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Baryon Electromagnetic Mass Differences and Octet Enhancement in Broken $SU(3)$ Symmetry*†

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The contribution of self-consistent terms arising from strong-interaction reciprocal bootstraps to the electromagnetic mass differences of the Σ, Y_1^*, Ξ, and Ξ^* are considered within the formalism developed by Dashen and Frautschi. It is shown that the self-consistent terms from a Σ-Y_1^* reciprocal bootstrap lead to an enhancement of the Σ and Y_1^* electromagnetic mass differences transforming as I_z, but not of those transforming as I_z^2. The Ξ and Ξ^* electromagnetic mass differences are found not to be enhanced by a Ξ-Ξ^* reciprocal bootstrap mechanism, but the large Ξ^0-Ξ^- mass difference is found to be explained by terms arising from external Σ mass shifts in the $\bar{K}\Sigma$ channel of a Ξ bootstrap calculation.

I. INTRODUCTION

Within the past few years there has been much interest, both theoretical and experimental, in electromagnetic mass differences within baryon isospin multiplets. Part of this interest stems from the fact that theories of strong interaction symmetries can be used to obtain relations between the electromagnetic mass splittings in one isospin multiplet and those in another. The subject of electromagnetic mass splittings can thus be looked upon as forming a testing ground for conjectures about strong interactions and strong interaction symmetries.

One of the most interesting of these conjectures pertains to the violations of $SU(3)$. On the basis of present experimental evidence, it would appear that the strong, electromagnetic, and weak violations of $SU(3)$ follow a characteristic pattern in that those violations of $SU(3)$ which transform like the components of an octet seem to predominate in nature.[1] It has been proposed[2-4] that this octet pattern follows from the operation of bootstrap dynamics. This suggestion has been systematically explored in the case of the strong and electromagnetic mass splittings[4] between members of the $\frac{1}{2}^+$ baryon octet (B) and the $\frac{3}{2}^+$ baryon decuplet (Δ), and for the strong, electromagnetic, and weak shifts[5,6] in the $BB\Pi$ and $B\Delta\Pi$ couplings from their $SU(3)$ symmetric values.[7]

In the case of the mass splittings,[4] the calculation begins by assuming an $SU(3)$ symmetric reciprocal bootstrap model for the B and Δ. That is, one assumes that the exchange of the baryon octet and decuplet in pseudoscalar meson-baryon scattering results in the octet and decuplet appearing as direct channel poles associated with zeros of the relevant denominator functions.

In studying the perturbations about this $SU(3)$ symmetric problem, one describes the mass splittings by a mass shift operator which is decomposed into irreducible representations of $SU(3)$. One supposes that one can work to first order in the symmetry violation of interest—neglecting, for example, the effect of medium strong symmetry-breaking on the electromagnetic mass shifts. The various relevant mass ratios, and hence the *pattern* of symmetry violation, are then described by a matrix, the A matrix, whose structure is determined by self-consistent terms coming from the $SU(3)$ symmetric bootstrap equations.[3,4,7] The approximate evaluation of this matrix yields the result[3,4] that mass shifts transforming like an octet are enhanced compared to those transforming like a 27-plet. Since this octet enhancement is determined by the $SU(3)$ symmetric A matrix [that is, the A matrix does not distinguish between different components of an $SU(3)$ representation], the electromagnetic mass shifts are characterized by the same ratios as the strong mass shifts.

More specifically, for electromagnetic mass differences, octet enhancement implies that mass differences transforming like I_z, i.e., $\Delta I=1$ mass differences, which in $SU(3)$ transform as the third compoenent of an octet, are strongly enhanced relative to mass differences transforming as $(3I_z^2-1)/2$, i.e., $\Delta I=2$ mass differences, which in $SU(3)$ transform as part of the 27-dimensional representation. A number of sum rules, originally given by Coleman and Glashow,[1,8] are then predicted. One of these is an equal spacing rule, with a definite value

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‡ National Science Foundation Postdoctoral Fellow, 1965–66.
[1] S. Coleman and S. L. Glashow, Phys. Rev. 134, B671 (1964).
[2] R. E. Cutkosky and P. Tarjanne, Phys. Rev. 132, 1355 (1963).
[3] R. F. Dashen and S. C. Frautschi, Phys. Rev. Letters 13, 497 (1964).
[4] R. F. Dashen and S. C. Frautschi, Phys. Rev. 137, B1331 (1965).
[5] R. F. Dashen, S. C. Frautschi, and D. H. Sharp, Phys. Rev. Letters 13, 777 (1964).
[6] R. F. Dashen, Y. Dothan, S. C. Frautschi, and D. H. Sharp, Phys. Rev. 143, 1185 (1966).
[7] For recent reviews of the methods and results of the bootstrap theory of octet enhancement see the lectures of D. H. Sharp in *Recent Developments in Particle Symmetries* (Academic Press Inc., New York, 1966); S. C. Frautschi, lectures given at the Pacific International Summer School in Physics, August 1965 (Gordon and Breach Science Publishers, New York, to be published).
[8] S. Coleman and S. L. Glashow, Phys. Rev. Letters 6, 423 (1961).

for the spacing, for all the electromagnetic mass splittings within the decuplet,[1,4] another relates the electromagnetic mass splittings within the baryon octet,[8] others relate strong to electromagnetic mass splittings.[1] The latter are a consequence of the universality of the octet enhancement mechanism. These predictions, insofar as they have been tested, are in moderately good agreement with experiment.

All the above results were obtained on the assumption that one can treat the electromagnetic violations of $SU(3)$ to first order (order e^2), with the effects of strong symmetry-breaking appearing as small second order corrections. One may well ask whether this is at all a reasonable approximation. After all, strong symmetry breaking has quite a violent effect on some of the properties of the baryons. In the case of the N and N^*, for instance, strong symmetry breaking forces the πN threshold far below that of the $K\Lambda$ and $K\Sigma$ channels. At the same time, in broken $SU(3)$ the couplings of the N and N^* to the higher mass channels are substantially reduced, compared to their $SU(3)$ symmetric couplings, while their couplings to the low mass channels (πN) are raised.[6] As a result of these features of strong symmetry breaking, a certain simplification is introduced in that the N and N^* can be considered as a separate, closed $SU(2)$ symmetric reciprocal bootstrap system, split off from the rest of the octet and decuplet. However, it is no longer clear that the pattern of electromagnetic mass splittings within such an $SU(2)$ multiplet will bear any resemblance to the pattern obtained on the basis of a first-order calculation of octet enhancement in which strong symmetry-breaking effects are neglected.

This point has already been investigated[4,9,10] in the case of the N-N^* system. A calculation of the A matrix elements using values for masses, coupling constants, and denominator functions which appear to be reasonable from the point of view of *broken* $SU(3)$ and *experiment* showed that in this case there is no enhancement of any of the mass differences due to self-consistent terms. Hence, in an $SU(2)$ symmetric model of the N-N^* system, the bootstrap mechanism does *not* operate so as the enhance the mass shifts transforming like $\Delta I=1$ over those transforming like $\Delta I=2$. In this case, then, the octet enhancement result is largely washed out by strong symmetry-breaking and an $SU(2)$ symmetric calculation, together with a careful study of the driving terms, is required to give an accurate description of the pattern as well as the magnitude of the mass shifts.

