\epsilon)$. Hence the essential problem is the extraction of the finite limiting value.

The divergent part of Eq. (B17) may be isolated by adding and subtracting a quantity $S(\epsilon)$, defined by

$$S(\epsilon)=\frac{2\pi}{\mu\mu'}\sum_{lm}{}'\frac{\exp[-4\pi^2\epsilon(l^2\mu^{-2}+m^2\mu'^{-2})]}{4\pi^2(l^2\mu^{-2}+m^2\mu'^{-2})}; \quad \text{(B19)}$$

Eq. (B17) then becomes

$$\Xi(\mu,\mu',\epsilon)=e^{-\epsilon}(2\pi/\mu\mu')+e^{-\epsilon}S(\epsilon)-\tfrac{1}{2}E_1(\epsilon)-(2\pi/\mu\mu')e^{-\epsilon}\sum_{lm}{}'\times\{\exp[-4\pi^2\epsilon(l^2\mu^{-2}+m^2\mu'^{-2})]\times[1+4\pi^2(l^2\mu^{-2}+m^2\mu'^{-2})]^{-1}\times[4\pi^2(l^2\mu^{-2}+m^2\mu'^{-2})]^{-1}\}\,. \quad \text{(B20)}$$

The last term converges as $\epsilon\to 0$, so that

$$\Xi(\mu,\mu')=\frac{2\pi}{\mu\mu'}+\lim_{\epsilon\to 0}[S(\epsilon)-\tfrac{1}{2}E_1(\epsilon)]-\frac{2\pi}{\mu\mu'}\sum_{lm}{}'\frac{1}{[1+4\pi^2(l^2\mu^{-2}+m^2\mu'^{-2})][4\pi^2(l^2\mu^{-2}+m^2\mu'^{-2})]}\,. \quad \text{(B21)}$$

Equation (B19) may be simplified by differentiating with respect to ϵ:

$$\partial S(\epsilon)/\partial\epsilon=-(2\pi/\mu\mu')\sum_{lm}{}'\exp[-4\pi^2\epsilon(l^2\mu^{-2}+m^2\mu'^{-2})]$$
$$=(2\pi/\mu\mu')-(2\epsilon)^{-1}-(2\epsilon)^{-1}\sum_{lm}{}'\times\exp[-(l^2\mu^2+m^2\mu'^2)/4\epsilon]\,, \quad \text{(B22)}$$

where the Poisson sum formula [Eq. (B16)] has been used in the last step. Integrate Eq. (B22) from ϵ to ϵ_0, where ϵ_0 is an arbitrary constant:

$$S(\epsilon_0)-S(\epsilon)=(2\pi/\mu\mu')(\epsilon_0-\epsilon)-\tfrac{1}{2}\ln(\epsilon_0/\epsilon)-\tfrac{1}{2}\sum_{lm}{}'\int_\epsilon^{\epsilon_0}d\epsilon\,\epsilon^{-1}\exp[-(l^2\mu^2+m^2\mu'^2)/4\epsilon]$$
$$=(2\pi/\mu\mu')(\epsilon_0-\epsilon)-\tfrac{1}{2}\ln(\epsilon_0/\epsilon)-\tfrac{1}{2}\sum_{lm}{}'\{E_1[(l^2\mu^2+m^2\mu'^2)/4\epsilon_0]-E_1[(l^2\mu^2+m^2\mu'^2)/4\epsilon]\}\,. \quad \text{(B23)}$$

The limiting behavior of the exponential integral is given by[37]

$$E_1(x)\approx-\ln x-\gamma \quad (x\to 0)\,,$$
$$E_1(x)\sim x^{-1}e^{-x} \quad (x\to\infty)\,, \quad \text{(B24)}$$

where γ is Euler's constant, $\gamma=0.5772\cdots$.

