Exchange Contribution of the π-π $J=0$, $T=0$ Antibound State*

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It is shown that the exchange of the π-π $J=0$, $T=0$ antibound state in the crossed channel for π-π and π-N scattering gives a contribution similar to that due to the exchange of a scalar particle but with opposite sign of the residue. In the direct channel, this antibound state gives a large $T=0$ s-wave π-π scattering length, as suggested by Atkinson to explain the ABC phenomenon. The contribution of the antibound state to the physical amplitude is evaluated using dispersion relations involving the physical sheet and the second Riemann sheet.

IT was suggested by Atkinson[1] that for π-π $J=0$, $T=0$ amplitude an antibound state should occur. The antibound state was defined as a pole between $s=0$ and $s=4\mu^2$ on the second Riemann sheet reached by analytic continuation[2-5] through the elastic unitarity cut. Atkinson further suggested that this antibound state could give a large π-π $T=0$, s-wave scattering length, as indicated by the ABC phenomenon.[6] In this work, we shall show that the antibound state not only gives a large scattering length in the direct channel, but also gives a contribution in the crossed channel similar to that due to the exchange of a scalar particle, but with a residue opposite in sign, i.e., positive. For this purpose we consider dispersion relations of the type suggested by Oehme[3] involving the physical sheet as well as the second sheet. In these dispersion relations the unitarity cut is removed in favor of contributions from the poles on the second sheet.

We use Oehme's notation. The elastic scattering amplitude $F(s_+)$ is the boundary value of an analytic function $F(z)$ which has a right-hand branch cut for $4\mu^2\leqslant z<\infty$ and a left-hand branch cut for $0\geqslant z>-\infty$. The analytic continuation of $F(z)$ on the second Riemann sheet is given by

$$F^{\mathrm{II}}(z)=F^{\mathrm{I}}(z)/(1+2i\rho(z)F^{\mathrm{I}}(z))\,, \tag{1}$$

where the superscripts refer to the relevant sheets and $\rho(z)=[(z-4\mu^2)/z]^{1/2}$. $\rho(z)$ has a cut for $4\mu^2\leqslant z<\infty$ and a cut for $0\geqslant z>-\infty$, and on the physical sheet is positive imaginary for $0<z<4\mu^2$. Equation (1) shows that a zero of the S matrix $1+2i\rho(z)F^{\mathrm{I}}(z)$ gives a pole on the second sheet. If we consider the Cauchy integrals

$$\frac{1}{2\pi i}\oint\frac{F(z')dz'}{z'-z}$$

and

$$\frac{1}{2\pi i}\oint\frac{F(z')dz'}{\rho(z')(z'-z)},$$

where the contours are taken around the branch cuts on both sheets, we obtain

$$F^{\mathrm{I}}(z)+F^{\mathrm{II}}(z)=-\frac{\gamma_0}{s_0-z}+\frac{1}{\pi}\left[\int_{s_i}^{\infty}+\int_{-\infty}^{0}\right]\frac{\mathrm{Im}F^{\mathrm{I}}(z_+')+\mathrm{Im}F^{\mathrm{II}}(z_+')}{z'-z}dz', \tag{2}$$

and

$$\frac{1}{\rho^{\mathrm{I}}(z)}[F^{\mathrm{I}}(z)-F^{\mathrm{II}}(z)]=\frac{\gamma_0}{\rho^{\mathrm{I}}(s_0)(s_0-z)}+\frac{1}{2\pi i}\left[\int_{s_i}^{\infty}+\int_{-\infty}^{0}\right]\frac{F^{\mathrm{I}}(z_+')+F^{\mathrm{I}}(z_-')-F^{\mathrm{II}}(z_+')-F^{\mathrm{II}}(z_-')}{\rho^{\mathrm{I}}(z_+')(z'-z)}dz'. \tag{3}$$