It is our purpose in this paper to carry out such calculations for the remaining members of the octet and decuplet; the Σ, Y_1^*, Ξ, and Ξ^* (the Λ and Ω^-, being isosinglets, need not be considered). Here we shall concentrate on the A matrix, whose eigenvalues, together with the associated eigenvectors, will determine the pattern of the electromagnetic mass shifts as long as there is an enhanced eigenvector. A study of the driving terms, which determines the magnitude of the mass shifts (as well as the ratios of mass shifts in those cases where no enhancement emerges) will form the subject of a separate paper.[11]

In Sec. II of this paper, we briefly summarize the relevant kinematics pertaining to meson-baryon scattering, and write down the formulas to be used for calculating the A matrix.

Section III contains a discussion of the electromagnetic mass differences in the Σ and Y_1^* isomultiplets. Like the N and N^*, these particles form a reciptocal bootstrap system, split off from the rest of the octet and decuplet by the breaking of $SU(3)$. In this case it appears to be a reasonable approximation to consider the Σ and Y_1^* as bound states coupled mainly to the $\pi\Lambda$ and $\pi\Sigma$ channels, with Λ, Σ, and Y_1^* exchange providing the important binding forces. The Σ^+-Σ^- and Y_1^{*+}-Y_1^{*-} mass differences transform as $\Delta I=1$, while the $\Sigma^++\Sigma^--2\Sigma^0$ and $Y_1^{*+}+Y_1^{*-}-2Y_1^{*0}$ mass differences transform as $\Delta I=2$. We find that the structure of the A matrix for the Σ-Y_1^* system leads, in each isomultiplet, to an enhancement of the $\Delta I=1$ mass shifts over those transforming like $\Delta I=2$ by an amount in reasonably good agreement with experiment. Thus, in this case the result expected on the basis of octet enhancement is left intact.

In Sec. IV, a similar calculation is carried out for the electromagnetic mass differences in the Ξ and Ξ^* isomultiplets. It is found that the contributions to the mass differences arising from changes in the mass of the Ξ or Ξ^* entering as exchanged or external particles do not lead to an enhancement of the Ξ or Ξ^* mass differences. In this respect, the Ξ-Ξ^* system is similar to the N-N^* system. However, in contrast to the N-N^* system, the Ξ-Ξ^* system is not a closed system split off from the rest of the octet and decuplet. In particular, the coupling to the $K\Sigma$ channel is large, and the change in the external Σ masses gives a contribution which largely accounts for most of the observed Ξ^0-Ξ^- mass difference.

II. KINEMATICS AND METHOD OF CALCULATION

In the electromagnetic mass difference calculations in this paper, it will be assumed that the baryons of the unperturbed situation can be considered as bound-state poles in two-particle pseudoscalar meson-baryon scattering amplitudes. In particular, we shall be considering the $l=1$, $J^P=(\frac{1}{2})^+$, and $J^P=(\frac{3}{2})^+$ partial-wave amplitudes as functions of W, the total center-of-mass energy of the baryon and meson. We shall not give a detailed treatment here of the analytic properties of the partial-wave amplitudes in the W plane, but as we proceed we

[9] R. F. Dashen, Phys. Rev. **135**, B1196 (1964).

[10] S. Biswas, S. Bose, and L. Pande, Phys. Rev. **138**, B163 (1965).

[11] F. J. Gilman (to be published).

shall refer for needed properties to the excellent general discussions of Frazer and Fulco[12] and Frautschi and Walecka,[13] and to the specific discussions of pion-hyperon scattering in Feldman and Hwa[14] and Kayser.[15]

We shall thus consider meson-baryon scattering with M_i (M_f), E_i (E_f), m_i (m_f), and q_i (q_f), the initial (final) baryon mass, baryon energy, meson mass, and meson momentum in the center-of-mass system where W is the total energy. We take the $l=1$, $J^P=(\frac{1}{2})^+$, and $J^P=(\frac{3}{2})^+$ partial-wave amplitudes to be

$$T_{fi}^J(W)=\frac{2W}{((E_f-M_f)(E_i-M_i))^{1/2}}\frac{e^{2i\eta_J}-1}{2i(q_iq_f)^{1/2}}, \quad (1)$$

where

$$q_i^2(W)=[(W+M_i)^2-m_i^2][(W-M_i)^2-m_i^2]/4W^2. \quad (2)$$

The amplitudes $T_{fi}^J(W)$ are free of kinematic singularities in the W plane and have the proper threshold behavior factored out.[12–15]

In general, we must consider an n-channel unperturbed scattering amplitude $\mathbf{T}(W)$,[16] where $\mathbf{T}(W)$ is an $n\times n$ symmetric partial-wave scattering matrix which has a bound-state pole. We assume further that the unperturbed amplitude has been written in the form[17]

$$\mathbf{T}(W)=\mathbf{N}(W)\mathbf{D}(W)^{-1}, \quad (3)$$

where $\mathbf{N}(W)$ is an $n\times n$ matrix whose elements are analytic in W except for left-hand cuts (LHC) and $\mathbf{D}(W)$ is an $n\times n$ matrix whose elements are analytic except for right-hand cuts (RHC) present in the the partial-wave amplitude $\mathbf{T}(W)$.

On the right-hand cut, we assume elastic two-particle unitarity before the perturbation is introduced:

$$\text{Im}\mathbf{T}(W)=\mathbf{T}(W)\boldsymbol{\rho}(W)\mathbf{T}(W)^\dagger, \quad (4)$$

where $\boldsymbol{\rho}(W)$ is a diagonal $n\times n$ matrix containing phase-space factors which are functions of the total center-of-mass energy W. For partial-wave amplitudes defined as in Eq. (1), the elements of the diagonal $\boldsymbol{\rho}(W)$ matrix are

$$\rho_{ii}(W)=((E_i-M_i)/2W)q_i(W)\theta(W^2-(M_i+m_i)^2). \quad (5)$$

The pole in $\mathbf{T}(W)$ is assumed to be due to the vanishing of $\det[\mathbf{D}(W)]$ at $W=M_B$, the bound-state mass. The residue at the bound-state pole is then defined by

$$\mathbf{R}=\mathbf{N}(M_B)\boldsymbol{\Delta}, \quad (6)$$

where

$$\boldsymbol{\Delta}=\lim_{W\to M_B}(W-M_B)\mathbf{D}(W)^{-1}. \quad (7)$$

When the perturbing electromagnetic interaction is introduced, $\mathbf{T}(W)\to\mathbf{T}(W)+\delta\mathbf{T}(W)$ and $M_B\to M_B+\delta M_B$. Following Dashen and Frautschi,[18] to lowest order in the perturbation we have

$$\delta M_B=\text{Tr}\left[\mathbf{R}\boldsymbol{\Delta}-\frac{1}{\pi}\int_{\text{cuts}}dW'\frac{\text{Im}\mathbf{D}^T(W)\delta\mathbf{T}(W)\mathbf{D}(W)}{W'-M_B}\boldsymbol{\Delta}\right]\Big/\text{Tr}[\mathbf{RR}]. \quad (8)$$

We shall separate contributions to Eq. (8) for the change in the baryon masses into two parts. The first part consists of contributions due to changes in the masses of the baryons themselves when acting as exchanged or external particles in the strong-interaction bootstrap. We call these the self-consistent terms. The second part consists of all other contributions due to changes in other exchanged or external particle masses, changes in coupling constants, or changes in the discontinuity across the left- or right-hand cut in Eq. (8) due to additional diagrams with explicit photons. We lump together all contributions of the second kind and call them driving terms.