The logarithmic divergence of $S(\epsilon)$ is made explicit in Eq. (B23), and the finite contribution to Eq. (B21) is

$$\lim_{\epsilon\to 0}[S(\epsilon)-\tfrac{1}{2}E_1(\epsilon)]=\tfrac{1}{2}\ln\epsilon_0-(2\pi\epsilon_0/\mu\mu')+\tfrac{1}{2}\gamma+\tfrac{1}{2}\sum_{lm}{}'E_1[(l^2\mu^2+m^2\mu'^2)/4\epsilon_0]+S(\epsilon_0)\,. \quad \text{(B25)}$$

Equation (B25) is valid for arbitrary values of ϵ_0, but a careful choice can simplify the subsequent calculations. With

$$\epsilon_0=\mu\mu'/4\pi\,, \quad \text{(B26)}$$

Eq. (B25) reduces to

$$\lim_{\epsilon\to 0}[S(\epsilon)-\tfrac{1}{2}E_1(\epsilon)]=\tfrac{1}{2}\ln(\mu\mu'/4\pi)-\tfrac{1}{2}+\tfrac{1}{2}\gamma+\tfrac{1}{2}\sum_{lm}{}'\{E_1[\pi(l^2\mu/\mu'+m^2\mu'/\mu)]+[\pi(l^2\mu'/\mu+m^2\mu/\mu')]^{-1}\times\exp[-\pi(l^2\mu'/\mu+m^2\mu/\mu')]\}\,. \quad \text{(B27)}$$

Substitution of Eq. (B27) into Eq. (B21) yields an *exact* expression for $\Xi(\mu,\mu')$.

The evaluation of the various lattice sums is now straightforward:

$$\Sigma_{24}=16\pi/\mu^4-2/\mu^2+0(\mu^2)\,,$$
$$\Sigma_{x4}=\Sigma_{y4}=6\pi/\mu^4-1/\mu^2+2\pi C/\mu^2+0(1)\,, \quad \text{(B28)}$$
$$\Sigma_{xy4}=2\pi/\mu^4-2\pi C/\mu^2+0(1)\,,$$

where the constant C is defined as

$$C=\frac{1}{4\pi}\sum_{lm}{}'\exp[-\pi(l^2+m^2)]\left[\frac{2\pi(l^2-m^2)^2}{l^2+m^2}+\frac{(l^4-10l^2m^2+m^4)}{(l^2+m^2)^2}+\frac{(l^4-6l^2m^2+m^4)}{\pi(l^2+m^2)^3}\right] \quad \text{(B29)}$$
$$\approx 0.10331\,.$$

This series converges rapidly, so that third-nearest

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[37] We follow the notation of *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and I. A. Stegun (U. S. Government Printing Office, Washington, 1964), Natl. Bur. Std. Appl. Math. Ser. 55, p. 228.

neighbors are sufficient to obtain an accuracy of five significant figures. Equation (B2) can be verified immediately with Eqs. (B28).

The only remaining sum for the square lattice is Σ_{14}, which has no divergence; its evaluation is therefore described only briefly:

$$\begin{aligned}\Sigma_{14}&=\sum_{lm}{}' (l^2+m^2)K_0[\mu(l^2+m^2)^{1/2}]\\&=-2(\partial/\partial\mu^2)\int_0^\infty d\tau e^{-\tau}\sum_{lm}{}'\exp[-\mu^2(l^2+m^2)/4\tau]\\&=-8\pi(\partial/\partial\mu^2)\mu^{-2}\sum_{lm}[1+4\pi^2\mu^{-2}(l^2+m^2)]^{-2} \qquad \text{(B30)}\\&=8\pi/\mu^4+8\pi\sum_{lm}{}'[\mu^2-4\pi^2(l^2+m^2)]\\&\qquad\times[\mu^2+4\pi^2(l^2+m^2)]^{-3}\\&=8\pi/\mu^4+0(1).\end{aligned}$$

Here Eqs. (B13) and (B16) have been used in the derivation of the second and third lines, respectively. Equations (B28) and (B30) suffice for the calculation of the properties of a square lattice of flux lines in the intermediate-density regime ($\mu\lesssim 1$).