Eliminating $F^{\mathrm{II}}(z)$ between Eqs. (2) and (3) and rearranging the terms, we have

$$\begin{aligned}F^{\mathrm{I}}(z)=R_-(z,s_0)\frac{\gamma_0}{s_0-z}&+\frac{1}{\pi}\left[\int_{s_i}^{\infty}+\int_{-\infty}^{0}\right]\frac{\mathrm{Im}F^{\mathrm{I}}(z_+')}{z'-z}dz'\\&+\frac{1}{2\pi i}\left[\int_{s_i}^{\infty}+\int_{-\infty}^{0}\right]\{R_-(z,z_+')[F^{\mathrm{I}}(z_+')-F^{\mathrm{II}}(z_+')]+R_+(z,z_+')[F^{\mathrm{I}}(z_-')-F^{\mathrm{II}}(z_-')]\}\frac{dz'}{z'-z},\end{aligned} \tag{4}$$

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[1] D. Atkinson, Phys. Letters 9, 69 (1964).
[2] J. Gunson and J. G. Taylor, Phys. Rev. 119, 1121 (1961).
[3] R. Oehme, Phys. Rev. 121, 1840 (1961).
[4] R. Blankenbecler, M. L. Goldberger, S. W. MacDowell, and S. B. Treiman, Phys. Rev. 123, 692 (1961).
[5] W. Zimmermann, Nuovo Cimento 21, 249 (1961).
[6] A. Abashian, N. E. Booth, and K. M. Crowe, Phys. Rev. Letters 5, 258 (1960); N. E. Booth and A. Abashian, Phys. Rev. 132, 2314 (1963).

where

$$R_{\mp}(z,z')=\frac{1}{2}\left[\mp 1+\frac{\rho(z)}{\rho(z')}\right], \tag{5}$$

and

$$\rho(z)=\rho^{\mathrm{I}}(z).$$

In Eq. (4) we have assumed that there is just one pole at $z=s_0$ on the second sheet corresponding to the antibound state. In general, however, we can have a number of poles and the first term on the right-hand side of (4) then should be replaced by

$$\sum_n \left[R_-(z,z_n)\frac{\gamma_n}{z_n-z}+R_-(z,z_n^*)\frac{\gamma_n^*}{z_n^*-z}\right]. \tag{5a}$$

γ_0 in Eq. (4) is the residue of $F^{\mathrm{II}}(z)$ at the pole. The function $R_-(z,s_0)$ develops a zero at $z=s_0$ and

$$\lim_{z\to s_0} R_-(z,s_0)\frac{\gamma_0}{s_0-z}=\frac{\mu^2\gamma_0}{(4\mu^2-s_0)s_0}. \tag{6}$$

Again,

$$\lim_{z\to s_0} F^{\mathrm{I}}(z)=\frac{-1}{2i\rho(s_0)}=\frac{1}{2}\left[\frac{s_0}{4\mu^2-s_0}\right]^{1/2}. \tag{7}$$

If we now consider that for z near s_0, the antibound-state term in Eq. (4) is the dominant term,[7] then we have,[8] equating (6) and (7),

$$\gamma_0=s_0^{3/2}[4\mu^2-s_0]^{1/2}/2\mu^2. \tag{8}$$

Let us examine the antibound-state contribution for $s\gtrsim 4\mu^2$. We have

$$\begin{aligned}R_-(s,s_0)\frac{\gamma_0}{s_0-s}&=-\frac{1}{2}\left[1+i\rho(s)\left(\frac{s_0}{4\mu^2-s_0}\right)^{1/2}\right]\frac{\gamma_0}{s_0-s}\\&=\left(\frac{s_0}{4\mu^2-s_0}\right)^{1/2}\frac{2\mu^2}{s_0 s}\gamma_0\frac{1}{[(4\mu^2-s_0)/s_0]^{1/2}-i\rho(s)}\\&=\frac{s_0}{s}\frac{1}{[(4\mu^2-s_0)/s_0]^{1/2}-i\rho(s)}\quad[\text{using (8)}].\end{aligned} \tag{9}$$

If s_0 is near threshold and if the antibound-state term in (4) dominates, then we get

$$F^{\mathrm{I}}(s)\approx\frac{1}{[(4\mu^2-s_0)/s_0]^{1/2}-i\rho(s)}. \tag{10}$$

This predicts a $T=0$ s-wave scattering length

$$a=\left(\frac{s_0}{4\mu^2-s_0}\right)^{1/2}\mu^{-1}. \tag{11}$$

For s_0 near $4\mu^2$, Eq. (11) gives a large scattering length[9] and thus the antibound state can be identified with the ABC particle. If $s_0=(0.25\text{ BeV})^2$, $a\approx 2\mu^{-1}$.