We are thus led to first order in the perturbation to write equations of the form[2–4]

$$\delta M_i=\sum_j A_{ij}\delta M_j+d_i, \quad (9)$$

where the A's are numerical coefficients which represent the effect of the self-consistent terms and the d's are the driving terms. Now the solution of the linear equations (9) for the δM_i involves the quantity $\det[\mathbf{1}-\mathbf{A}]$ in the denominator, so that if $\det[\mathbf{1}-\mathbf{A}]$ is small, i.e., if A has an eigenvalue near one, then the corresponding mass shifts are enhanced by the self-consistent terms coming from the reciprocal bootstrap dynamics.

For the problem we are interested in here, we might write the $\mathbf{A}$ matrix symbolically as

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}^{NN} & \mathbf{A}^{N\Sigma} & \mathbf{A}^{N\Xi}\\ \mathbf{A}^{\Sigma N} & \mathbf{A}^{\Sigma\Sigma} & \mathbf{A}^{\Sigma\Xi}\\ \mathbf{A}^{\Xi N} & \mathbf{A}^{\Xi\Sigma} & \mathbf{A}^{\Xi\Xi}\end{bmatrix}. \quad (10)$$

Here, $\mathbf{A}^{NN}$ is a submatrix which represents the effect of a shift in the mass of a strangeness zero (N or N^*) baryon acting as an external or exchanged particle on the N and N^* bound-state masses. Similarly, $\mathbf{A}^{\Sigma\Xi}$ is a submatrix which represents the effect of a shift in mass of the strangeness $=-2$ baryons, Ξ and

[12] W. R. Frazer and J. R. Fulco, Phys. Rev. **119**, 1420 (1960).
[13] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).
[14] D. Feldman and R. Hwa, Ann. Phys. (N. Y.) 21, 453 (1963).
[15] B. Kayser, Phys. Rev. **138**, B1244 (1965).
[16] $\mathbf{A}$ denotes a matrix; $(\mathbf{A}^T)_{ij}=(\mathbf{A})_{ji}$; $(\mathbf{A}^\dagger)_{ij}=(\mathbf{A})_{ji}^*$; $\det(\mathbf{A})=|\mathbf{A}|=$ the determinant of A; and $\text{Tr}(\mathbf{A})=\Sigma_i(\mathbf{A})_{ii}=$ the trace of A.
[17] J. D. Bjorken, Phys. Rev. Letters 4, 473 (1960).
[18] R. F. Dashen and S. C. Frautschi, Phys. Rev. **137**, B1318 (1965).

Ξ^*, acting as exchanged or external particles on the strangeness $=-1$ (Σ and Y_1^*) bound state masses.

In the $SU(3)$ symmetric calculation, all these submatrices are, of course, related to one another by suitable $SU(3)$ Clebsch-Gordan coefficients and none is zero or generally neglectable. In the case of broken $SU(3)$, the submatrices are no longer simply related, but there is an important *asymmetry of the large A matrix*.[19] As noted before, the breaking of $SU(3)$ results in the N and N^* splitting off from the rest of the octet and decuplet bootstrap and forming their own reciprocal bootstrap subsystem. Stated in terms of the A matrix of Eq. (10), this means that in broken $SU(3)$, $\mathbf{A}^{N\Sigma}$ and $\mathbf{A}^{N\Xi}\approx 0$. Similarly, inasmuch as in broken $SU(3)$ we can consider the Σ and Y_1^* as only $\pi\Lambda$ and $\pi\Sigma$ bound states, they also form their own reciprocal bootstrap subsystem, i.e., $\mathbf{A}^{\Sigma N}$ and $\mathbf{A}^{\Sigma\Xi}\approx 0$. So in broken $SU(3)$, the large A matrix looks like

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}^{NN} & 0 & 0\\ 0 & \mathbf{A}^{\Sigma\Sigma} & 0\\ \mathbf{A}^{\Xi N} & \mathbf{A}^{\Xi\Sigma} & \mathbf{A}^{\Xi\Xi}\end{bmatrix}.$$

Now it is clear that the eigenvalues of such a matrix are simply the eigenvalues of the diagonal submatrices. Thus, to determine if there is an enhancement, i.e., an eigenvalue of $\mathbf{A}$ near one, we need only calculate the diagonal submatrices of $\mathbf{A}$, $\mathbf{A}^{NN}$, $\mathbf{A}^{\Sigma\Sigma}$, and $\mathbf{A}^{\Xi\Xi}$. Furthermore, from solving the linear equations (9), it is apparent that for the asymmetric A matrix above,

$$\delta M^N=d^N/(1-A^{NN}),\tag{11a}$$

$$\delta M^\Sigma=d^\Sigma/(1-A^{\Sigma\Sigma}),\tag{11b}$$

and

$$\delta M^\Xi=(A^{\Xi N}\delta M^N+A^{\Xi\Sigma}\delta M^\Sigma+d^\Xi)/(1-A^{\Xi\Xi}).\tag{11c}$$

The question of whether there is an enhancement of the N-N^* or Σ-Y_1^* mass differences is then answered by just calculating their respective diagonal submatrices of the large A matrix. The Ξ-Ξ^* mass differences, however, depend both on the diagonal $\mathbf{A}^{\Xi\Xi}$ submatrix and the off-diagonal $\mathbf{A}^{\Xi N}$ and $\mathbf{A}^{\Xi\Sigma}$ submatrices. From previous calculations,[4] we know that A^{NN} has no eigenvalue near unity in broken $SU(3)$ and the N-N^* mass differences are thus not enhanced by self-consistent terms. As stated in the Introduction, we find that $A^{\Xi\Xi}$ also has no eigenvalues near one in broken $SU(3)$, but that there is an eigenvalue of $A^{\Sigma\Sigma}$ near one, corresponding to an enhancement of $\Delta I=1$ mass differences. Even through they are not directly enhanced, the Ξ mass differences are still large, since we find $A^{\Xi\Sigma}\sim 1$, so that the Ξ mass differences *follow the enhanced eigenvector* of the Σ mass differences.

Up to this point, our comments on the effects of the breaking of $SU(3)$ upon baryon electromagnetic mass differences have been rather general and would be expected to hold independently of any particular bootstrap model or perturbation formalism. We now proceed, however, to calculate in detail the values of the various A matrix elements discussed above in a specific model and using the S-matrix perturbation theory developed by Dashen and Frautschi.[4,18]

[19] Some of the effects of the asymmetry of the A matrix are discussed at length in Ref. 4.

III. THE Σ AND Y_1^* SYSTEM

Let us now consider the electromagnetic mass differences of the Σ^+, Σ^0, Σ^-, and Y_1^{*+}, Y_1^{*0}, Y_1^{*-} baryons within a reciprocal bootstrap model where both the Σ and Y_1^* are regarded as $\pi\Lambda$ and $\pi\Sigma$ bound states. Using approximations very similar to those used in the octet enhancement calculation of Dashen and Frautschi,[4] we shall see that the strong-interaction reciprocal bootstrap leads to an enhancement of the electromagnetic mass differences transforming as $\Delta I=1$ over those transforming as $\Delta I=2$.

We start by assuming that the unperturbed (by electromagnetism) two-channel, strong interaction problem of obtaining the Σ with a mass of 1190 MeV as an $I=1$, $J^P=(\frac{1}{2})^+$ bound state[20] of $\pi\Lambda$ and $\pi\Sigma$ has been solved, and that the unperturbed amplitudes have been obtained in the form $\mathbf{T}=\mathbf{N}\mathbf{D}^{-1}$. Exactly this problem has recently been treated by Kayser.[15] Possible exchanged baryons which are members of the octet and decuplet are the Λ, Σ, and Y_1^*. It is the exchange of these particles which results in a large part of the "binding force" in Kayser's calculation. The inclusion of both channels turns out to be essential for a successful Σ bootstrap.