Triangular Lattice

The triangular lattice is simpler than the square lattice because the sixfold symmetry implies certain additional relations between the lattice sums [Eq. (B4)]. In contrast to the square lattice, we need only consider a lattice of equilateral triangles, with basis vectors

$$\mathbf{a}_1=d\hat{x},\quad \mathbf{a}_2=\tfrac{1}{2}d\hat{x}+\tfrac{1}{2}\sqrt{3}d\hat{y}. \qquad \text{(B31)}$$

An arbitrary lattice vector may therefore be written as

$$\mathbf{r}_L=l\mathbf{a}_1+m\mathbf{a}_2, \qquad \text{(B32)}$$

exactly as in Eq. (B10). With the definition

$$Z(\mu)=\sum_{lm}{}' K_0[\mu(l^2+lm+m^2)^{1/2}], \qquad \text{(B33)}$$

the lattice sum Σ_{26} is expressible as

$$\Sigma_{26}=4\mu^2(\partial/\partial\mu^2)^2Z(\mu). \qquad \text{(B34)}$$

The calculation of $Z(\mu)$ is almost identical with that of $\Xi(\mu,\mu')$; the necessary Poisson sum formula for the triangular lattice is[38]

$$\sum_{lm}\exp[-\mu^2(l^2+lm+m^2)/4\tau]=(8\pi\tau/\sqrt{3}\mu^2)\sum_{ml}\times\exp[-16\pi^2\tau(l^2+lm+m^2)/3\mu^2], \qquad \text{(B35)}$$

where the sum is over all l and m. We find

$$\Sigma_{26}=32\pi/\sqrt{3}\mu^4-2/\mu^2+0(\mu^2). \qquad \text{(B36)}$$

[38] The derivation of Eq. (B35) is straightforward using the method described in Ref. 35.

The other sum Σ_{16} may be evaluated as in Eq. (B30); the Poisson sum formula (B35) is used to obtain the result

$$\Sigma_{16}=16\pi/\sqrt{3}\mu^4+0(1). \qquad \text{(B37)}$$

The triangular lattice of flux lines in the intermediate-density regime can now be completely described with Eqs. (B36) and (B37).

Evaluation of Vibration Frequencies

Equations (B8) may be evaluated exactly in the limit of small d/λ. A combination of Eqs. (B6)–(B8) with the explicit lattice sums yields

$$\begin{aligned}\omega^2&=(\bar\kappa n)^2q^4\lambda^2d^2\{(16\pi)^{-1}-\cos4\chi[\tfrac{1}{2}C-(16\pi)^{-1}]\},\\ \omega^2&=(\bar\kappa n)^2q^4\lambda^2d^2(\sqrt{3}/32\pi),\end{aligned} \qquad \text{(B38)}$$

for the square and triangular lattice, respectively. The following relation between the lattice spacing and the vortex density has been used in the above derivation: $n=d^{-2}$ (square), $n=(2/\sqrt{3})d^{-2}$ (triangular). The square lattice is unstable for directions in which ω^2 is negative.

The next order terms in the expansion in d^2/λ^2 may be obtained with little difficulty. Such results are of interest only for the stable triangular lattice. A detailed calculation shows that the square frequency may be written as

$$\omega^2=(eB/mc)^2q^4\lambda^2d^2(\sqrt{3}/32\pi)[1-(3\sqrt{3}/16\pi^3)G\mu^2], \qquad \text{(B39)}$$

where the constant G is given by

$$G=\sum_{lm}{}'(l^2+lm+m^2)^{-2}. \qquad \text{(B40)}$$

The correction term is approximately 0.1 for $\mu^2\approx 1$, so that the expansion appears to converge reasonably well, even for $d\approx\lambda$. Thus our evaluation is valid for intermediate values of the magnetic field strength $H_{c_1}\lesssim H \ll H_{c_2}$ and fails only in a narrow range near H_{c_1}, where a different approximation method must be used.[4] In this low-density region the interaction between vortices is so weak that crystal imperfections will be dominant.