Let us now examine the antibound-state contribution for $s<0$, i.e., when this state is exchanged in the crossed channel. We find that it becomes complex, the imaginary part coming from $\rho(s_0)=i[(4\mu^2-s_0)/s_0]^{1/2}$. However, in Eq. (4) the imaginary part of the left-hand side comes from the second term on the right-hand side, and so the imaginary part of the antibound-state term must be cancelled by the imaginary part coming from the last term in (4). Thus, the effective contribution of the antibound state to the elastic amplitude $F(s_+)$ for $s<0$ is $-\gamma_0/2(s_0-s)$, i.e., it is similar to the contribution due to the exchange of a scalar particle but with a positive residue.[10] The details of the above argument are given in the Appendix.

Let us next examine the production amplitude $G(s)$ for $\pi\pi\to N\bar{N}$ in the $J=0$, $T=0$ state. The analytic continuation of this amplitude to the second sheet is given by

$$G^{\mathrm{II}}(z)=\frac{G^{\mathrm{I}}(z)}{1+2i\rho(z)F^{\mathrm{I}}(z)}. \tag{12}$$

The amplitude $G(z)$ has cuts along the real axis for $4\mu^2\leq z<\infty$ coming from unitarity cuts in the direct channel and for $-\infty<z\leq 0$ coming from unitarity cuts in the crossed channels.[11] $G(z)$ has an additional left-hand cut for $-\infty<z\leq a$ coming from the single nucleon pole terms; here $a=4\mu^2(1-\mu^2/4m^2)$. The discontinuity across this latter cut is known in terms of the π-N coupling constant and the nucleon mass and its contribution to $G(z)$ can be explicitly obtained. We denote this contribution by $B(z)$. The second sheet function $G^{\mathrm{II}}(z)$ has now an additional branch cut for $0<z\leq a$ and Cauchy integrals of the type which we have considered before are not suitable for obtaining the antibound-state contribution. Instead, we first define a new function

$$\phi(z)=G(z)-B(z)[1+2i\rho(z)F(z)]. \tag{13}$$

[7] It is known that if an antibound state exists near threshold, then it can dominate the scattering amplitude and give large scattering length. The most familiar example is the n-p singlet state. Here the large scattering length is explained in terms of the dominance of a pole on the second sheet near threshold [see, for example, W. R. Frazer and A. W. Hendry, Phys. Rev. **134**, B1307 (1964)]. Now, if we believe that the large π-π $T=0$, s-wave scattering length is also due to the existence of an antibound state near threshold, then its contribution dominates over that of the cuts in this region and this leads to Eq. (8). The procedure is justified *a posteriori* by the fact that a large scattering length is then predicted by the theory.

[8] The appearance of the factor $[4\mu^2-s_0]^{1/2}$ in the residue γ_0 can be physically understood from the following argument: The position s_0 of the antibound state corresponds to a zero of the S matrix $S_l=1+2i\rho F_l$. However, $S_l=e^{2i\delta_l}$ is unity at threshold and therefore no antibound state can occur at $s=4\mu^2$. Thus, $F^{\mathrm{II}}(z)$ has no pole at the elastic threshold and this is indicated by the fact that the residue γ_0 of the antibound-state term vanishes when $s_0\to 4\mu^2$.

[9] Our result for the scattering length, viz., Eq. (11), is larger by a factor of 2 than the estimate of Atkinson.

[10] The contribution due to the exchange of a scalar particle in the s-channel is $(1/16\pi)g_{\pi\pi s}^2/(m_s^2-s)$, where m_s is the mass and $g_{\pi\pi s}$ is the coupling constant of the particle.

[11] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960).