We shall further assume that the Y_1^* has been obtained as an $I=1$, $J^P=(\frac{3}{2})^+$ resonant state of $\pi\Lambda$ and $\pi\Sigma$ scattering with a mass of 1385 MeV and a width of 50 MeV in a calculation similar to that for the Σ bound state. This strong interaction problem has been considered recently by Kayser and Bloom,[21] who include both a $\pi\Lambda$ and $\pi\Sigma$ channel, and also by Martin,[22] who considers only the $\pi\Lambda$ channel. Recent experimental indications are that the coupling of the Y_1^* to the $\pi\Sigma$ channel is about a factor of 2 smaller than the coupling to the $\pi\Lambda$ channel,[23] so we shall include the $\pi\Sigma$ as well as $\pi\Lambda$ channel in our calculation.

The two-body channel with the next highest threshold is the $\bar{K}N$ channel. As we shall see later, predictions based on $SU(3)$ [and also broken $SU(3)$] give a small value for the $\bar{K}N\Sigma$ coupling constant. For the Y_1^*, one may argue from the lack of appreciable P-wave K^-p scattering that this channel can be neglected in the Y_1^* bootstrap.[22] It is precisely in the neglect of this and the higher threshold $\eta\Sigma$ and $K\Xi$ channels that we are first

[20] For data on particle and resonance masses, spins, and decay widths, we shall refer, unless otherwise indicated, to A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **36**, 977 (1964).

[21] B. Kayser and E. Bloom, Phys. Rev. **144**, 1176 (1966).

[22] B. Martin, Phys. Rev. **138**, B1136 (1965).

[23] D. O. Huwe, University of California Radiation Laboratory Report No. UCRL-11291, 1964 (unpublished).

breaking the $SU(3)$ symmetric reciprocal bootstrap of the octet and decuplet. In their analysis of coupling constants in broken $SU(3)$, Dashen *et al.*[6] have found that the breaking of $SU(3)$ generally raises the coupling strengths to low-lying channels and decreases the coupling strengths to higher mass channels, thus providing an added justification for our neglect of higher mass channels. For the particular case of interest here, they find on the basis of coupling strengths alone that the Σ and Y_1^* are both 50–60% $\pi\Lambda$ and $\pi\Sigma$ in broken $SU(3)$, while they are only 30–40% $\pi\Lambda$ and $\pi\Sigma$ in unbroken $SU(3)$. In any case, in the calculations to follow, we shall neglect the $\bar{K}N$, $\eta\Sigma$, and $K\Xi$ channels, as well as three-particle channels, but we shall estimate their effect on the calculation at the end. We again stress that in taking only the $\pi\Lambda$ and $\pi\Sigma$ channels, we are assuming a badly broken $SU(3)$ symmetry.

The partial-wave-scattering matrix will then be 2×2 with channel one being $\pi\Lambda$ scattering and channel two $\pi\Sigma$ scattering. Since the Λ is an isospin singlet while the pion has isospin 1, the $\pi\Lambda$ system can only have total isospin $I=1$. If we let $i=1, 2, 3$ be a Cartesian isospin index with $\pi^+=(\pi_1+i\pi_2)/\sqrt{2}$, $\pi^0=\pi_3$, and $\pi^-=(\pi_1-i\pi_2)/\sqrt{2}$, then we define the $I=1$ $\pi\Lambda$ states to be

$$|\pi\Lambda, I=1, i\rangle = |\Lambda\pi_i\rangle. \tag{12}$$

As the Σ has isospin one, we define $\Sigma^+=(\Sigma_1+i\Sigma_2)/\sqrt{2}$, $\Sigma^0=\Sigma_3$, and $\Sigma^-=(\Sigma_1-i\Sigma_2)/\sqrt{2}$ analogously to the pion states. The $I=1$, $\pi\Sigma$ states are then defined by

$$|\pi\Sigma, I=1, i\rangle = |i(\mathbf{\Pi}\times\mathbf{\Sigma}/\sqrt{2})_i\rangle. \tag{13}$$

Since the Σ and Y_1^* have the same isospin, we can read off the proper signs of the residue and isospin crossing matrices for the Y_1^* directly from the corresponding Σ channels.

With the above definitions, the Σ^+, Σ^0, and Σ^- will occur as bound states in the $J^P=(\frac{1}{2})^+$, $I=1$, $I_z=1$, 0, and -1 channels, respectively. We denote the partial-wave amplitudes in these channels as $\mathbf{T}^{\Sigma^+}(W)$, $\mathbf{T}^{\Sigma^0}(W)$, and $\mathbf{T}^{\Sigma^-}(W)$, i.e., we label the partial-wave amplitudes for a given channel by the bound state which occurs in that channel. In the absence of electromagnetism, we assume that the strong interactions conserve isospin, so that $\mathbf{T}^{\Sigma^+}(W)$, $\mathbf{T}^{\Sigma^0}(W)$, and $\mathbf{T}^{\Sigma^-}(W)$ all have poles with the same residue at the unperturbed Σ mass, M^Σ, of the form

$$\mathbf{T}^\Sigma = \mathbf{R}^\Sigma/(W-M^\Sigma). \tag{14}$$

The residue matrix R^Σ is related to the coupling constants defined in Appendix A by

$$\mathbf{R}^\Sigma = \begin{pmatrix} R_{11}^\Sigma & R_{12}^\Sigma \\ R_{21}^\Sigma & R_{22}^\Sigma \end{pmatrix} = \frac{1}{4\pi}\begin{pmatrix} -g_{\pi\Lambda\Sigma}^2 & \sqrt{2}g_{\pi\Lambda\Sigma}g_{\pi\Sigma\Sigma} \\ \sqrt{2}g_{\pi\Lambda\Sigma}g_{\pi\Sigma\Sigma} & -2g_{\pi\Sigma\Sigma}^2 \end{pmatrix}. \tag{15}$$

We have defined the Lagrangian density in Appendix A so that $g_{\pi\Lambda\Sigma}$ and $g_{\pi\Sigma\Sigma}$ are the rationalized, renormalized pseudoscalar coupling constants ($g_{\pi NN}^2/4\pi=14.8$).