APPENDIX C

In Sec. V, the elasticity theory of a two-dimensional lattice has been developed for an arbitrary central-force potential ψ. Here the method will be applied to the special form of the potential appropriate for flux lines in type-II superconductors,

$$\begin{aligned}\psi(r^2)&=(2\pi)^{-1}\rho\bar\kappa^2K_0(r/\lambda)\\&=(\varphi_0{}^2/8\pi^2\lambda^2)K_0(r/\lambda).\end{aligned} \qquad \text{(C1)}$$

The derivatives of ψ are easily computed, and the elastic constants of the lattice may be evaluated in terms of three constants

$$\begin{aligned}\bar K&=(\rho\bar\kappa^2d^2/16\pi\lambda^2)[\Sigma_1-\Sigma_2],\\ \bar L&=(\rho\bar\kappa^2d^2/8\pi\lambda^2)\Sigma_x,\\ \bar M&=(\rho\bar\kappa^2d^2/8\pi\lambda^2)\Sigma_{xy},\end{aligned} \qquad \text{(C2)}$$

where the notation of Appendix B has been used for the lattice sums. The symmetry properties discussed in Appendix B show that $\{ij\}$ [defined in Eq. (43)] vanishes unless $i=j$, while $\{ijkl\}$ [also defined in Eq. (43)] vanishes unless its indices are equal in pairs. The only nonvanishing elements are

$$\begin{aligned}\{11\}&=\{22\}=n\bar{K}\,,\\ \{1111\}&=\{2222\}=n\bar{L}\,,\\ \{1122\}&=n\bar{M}\,,\end{aligned}\tag{C3}$$

and those obtained by interchanging indices.

The energy density $\mathcal{E}$ may be written [Eqs. (45) and (46)] as

$$\mathcal{E}=\sum_{ij} S_{ij}u_{ij}+\sum_{ijkl} S_{ij,kl}u_{ij}u_{kl}\,.\tag{C4}$$

Only the quadratic terms affect the equations of motion, and it is these terms which must be compared with Matricon's[5] expansion of the energy. A straightforward comparison shows that his elastic constants K_x, K_y, and L may be written in terms of our elastic moduli as

$$\begin{aligned}K_x=K_y=S_{11,11}=S_{22,22}&=n(\bar{K}+2\bar{L})\,,\\ L=S_{12,12}&=n(\bar{K}+2\bar{M})\,.\end{aligned}\tag{C5}$$

These relations can also be verified by direct calculation based on the formulas given in Ref. 5. Numerical evaluation of the shear modulus L using Eqs. (B28)–(B30), (B36), and (B37) yields

$$L=-B\varphi_0(4\pi\lambda)^{-2}(0.14910)\,,\tag{C6a}$$

$$L=\tfrac{1}{4}B\varphi_0(4\pi\lambda)^{-2}\,,\tag{C6b}$$

for the square and triangular lattice, respectively. Equation (C6a) agrees precisely with Matricon's result, which provides a check on the accuracy of our summation method; Eq. (C6b) is smaller than Matricon's[5] by a factor ≈ 6. The additional elastic constant $S_{21,12}$, which would be identical with $S_{12,12}$ in the usual theory, is given by $S_{21,12}=2n\bar{M}$.

The equation of motion of an element of the lattice is determined by the vanishing of the elastic force plus the Magnus force

$$n\rho\bar{\kappa}(\hat{z}\times\dot{\mathbf{u}})_i+\sum_{jkl} S_{ij,kl}(\partial^2 u_k/\partial r_j\partial r_l)=0\,.\tag{C7}$$

If the displacement $\mathbf{u}$ is a propagating wave [Eq. (6)], the equation of motion leads to an eigenvalue condition for the vibration frequencies

$$\omega^2\rho^2\bar{\kappa}^2=q^4[\bar{K}^2+2\bar{K}(\bar{L}+\bar{M})+4\bar{L}\bar{M}]+4q_x^2q_y^2(\bar{L}+\bar{M})(\bar{L}-3\bar{M})\,.\tag{C8}$$

The second term vanishes identically for a triangular lattice, since $\Sigma_{x6}=3\Sigma_{xy6}$, which reproduces the standard result that a lattice with a sixfold symmetry axis behaves like an isotropic medium for propagation perpendicular to the axis.[39] Detailed evaluation of Eq. (C8) gives identical vibration frequencies with those derived from the theory of lattice dynamics [Eqs. (36) and (37) or Eqs. (B38)].

[39] See, for example, Ref. 25, p. 40.