Then, the analytic continuation of $\phi(z)$ through the branch cut $4\mu^2 \leq z < 16\mu^2$ is

$$\phi^{\rm II}(z) = G^{\rm II}(z) - B(z)[1 - 2i\rho^{\rm I}(z)F^{\rm II}(z)] = \frac{G(z) - B(z)}{1 + 2i\rho(z)F(z)}. \qquad (14)$$

Now, $\text{Im}G(z) = \text{Im}B(z)$ for $0 < z < a$. Therefore, $\phi^{\rm II}(z)$ does not have any cut from $z=0$ to $z=a$. The only singularity $\phi^{\rm II}(z)$ has for $0 < z < 4\mu^2$ is the antibound-state pole. We now consider the Cauchy integrals

$$\frac{1}{2\pi i}\oint \frac{\phi(z')dz'}{z'-z} \quad \text{and} \quad \frac{1}{2\pi i}\oint \frac{\phi(z')dz'}{\rho(z')(z'-z)},$$

involving both Riemann sheets and then eliminate $\phi^{\rm II}(z)$. A dispersion relation similar to (4) is obtained with additional terms coming from the extra branch cut $0 < z \leq a$. The final result can be written as

$$\begin{aligned} G^{\rm I}(z) &= \phi^{\rm I}(z) + B(z)[1 + 2i\rho(z)F^{\rm I}(z)] \\ &= R_-(z,s_0)\frac{\lambda_0}{s_0 - z} + B(z)[1 + 2i\rho(z)F^{\rm I}(z)] \\ &\quad + \frac{1}{2\pi}\int_0^a \frac{\text{Im}\phi^{\rm I}(z_+')dz'}{z'-z} + \frac{\rho(z)}{2\pi}\int_0^a \frac{\text{Im}\phi^{\rm I}(z_+')dz'}{\rho(z')(z'-z)} + \frac{1}{\pi}\left[\int_{s_i}^\infty + \int_{-\infty}^0\right]\frac{\text{Im}\phi^{\rm I}(z_+')dz'}{z'-z} \\ &\quad + \frac{1}{2\pi i}\left[\int_{s_i}^\infty + \int_{-\infty}^0\right]\{R_-(z,z_+')[\phi^{\rm I}(z_+') - \phi^{\rm II}(z_+')] + R_+(z,z_+')[\phi^{\rm I}(z_-') - \phi^{\rm II}(z_-')]\}\frac{dz'}{z'-z}. \end{aligned} \qquad (15)$$

λ_0 in (15) is the residue of $\phi^{\rm II}(z)$ at the pole $z = s_0$. We first want to show that λ_0 is positive. If for s near the threshold, $G(s_+)$ is dominated by the antibound-state term, then from (15)

$$G^{\rm I}(s_+) \approx -\frac{1}{2}\left[1 + i\rho(s_+)\left(\frac{s_0}{4\mu^2 - s_0}\right)^{1/2}\right]\frac{\lambda_0}{s_0 - s}, \qquad (16)$$

and

$$\text{Im}G^{\rm I}(s_+) = \tfrac{1}{2}\rho(s_+)\left(\frac{s_0}{4\mu^2 - s_0}\right)^{1/2}\frac{\lambda_0}{s - s_0} \quad (s > 4\mu^2). \qquad (17)$$

Again, $G(s_+)$ has the same phase as the elastic amplitude $F(s_+)$ for $4\mu^2 < s < 16\mu^2$, so that

$$\text{Im}G^{\rm I}(s_+) = \left|\frac{G^{\rm I}(s_+)}{F^{\rm I}(s_+)}\right| \text{Im}F^{\rm I}(s_+). \qquad (18)$$

Since $\text{Im}F^{\rm I}(s_+)$ is positive, as can be seen, say, from Eq. (10), therefore, $\text{Im}G^{\rm I}(s_+)$ is positive. From (17) we then have λ_0 positive. For $s < 0$, the antibound-state term in Eq. (15) has an imaginary part. However, as in the case of $F^{\rm I}(z)$, this imaginary part has to be exactly cancelled by the imaginary part coming from the last term in (15). Thus, the effective contribution of the antibound state to the amplitude $G^{\rm I}(s_+)$ for $s < 0$ is

$$-\frac{\lambda_0}{2(s_0 - s)}. \qquad (19)$$

Identifying the production amplitude $G(t)$ with the helicity amplitude $f_+^{(+)J=0}(t)$ of Frazer and Fulco,[11] we find the contribution of the π-π $J=0$, $T=0$ antibound state to the invariant amplitudes $A^{(+)}(s,t)$, $B^{(+)}(s,t)$ of π-N scattering to be