Similarly, in the $J^P=(\frac{3}{2})^+$, $I=1$, $I_z=1$, 0, and -1 channels, the corresponding partial-wave amplitudes, $\mathbf{T}^{Y^+}(W)$, $\mathbf{T}^{Y^0}(W)$, and $\mathbf{T}^{Y^-}(W)$ all have poles at the unperturbed Y_1^* mass, M^Y, of the form[24]

$$\mathbf{T}^Y = \frac{\mathbf{R}^Y}{W-M^Y} = \frac{1}{W-M^Y}\begin{pmatrix} R_{11}^Y & R_{12}^Y \\ R_{21}^Y & R_{22}^Y \end{pmatrix}. \tag{16}$$

The partial width for the decay of the Y_1^* in channel i is given by[25]

$$\Gamma_i = -2\rho_{ii}(M^Y)R_{ii}^Y. \tag{17}$$

The magnitude of off-diagonal residues is fixed by $|R_{ij}|=(R_{ii}R_{jj})^{1/2}$. Note that R_{11}^Σ, R_{22}^Σ, R_{11}^Y, and R_{22}^Y must all be negative with our conventions, but that the signs of off-diagonal residues depend on relative coupling constant signs which will be taken from $SU(3)$ predictions.[26]

In the presence of the electromagnetic interaction, the poles in $\mathbf{T}^{\Sigma^+}(W)$, $\mathbf{T}^{\Sigma^0}(W)$, and $\mathbf{T}^{\Sigma^-}(W)$ will be shifted from their common unperturbed value, M^Σ, to M^{Σ^+}, M^{Σ^0}, and M^{Σ^-}, the observed masses of the Σ^+, Σ^0, and Σ^- baryons, respectively. Similarly, the poles in $\mathbf{T}^{Y^+}(W)$, $\mathbf{T}^{Y^0}(W)$, and $\mathbf{T}^{Y^-}(W)$ will be shifted to M^{Y^+}, M^{Y^0}, and M^{Y^-}, the observed masses of the Y_1^{*+}, Y_1^{*0}, and Y_1^{*-}, respectively. We write $\delta M^{\Sigma^+}=M^{\Sigma^+}-M^\Sigma$, $\delta M^{Y^+}=M^{Y^+}-M^Y$, etc.

Since the electromagnetic perturbation breaks the invariance of the strong interactions under isospin rotations, it is convenient to write mass operators in isospin space, the diagonal expectation values of which are the observed masses, and which are expressed as sums of parts which transform according to irreducible representations of the group of isospin rotations:

$$\begin{aligned} M_{op}^\Sigma &= M^\Sigma+\delta M_0^\Sigma+\tfrac{1}{2}\delta M_1^\Sigma I_z+\tfrac{1}{3}\delta M_2^\Sigma(\tfrac{1}{2}(3I_z^2-1)), \\ M_{op}^Y &= M^Y+\delta M_0^Y+\tfrac{1}{2}\delta M_1^Y I_z +\tfrac{1}{3}\delta M_2^Y(\tfrac{1}{2}(3I_z^2-1)), \end{aligned} \tag{18}$$

where

$$\begin{aligned} M^\Sigma+\delta M_0^\Sigma &= [M^{\Sigma^+}+4M^{\Sigma^0}+M^{\Sigma^-}]/6, \\ M^Y+\delta M_0^Y &= [M^{Y^+}+4M^{Y^0}+M^{Y^-}]/6, \\ \delta M_1^\Sigma &= [M^{\Sigma^+}-M^{\Sigma^-}] = \delta M^{\Sigma^+}-\delta M^{\Sigma^-}, \\ \delta M_1^Y &= [M^{Y^+}-M^{Y^-}] = \delta M^{Y^+}-\delta M^{Y^-}, \\ \delta M_2^\Sigma &= M^{\Sigma^+}+M^{\Sigma^-}-2M^{\Sigma^0} \\ &= \delta M^{\Sigma^+}+\delta M^{\Sigma^-}-2\delta M^{\Sigma^0}, \\ \delta M_2^Y &= M^{Y^+}+M^{Y^-}-2M^{Y^0} \\ &= \delta M^{Y^+}+\delta M^{Y^-}-2\delta M^{Y^0}. \end{aligned} \tag{19}$$

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[24] Since the Y_1^* is a resonance, the pole in $\mathbf{T}^Y(W)$ is in fact not on the real axis in the W plane. However, in all calculations involving members of the decuplet in this paper, we shall treat them as stable particles.

[25] E. Abers and C. Zemach, Phys. Rev. 131, 2305 (1963).

[26] In the calculation of coupling constants in broken $SU(3)$ in Ref. 6, it was found that while there are large changes in the magnitudes of the coupling constants, there are no changes in signs.

The subscripts denote the transformation properties ($\Delta I=0$, 1, 2) of the corresponding parts of the mass operator under the group of isospin rotations.

We now note that if we restrict ourselves to first order in the mass shifts, then due to the invariance of the strong ingeractions under isospin rotations a shift in an exchanged or external mass transforming according to a given irreducible isospin representation can only give rise to shifts in the bound-state mass transforming according to the same representation.[27,28] The problem of determining the mass shifts thus splits up into a set of problems which are disconnected from each other, one for each irreducible representation of the isotropic spin group. We can thus write[18]

$$\delta M_1{}^\Sigma = A_{1x}{}^{\Sigma\Sigma}\delta M_1{}^\Sigma + A_{1x}{}^{\Sigma Y}\delta M_1{}^Y + A_{1e}{}^{\Sigma\Sigma}\delta M_1{}^\Sigma + d_1{}^\Sigma,$$

$$\delta M_1{}^Y = A_{1x}{}^{Y\Sigma}\delta M_1{}^\Sigma + A_{1x}{}^{YY}\delta M_1{}^Y + A_{1e}{}^{Y\Sigma}\delta M_1{}^\Sigma + d_1{}^Y, \tag{20a}$$

and

$$\delta M_2{}^\Sigma = A_{2x}{}^{\Sigma\Sigma}\delta M_2{}^\Sigma + A_{2x}{}^{\Sigma Y}\delta M_2{}^Y + A_{2e}{}^{\Sigma\Sigma}\delta M_2{}^\Sigma + d_2{}^\Sigma,$$

$$\delta M_2{}^Y = A_{2x}{}^{Y\Sigma}\delta M_2{}^\Sigma + A_{2x}{}^{YY}\delta M_2{}^Y + A_{2e}{}^{Y\Sigma}\delta M_2{}^\Sigma + d_2{}^Y, \tag{20b}$$

where $A_{1x}{}^{\Sigma\Sigma}$ is a numerical coefficient giving the effect of a $\Delta I=1$ shift in the mass of the Σ acting as an exchanged particle on the position of the Σ bound state. Similarly, $A_{1e}{}^{Y\Sigma}$ is a numerical coefficient giving the effect of a shift in the mass of the Σ acting as an external particle on the position of the $Y_1{}^*$ resonance, with both shifts transforming as $\Delta I=1$. The d's are the corresponding driving terms. There is no term containing an $A_e{}^{\Sigma Y}$ since the $Y_1{}^*$ never appears as an external particle. $\delta M_0{}^\Sigma$ and $\delta M_0{}^Y$ are not of interest to us since they involve a symmetric increase in all the masses with no relative shifts.

The A's are now the objects of interest and will be calculated using Eq. (8). We shall write the results in matrix form as

$$\mathbf{A}_1 = \begin{pmatrix} A_1{}^{\Sigma\Sigma} & A_1{}^{\Sigma Y} \\ A_1{}^{Y\Sigma} & A_1{}^{YY} \end{pmatrix} = \begin{pmatrix} A_{1x}{}^{\Sigma\Sigma}+A_{1e}{}^{\Sigma\Sigma} & A_{1x}{}^{\Sigma Y} \\ A_{1x}{}^{Y\Sigma}+A_{1e}{}^{Y\Sigma} & A_{1x}{}^{YY} \end{pmatrix}, \tag{21}$$

with a similar expression for $\mathbf{A}_2$. As observed in Sec. II, there will be an enhancement of the $\Delta I=1$ or $\Delta I=2$ mass differences if $|1-\mathbf{A}_1|$ or $|1-\mathbf{A}_2|$ is small, respectively. We now turn to the detailed calculation of the elements of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$.

In order to calculate the A_x's, we must know the individual contributions of the various Σ and $Y_1{}^*$ charge states to the amplitudes which contain the Σ^+, Σ^0, Σ^- and $Y_1{}^{*+}$, $Y_1{}^{*0}$, $Y_1{}^{*-}$ bound states and resonance poles, respectively. We label the resulting amplitudes due to Σ exchange, $\mathbf{T}^{\Sigma^+}(W,\Sigma\text{ exch.})$, $\mathbf{T}^{\Sigma^0}(W,\Sigma\text{ exch.})$, etc. In calculating the effect of changes in the masses of exchanged particles, we closely follow Dashen and Frautschi[4] and use the approximation of keeping only the nearby (to $W=M^\Sigma$) short cuts of the u-channel exchange amplitudes[12,13] and approximate these short cuts by pseudopoles, i.e., we shall use the static approximation. Within the static approximation, exchange of the Σ^+, Σ^0, and Σ^- gives rise to pseudopoles in $\mathbf{T}^\Sigma(W)$ and $\mathbf{T}^Y(W)$ at $W=2M^\Sigma - M^{\Sigma\text{ exch.}}$.[29] From some long but straightforward arithmetic using the crossing relations for partial-wave amplitudes in the static approximation,[30] we find

$$T^{\Sigma^+}(W,\Sigma\text{ exch.}) = +\frac{1}{3}\frac{1}{W-2M^\Sigma+M^{\Sigma^+}}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{4}R_{22}{}^\Sigma \end{pmatrix}\ (\Sigma^+\text{ exchange})$$
$$+\frac{1}{3}\frac{1}{W-2M^\Sigma+M^{\Sigma^0}}\begin{pmatrix} 0 & -\frac{1}{2}R_{12}{}^\Sigma \\ -\frac{1}{2}R_{12}{}^\Sigma & 0 \end{pmatrix}\ (\Sigma^0\text{ exchange})$$
$$+\frac{1}{3}\frac{1}{W-2M^\Sigma+M^{\Sigma^-}}\begin{pmatrix} R_{11}{}^\Sigma & -\frac{1}{2}R_{12}{}^\Sigma \\ -\frac{1}{2}R_{12}{}^\Sigma & \frac{1}{4}R_{22}{}^\Sigma \end{pmatrix}\ (\Sigma^-\text{ exchange}),$$

$$T^{\Sigma^0}(W,\Sigma\text{ exch.}) = +\frac{1}{3}\frac{1}{W-2M^\Sigma+M^{\Sigma^+}}\begin{pmatrix} 0 & -\frac{1}{2}R_{12}{}^\Sigma \\ -\frac{1}{2}R_{12}{}^\Sigma & 0 \end{pmatrix} + \frac{1}{3}\frac{1}{W-2M^\Sigma+M^{\Sigma^0}}\begin{pmatrix} R_{11}{}^\Sigma & 0 \\ 0 & \frac{1}{2}R_{22}{}^\Sigma \end{pmatrix}$$
$$+\frac{1}{3}\frac{1}{W-2M^\Sigma+M^{\Sigma^-}}\begin{pmatrix} 0 & -\frac{1}{2}R_{12}{}^\Sigma \\ -\frac{1}{2}R_{12}{}^\Sigma & 0 \end{pmatrix}, \tag{22a}$$

[27] S. Glashow, Phys. Rev. **130**, 2132 (1963).

[28] R. Cutkosky and P. Tarjanne, Phys. Rev. **132**, 1355 (1963).

[29] For the purpose of calculating the effect of exchanged particle mass shifts, we have set both the external Λ and Σ masses equal to $M^\Sigma=1190$ MeV.

[30] For a derivation of the crossing relations within the static model, see for example, E. Henley and W. Thirring, *Elementary Quantum Field Theory* (McGraw-Hill Book Company, Inc., New York, 1962), Chap. 18. Also, see the approximation of the u-channel cuts of the relativistic partial-wave amplitudes by poles in Ref. 13.

$$T^{\Sigma^-}(W,\Sigma\text{ exch.})=+\frac{1}{3}\frac{1}{W-2M^\Sigma+M^{\Sigma^+}}\begin{pmatrix}R_{11}^\Sigma & -\frac{1}{2}R_{12}^\Sigma\\ -\frac{1}{2}R_{12}^\Sigma & \frac{1}{4}R_{22}^\Sigma\end{pmatrix}+\frac{1}{3}\frac{1}{W-2M^\Sigma+M^{\Sigma^0}}\begin{pmatrix}0 & -\frac{1}{2}R_{12}^\Sigma\\ -\frac{1}{2}R_{12}^\Sigma & 0\end{pmatrix}+\frac{1}{3}\frac{1}{W-2M^\Sigma+M^{\Sigma^-}}\begin{pmatrix}0 & 0\\ 0 & \frac{1}{4}R_{22}^\Sigma\end{pmatrix},$$

and

$$T^{Y^+,Y^0,Y^-}(W,\Sigma\text{ exch.})=-2T^{\Sigma^+,\Sigma^0,\Sigma^-}(W\Sigma,\text{exch.}).\tag{22b}$$

Similarly,

$$T^{\Sigma^+}(W,Y_1^*\text{ exch.})=-\frac{4}{3}\frac{1}{W-2M^\Sigma+M^{Y^+}}\begin{pmatrix}0 & 0\\ 0 & \frac{1}{4}R_{22}^Y\end{pmatrix}-\frac{4}{3}\frac{1}{W-2M^\Sigma+M^{Y^0}}\begin{pmatrix}0 & -\frac{1}{2}R_{12}^Y\\ -\frac{1}{2}R_{12}^Y & 0\end{pmatrix}-\frac{4}{3}\frac{1}{W-2M^\Sigma+M^{Y^-}}\begin{pmatrix}R_{11}^Y & -\frac{1}{2}R_{12}^Y\\ -\frac{1}{2}R_{12}^Y & \frac{1}{4}R_{22}^Y\end{pmatrix},$$

$$T^{\Sigma^0}(W,Y_1^*\text{ exch.})=-\frac{4}{3}\frac{1}{W-2M^\Sigma+M^{Y^+}}\begin{pmatrix}0 & -\frac{1}{2}R_{12}^Y\\ -\frac{1}{2}R_{12}^Y & 0\end{pmatrix}-\frac{4}{3}\frac{1}{W-2M^\Sigma+M^{Y^0}}\begin{pmatrix}R_{11}^Y & 0\\ 0 & \frac{1}{4}R_{22}^Y\end{pmatrix}-\frac{4}{3}\frac{1}{W-2M^\Sigma+M^{Y^-}}\begin{pmatrix}0 & -\frac{1}{2}R_{12}^Y\\ -\frac{1}{2}R_{12}^Y & 0\end{pmatrix},\tag{22c}$$

$$T^{\Sigma^-}(W,Y_1^*\text{ exch.})=-\frac{4}{3}\frac{1}{W-2M^\Sigma+M^{Y^+}}\begin{pmatrix}R_{11}^Y & -\frac{1}{2}R_{12}^Y\\ -\frac{1}{2}R_{12}^Y & \frac{1}{4}R_{22}^Y\end{pmatrix}-\frac{4}{3}\frac{1}{W-2M^\Sigma+M^{Y^0}}\begin{pmatrix}0 & -\frac{1}{2}R_{12}^Y\\ -\frac{1}{2}R_{12}^Y & 0\end{pmatrix}-\frac{4}{3}\frac{1}{W-2M^\Sigma+M^{Y^-}}\begin{pmatrix}0 & 0\\ 0 & \frac{1}{4}R_{22}^Y\end{pmatrix},$$

and

$$\mathbf{T}^{Y^+,Y^0,Y^-}(W,Y_1^*\text{ exch.})=\tfrac{1}{4}\mathbf{T}^{\Sigma^+,\Sigma^0,\Sigma^-}(W,\ Y_1^*\text{ exch.}).\tag{22d}$$