$$\begin{aligned} A^{(+)}(s,t) &= \frac{8\pi}{m^2 - t/4} f_+^{(+)J=0}(t) = -\frac{8\pi}{(m^2 - t/4)}\frac{\lambda_0}{2(s_0 - t)} \\ &= -\frac{16\pi\lambda_0}{4m^2 - s_0}\left[\frac{1}{s_0 - t} - \frac{1}{4m^2 - t}\right] \\ &\approx -\frac{16\pi\lambda_0}{(4m^2 - s_0)}\frac{1}{s_0 - t}, \end{aligned} \qquad (20a)$$

$$B^{(+)}(s,t) = 0. \qquad (20b)$$

In (20a) we have neglected the term which would correspond to the exchange of a particle of mass $2m$ and therefore, to a very short-range force.

Comparing Eq. (20) above with the corresponding equation in Ref. 12, we now see why the parameter λ there is negative[13] and why the parameter m_0 is nearly equal to 2μ. Thus, the force we have obtained in π-N scattering can be explained as due to the exchange of the π-π $J=0$, $T=0$ antibound state. This force, as we have found,[12] gives the repulsive real part of the forward scattering amplitude for π-N scattering. The close similarity of the forces and the results obtained for π-N scattering and spinless p-p scattering[14] suggest that,

[12] M. M. Islam, preceding paper, Phys. Rev. **147**, 1144 (1966).

[13] It is worth pointing out that the contribution to π-N scattering due to a $T=0$ scalar particle exchanged in the t channel is given by $A^{(+)}(s,t) = g_{\pi\pi s}g_{NNs}/(m_s^2 - t)$. Since the exchange of such a particle is expected to give an attractive force for s-channel scattering, $(g_{\pi\pi s}g_{NNs})$ should be taken as positive.

[14] M. M. Islam, Phys. Rev. **141**, 1524 (1966).

for actual pp scattering also, the repulsive real part of the forward nuclear amplitude at high energy[15–17] is due to the exchange of this π-π antibound state.

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APPENDIX

In this Appendix, we shall discuss how the force in π-π scattering arising from the exchange of the antibound state is calculated. One comment seems worthwhile to make. Phenomenological partial-wave calculations are done by considering contributions from a few resonances of definite l in the s channel as giving rise to important forces (or equivalently, left-hand-cut contributions) in the t channel. This *ad hoc* procedure has produced meaningful results, even though it is well known that the partial-wave expansion in the s channel does not converge in the physical region of the t channel. On this basis, we shall assume that the force in the t channel due to the exchange of a system of definite l in the s channel is a meaningful concept, provided we have phenomenological reasons to consider that the exchange of the system gives a strong effect.

One way of calculating the force in the t channel due to the exchange of a system (say, a resonance) in the s channel is to consider first the diagram corresponding to this exchange (Fig. 1). The invariant amplitude for this process can then be calculated by considering the partial wave in which this system occurs and next by separating the contribution corresponding to this diagram from other contributions in that partial wave. We can now continue this invariant amplitude in s and t variables till we reach the region $s<0$ and $t>4\mu^2$ and this gives us the contribution of the diagram in the physical region of the t channel. If partial waves in the t channel are projected out from this contribution, we find that in each l a left-hand-cut contribution occurs. This we interpret as the force arising from the diagram in Fig. 1. This way of calculating the force is similar to the calculation of force by Feynman diagrams.

Now, in the present case, we have picked up an invariant contribution in the s-wave amplitude $F^1(s)$ and identified it as that due to the antibound state. This contribution can be associated with a diagram like Fig. 1 and can be continued to the physical t region. The continuation gives a real part $-\gamma_0/2(s_0-s)$ and an imaginary part $i[\gamma_0/2|\rho(s_0)|][\rho(s)/(s-s_0)]$. Since the force or left-hand contribution should be real in the physical t region, we expect the imaginary part to be cancelled by some other imaginary part. This indeed occurs (another equivalent way of saying this is that only the real part is relevant for calculating the force). Thus, the effective contribution of the antibound-state term which gives the force in the physical t region is $-\gamma_0/2(s_0-s)$.

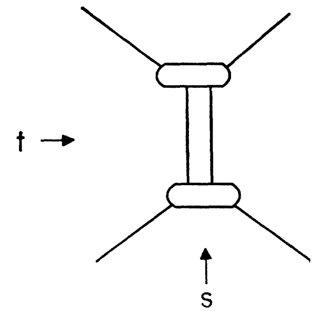

FIG. 1. Exchange of a system in the s channel.