All of our exchange amplitudes have the form $\mathbf{T}=\mathbf{R}/(W-2M^\Sigma+M^x)$. When the electromagnetic interaction is turned on, the change in M^x results in a change in the amplitudes to first order of the form

$$\delta\mathbf{T}=\mathbf{R}\delta M^x/(W-2M^\Sigma+M^x)^2.\tag{23}$$

In our choice of **D** function, we again closely follow the octet enhancement calculation of Dashen and Frautschi[4] by assuming that in a representation in which the multichannel **D** function is diagonal at the bound state or resonance mass, it is also approximately diagonal over the nearby parts of the left- and right-hand cuts.[31] Such a **D** function is

$$\mathbf{D}(W)=\mathbf{D}(M^*)+D'(M^*)[(W-M^*)/(W-W_0)](M^*-W_0)\mathbf{1}\tag{24}$$

where M^* is the bound state (resonance) mass, $\det[\mathbf{D}(M^*)]=0$, the prime denotes the derivative with respect to W, and W_0 is the position of a pole in $\mathbf{D}(W)$ which is taken to approximate the right-hand unitarity cut of $\mathbf{D}(W)$. We rewrite Eq. (24) as

$$\mathbf{D}^\Sigma(W)=\mathbf{D}^\Sigma(M^\Sigma)+D^\Sigma(W)\mathbf{1},\tag{25a}$$

and

$$\mathbf{D}^Y(W)=\mathbf{D}^Y(M^Y)+D^Y(W)\mathbf{1},\tag{25b}$$

where, for the purposes of this calculation,

$$D^\Sigma(W)=D^{\Sigma\prime}(M^\Sigma)[(W-M^\Sigma)/(W-W_1)](M^\Sigma-W_1),\tag{26a}$$

$$D^Y(W)=D^{Y\prime}(M^Y)[(W-M^Y)/(W-W_2)](M^Y-W_2).\tag{26b}$$

[31] This assumption on the behavior of $\mathbf{D}(W)$ would be expected to hold if the left-hand cut is dominated by a single pole and we only need take account of singularities close to the bound-state mass. Although one or both of these conditions hold in, say, a reciprocal bootstrap model of the Σ and Y_1^*, Eq. (24) must in general be treated as an additional assumption.

With D functions such as those in Eqs. (26), Eq. (8) simplifies[32] to

$$\delta M^{\Sigma^+,\Sigma^0,\Sigma^-}=\frac{\mathrm{Tr}\left[\mathbf{R}^{\Sigma}\frac{1}{\pi}\int_{\mathrm{cuts}}\frac{dW'}{W'-M^{\Sigma}}\,\mathrm{Im}\{[D^{\Sigma}(W')]^2\delta\mathbf{T}^{\Sigma^+,\Sigma^0,\Sigma^-}(W')\}\right]}{[D^{\Sigma\prime}(M^{\Sigma})]^2\,\mathrm{Tr}[\mathbf{R}^{\Sigma}\mathbf{R}^{\Sigma}]} \tag{27a}$$

for the Σ mass differences, and

$$\delta M^{Y^+,Y^0,Y^-}=\frac{\mathrm{Tr}\left[\mathbf{R}^{Y}\frac{1}{\pi}\int_{\mathrm{cuts}}\frac{dW'}{W'-M^{Y}}\,\mathrm{Im}\{[D^{Y}(W')]^2\delta\mathbf{T}^{Y^+,Y^0,Y^-}(W')\}\right]}{[D^{Y\prime}(M^{Y})]^2\,\mathrm{Tr}[\mathbf{R}^{Y}\mathbf{R}^{Y}]} \tag{27b}$$

for the Y_1^* mass differences.

After substituting Eqs. (22) in Eqs. (27), carrying out the integrals and traces, and taking account of the definitions of the A-matrix elements in Eq. (20), we find that

$$\begin{aligned}
A_{1x}^{\Sigma\Sigma}&=-\frac{1}{3}\frac{(R_{11}^{\Sigma})^2-(R_{12}^{\Sigma})^2}{(R_{11}^{\Sigma}+R_{22}^{\Sigma})^2}\left(\frac{[D^{\Sigma}(W)]^2}{[D^{\Sigma\prime}(M^{\Sigma})]^2(W-M^{\Sigma})}\right)'\Bigg|_{W=2M^{\Sigma}-M^{\Sigma}},\\
A_{1x}^{\Sigma Y}&=+\frac{4}{3}\frac{R_{11}^{\Sigma}R_{11}^{Y}-R_{12}^{\Sigma}R_{12}^{Y}}{(R_{11}^{\Sigma}+R_{22}^{\Sigma})^2}\left(\frac{[D^{\Sigma}(W)]^2}{[D^{\Sigma\prime}(M^{\Sigma})]^2(W-M^{\Sigma})}\right)'\Bigg|_{W=2M^{\Sigma}-M^{Y}},\\
A_{1x}^{Y\Sigma}&=+\frac{2}{3}\frac{R_{11}^{Y}R_{11}^{\Sigma}-R_{12}^{Y}R_{12}^{\Sigma}}{(R_{11}^{Y}+R_{22}^{Y})^2}\left(\frac{[D^{Y}(W)]^2}{[D^{Y\prime}(M^{Y})]^2(W-M^{Y})}\right)'\Bigg|_{W=2M^{\Sigma}-M^{\Sigma}},\\
A_{1x}^{YY}&=+\frac{1}{3}\frac{(R_{11}^{Y})^2-(R_{12}^{Y})^2}{(R_{11}^{Y}+R_{22}^{Y})^2}\left(\frac{[D^{Y}(W)]^2}{[D^{Y\prime}(M^{Y})]^2(W-M^{Y})}\right)'\Bigg|_{W=2M^{\Sigma}-M^{Y}},\\
A_{2x}^{\Sigma\Sigma}&=+\frac{1}{3}\frac{(R_{11}^{\Sigma})^2+(R_{12}^{\Sigma})^2+\frac{1}{2}(R_{22})^2}{(R_{11}^{\Sigma}+R_{22}^{\Sigma})^2}\left(\frac{[D^{\Sigma}(W)]^2}{[D^{\Sigma\prime}(M^{\Sigma})]^2(W-M^{\Sigma})}\right)'\Bigg|_{W=2M^{\Sigma}-M^{\Sigma}},\\
A_{2x}^{\Sigma Y}&=-\frac{4}{3}\frac{R_{11}^{\Sigma}R_{11}^{Y}+R_{12}^{\Sigma}R_{12}^{Y}+\frac{1}{2}R_{22}^{\Sigma}R_{22}^{Y}}{(R_{11}^{\Sigma}+R_{22}^{\Sigma})^2}\left(\frac{[D^{\Sigma}(W)]^2}{[D^{\Sigma\prime}(M^{\Sigma})]^2(W-M^{\Sigma})}\right)'\Bigg|_{W=2M^{\Sigma}-M^{Y}},\\
A_{2x}^{Y\Sigma}&=-\frac{2}{3}\frac{R_{11}^{Y}R_{11}^{\Sigma}+R_{12}^{Y}R_{12}^{\Sigma}+\frac{1}{2}R_{22}^{Y}R_{22}^{\Sigma}}{(R_{11}^{Y}+R_{22}^{Y})^2}\left(\frac{[D^{Y}(W)]^2}{[D^{Y\prime}(M^{Y})]^2(W-M^{Y})}\right)'\Bigg|_{W=2M^{\Sigma}-M^{\Sigma}},\\
A_{2x}^{YY}&=-\frac{1}{3}\frac{(R_{11}^{Y})^2+(R_{12}^{Y})^2+\frac{1}{2}(R_{22}^{Y})^2}{(R_{11}^{Y}+R_{22}^{Y})^2}\left(\frac{[D^{Y}(W)]^2}{[D^{Y\prime}(M^{Y})]^2(W-M^{Y})}\right)'\Bigg|_{W=2M^{\Sigma}-M^{Y}}.
\end{aligned} \tag{28}$$