Let us now evaluate the force due to the antibound-state exchange from a more formal point of view. For this purpose, it is helpful to examine its contribution a bit more closely. Denoting this contribution by $U(s)$, so that

$$U(s)=R_{-}(s,s_0)\frac{\gamma_0}{s_0-s}=\frac{1}{2}\left[-1-i\frac{\rho(s)}{|\rho(s_0)|}\right]\frac{\gamma_0}{s_0-s}, \tag{A1}$$

we find that on the physical sheet $U(s)$ has a right-hand cut from $s=4\mu^2$ to ∞ and a left-hand cut from $s=0$ to $-\infty$. It is analytic throughout the cut s plane and its discontinuity across the cuts is given by

$$\frac{1}{2i}[U(s_+)-U(s_-)]=\frac{\gamma_0}{2|\rho(s_0)|}\frac{\rho(s_+)}{(s-s_0)}, \tag{A2}$$
$$(4\mu^2<s<\infty,\ -\infty<s<0).$$

Thus, on the physical sheet, the antibound state contribution can be written as a dispersion relation similar to the partial-wave amplitude,

$$U(s)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\operatorname{Im}U(s')}{s'-s}ds'+\frac{1}{\pi}\int_{-\infty}^{0}\frac{\operatorname{Im}U(s')}{s'-s}ds' \tag{A3}$$

$$=\frac{\gamma_0}{2|\rho(s_0)|\pi}[I_1(s)+I_2(s)], \tag{A4}$$

where

$$I_1(s)=\int_{4\mu^2}^{\infty}\frac{\rho(s_+')ds'}{(s'-s_0)(s'-s)},\quad I_2(s)=\int_{-\infty}^{0}\frac{\rho(s_+')ds'}{(s'-s_0)(s'-s)}.$$

The integrals can be explicitly worked out and the

[15] L. Kirillova, L. Khristov, V. Nikitin, M. Shafranova, L. Strunov, V. Sviridov, Z. Korbel, L. Rob, P. Markov, Kh. Techernev, T. Todorov, and A. Zlateva, Phys. Letters 13, 93 (1965).

[16] G. Belletini, G. Cocconi, A. N. Diddens, E. Lillethum, J. Phal, J. P. Scanlor, J. Walters, A. M. Wetherell, and P. Zanella, Phys. Letters 14, 164 (1965).

[17] K. J. Foley, R. S. Gilmore, R. S. Jones, S. J. Lindenbaum, W. A. Love, S. Ozaki, E. H. Willen, R. Yamada, and L. C. L. Yuan, Phys. Rev. Letters 14, 74 (1965).

results are

$$I_1(s)=\left[-(s-4\mu^2)\frac{2}{s}F(1,\tfrac{1}{2};\tfrac{3}{2};1-4\mu^2/s)+i\pi\left(\frac{s-4\mu^2}{s}\right)^{1/2}+(s_0-4\mu^2)\frac{2}{s_0}F(1,\tfrac{1}{2};\tfrac{3}{2};1-4\mu^2/s_0)+\pi\left(\frac{4\mu^2-s_0}{s_0}\right)^{1/2}\right]\frac{1}{s-s_0},\quad \text{(A5)}$$

$$I_2(s)=\left[(s-4\mu^2)\frac{2}{s}F(1,\tfrac{1}{2};\tfrac{3}{2};1-4\mu^2/s)-(s_0-4\mu^2)\frac{2}{s_0}F(1,\tfrac{1}{2};\tfrac{3}{2};1-4\mu^2/s_0)\right]\frac{1}{s-s_0},\quad \text{(A6)}$$

where the F's are Gauss's hypergeometric functions. Substitution of (A5) and (A6) in (A4) gives us back (A1).

The dispersion relation (A3) shows an important dynamical difference between the antibound state contribution and that due to a narrow resonance. In the latter case, the amplitude in the physical region is approximately given by $\gamma_r(s_r-s-i\sigma_r)^{-1}$ and in the zero-width approximation, the imaginary part is given by $\pi\gamma_r\delta(s-s_r)$. Thus, the imaginary part essentially comes from a single point in the physical region. For the antibound state, on the other hand, as indicated by the dispersion relation (A3), both the right-hand and left-hand cuts are contributing, the imaginary part on these cuts being the same function.

Let us consider the isospin-zero invariant scattering amplitude in the s channel $A^{T=0}(s,t,u)$. Chew and Mandelstam[18] have written down the following fixed-energy dispersion relation for this amplitude with the s wave subtracted out:

$$A^{T=0}(s,t,u)=A_{l=0}{}^{T=0}(s)+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\tilde{A}_t{}^{T=0}(t',s)\times\left[\frac{1}{t'-t}+\frac{1}{t'+s+t-4\mu^2}-\frac{1}{2k^2}\ln\left(1+\frac{4k^2}{t'}\right)\right],\quad \text{(A7)}$$

where $\tilde{A}_t{}^T(t,s)=\chi_{TT'}A_t{}^{T'}(t,s)$ and $A_t{}^T(t,s)$ is the absorptive part in the t channel; $\chi_{TT'}$ is the crossing matrix

$$\begin{bmatrix}1/3 & 1 & 5/3\\ 1/3 & 1/2 & -5/6\\ 1/3 & -1/2 & 1/6\end{bmatrix}.$$

Now, the scattering amplitude in the t channel in a particular isospin state is related to the scattering amplitudes in the s channel by crossing symmetry[19]:

$$A^T(t,s,u)=\chi_{TT'}A^{T'}(s,t,u).\quad \text{(A8)}$$

From (A7) and (A8) we find that the contribution of $A_0{}^0(s)$ to a t-channel isospin amplitude is $\chi_{T0}A_0{}^0(s)$. The amplitude $A_0{}^0(s)$ is the amplitude $F^I(s)$ discussed in the main text. Let us write $A_0{}^0(s)=U(s)+V(s)$, so that $V(s)$ denotes the $T=0$, s-wave contribution with the antibound-state term subtracted out. The contribution of $A_0{}^0(s)$ to a t-channel isospin amplitude for $t>4\mu^2$ and $s<0$ can now be written as

$$\chi_{T0}[U(s)+V(s)]=\chi_{T0}[\mathrm{Re}U(s)+\mathrm{Re}V(s)+i\,\mathrm{Im}A_0{}^0(s)].\quad \text{(A9)}$$

If the total scattering amplitude is in a pure $T=0$ state in the s channel, then its isospin decomposition in the t channel is[19]

$$\tfrac{1}{3}A^{T=0}(t,s,u)+A^{T=1}(t,s,u)+(5/3)A^{T=2}(t,s,u).$$

To each of the above isospin amplitudes in the t channel, the antibound-state contribution, as seen from Eq. (A9), is $\frac{1}{3}\,\mathrm{Re}U(s)$. Thus, the effective contribution of the antibound-state term to the total amplitude in the physical t region is $\mathrm{Re}U(s)=-\gamma_0/2(s_0-s)$.

Perhaps, at this stage, we may wonder about one point regarding the contribution of the antibound state in the physical t channel. As seen from the dispersion relation (A3), $\mathrm{Re}U(s)$ for $s<0$ comes not only from the right-hand cut but also from the left-hand cut. On the other hand, our experience with resonances indicates that their contributions in the crossed channel essentially come from their unitarity cuts in the direct channel. Why does this difference occur? To answer this question let us examine the Chew-Mandelstam dispersion relation (A7). We notice that even when we are in the physical t channel ($s<0$), the s-wave contribution $A_0{}^0(s)$ comes from both right-hand and left-hand cuts in the s plane. Of course, no extra discontinuity of the total amplitude occurs, since $\mathrm{Im}A_0{}^0(s)$ in the physical t region will be cancelled by a corresponding imaginary part coming from the integral in (A7). The antibound state term on the physical sheet has the same analytic property as that of $A_0{}^0(s)$ and constitutes a part of it. Therefore, in the physical region of the t channel, the antibound-state contribution would come not only from its right-hand cut but also from its left-hand cut in the s plane. This result can, thus, be connected with the fact that to suppress the high-energy behavior of the absorptive part, we have used an s-wave subtracted dispersion relation (Eq. (A7).

[18] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[19] See, for example, K. Kang, Phys. Rev. **134**, B1324 (1964).