We now turn to the calculation of $A_{1e}^{\Sigma\Sigma}$, $A_{1e}^{Y\Sigma}$, $A_{2e}^{\Sigma\Sigma}$, and $A_{2e}^{Y\Sigma}$. In a calculation of perturbations to a single-channel bootstrap, one can use the invariance of the bootstrap solution under over-all scaling of all the masses in the problem to determine the A_e's in terms of the A_x's.[33] However, in a multichannel calculation the scaling invariance of the bootstrap solution alone is not sufficient to determine the A_e's, and one needs more information about the dynamics in the various channels.

Following our previous approximations, we shall obtain this dynamical information with which to calculate the A_e's by approximating the left-hand cut of the partial-wave amplitude by a sum of Born pseudopoles. Specifically, we shall take the static approximation Born terms due to the u-channel exchange of the Λ, Σ, and Y_1^*. The corresponding residue matrices are given in Appendix B. On the left-hand cut we then have

$$(\mathbf{T})_{ij}=\sum_{p=\Lambda,\Sigma,Y_1^*}\frac{[\mathbf{B}(p)]_{ij}}{W-M_i-M_j+M^p}, \tag{29}$$

where M_i and M_j are the masses of the external baryons in channels i and j.[34]

[32] For details of this simplification, see Ref. 4.

[33] For general discussions of the use of the bootstrap scaling invariance, see Refs. 18 and 28. The particular application to the octet and decuplet bootstrap is found in Ref. 4.

[34] For the purpose of explicitly exhibiting the dependence of the pseudopole position on the external masses, we have written the pole in $[\mathbf{T}(W)]_{ij}$ as being at $W=M_i+M_j-M^p$. After carrying out the differentiation, we again set $M_i=M_j=M^{\Sigma}$.

A shift in external masses changes $\mathbf{T}(W)$ on both the left- and right-hand cuts. In particular, when the external Σ mass changes, we have to first order on the left-hand cut,

$$(\delta\mathbf{T})_{ij}=\frac{\partial(\mathbf{T})_{ij}}{\partial M^{\Sigma\ \mathrm{ext}}}\delta M^{\Sigma\ \mathrm{ext}}=\frac{\partial(\mathbf{T})_{ij}}{\partial M_2}\delta M^{\Sigma\ \mathrm{ext}}=\sum_p\frac{[\mathbf{B}(p)]_{ij}(\delta_{2i}+\delta_{2j})}{(W-M_i-M_j+M^p)^2}\delta M^{\Sigma\ \mathrm{ext}}. \tag{30}$$

We are now in position to calculate the effect of the shift in $\mathbf{T}$ on the left-hand cut due to a change in external Σ masses. Using Eqs. (27), we find

$$\delta M^{\Sigma^+,\Sigma^0,\Sigma^-}=\sum_p\frac{-\mathrm{Tr}\left[\mathbf{R}^\Sigma\mathbf{B}^\Sigma(p)\begin{pmatrix}0&0\\0&2\end{pmatrix}\right]}{\mathrm{Tr}[\mathbf{R}^\Sigma\mathbf{R}^\Sigma]}\left(\frac{[D^\Sigma(W)]^2}{[D^{\Sigma\prime}(M^\Sigma)]^2(W-M^\Sigma)}\right)'\Bigg|_{W=2M^\Sigma-M^p}\delta M^{\Sigma\ \mathrm{ext}}(I_z=+1,0,-1),$$

$$\delta M^{Y^+,Y^0,Y^-}=\sum_p\frac{-\mathrm{Tr}\left[\mathbf{R}^Y\mathbf{B}^Y(p)\begin{pmatrix}0&0\\0&2\end{pmatrix}\right]}{\mathrm{Tr}[\mathbf{R}\mathbf{R}]}\left(\frac{[D^Y(W)]^2}{[D^{Y\prime}(M^Y)]^2(W-M^Y)}\right)'\Bigg|_{W=2M^\Sigma-M^p}\delta M^{\Sigma\ \mathrm{ext}}(I_z=+1,0,-1), \tag{31}$$

where $\delta M^{\Sigma\ \mathrm{ext}}(I_z=+1,0,-1)$ is the change in the external Σ masses in the $I=1$, $I_z=+1$, 0, and -1 $\pi\Sigma$ channels, respectively.

There is also a contribution to the dispersion integrals in Eq. (27) from the change in the external masses coming from the right-hand cut of $\delta T(W)$ of the form[18]

$$\delta M^\Sigma=\mathrm{Tr}\left[\mathbf{R}^\Sigma(\mathbf{\Delta}^\Sigma)^T\frac{1}{\pi}\int_{\mathrm{RHC}}dW'\frac{(\mathbf{N}^\Sigma)^T\delta\mathbf{p}\,\mathbf{N}^\Sigma}{W'-M^\Sigma}\mathbf{\Delta}^\Sigma\right]\Big/\mathrm{Tr}[\mathbf{R}^\Sigma\mathbf{R}^\Sigma],$$

$$\delta M^Y=\mathrm{Tr}\left[\mathbf{R}^Y(\mathbf{\Delta}^Y)^T\frac{1}{\pi}\int_{\mathrm{RHC}}dW'\frac{(\mathbf{N}^Y)^T\delta\mathbf{p}\,\mathbf{N}^Y}{W'-M^Y}\mathbf{\Delta}^Y\right]\Big/\mathrm{Tr}[\mathbf{R}^Y\mathbf{R}^Y]. \tag{32}$$

However, detailed numerical calculation of this contribution to δM using a left-hand cut approximated by pseudopoles yields the result that the right-hand-cut contribution of Eq. (32) is negligible compared with the contribution from the right-hand cut given in Eq. (31).[35]

Noting from the $I=1$ $\pi\Sigma$ states defined in Eq. (13) that

$$\delta M^{\Sigma\ \mathrm{ext}}(I_z=+1)=\tfrac{1}{2}\delta M^{\Sigma^+}+\tfrac{1}{2}\delta M^{\Sigma^0},\quad \delta M^{\Sigma\ \mathrm{ext}}(I_z=0)=\tfrac{1}{2}\delta M^{\Sigma^+}+\tfrac{1}{2}\delta M^{\Sigma^-},\quad \delta M^{\Sigma\ \mathrm{ext}}(I_z=-1)=\tfrac{1}{2}\delta M^{\Sigma^-}+\tfrac{1}{2}\delta M^{\Sigma^0}, \tag{33}$$

we find from taking the contribution from changes in the left-hand cut given in Eq. (31) that the resulting A_e matrix elements can be expressed as

$$A_{1e}{}^{\Sigma\Sigma}=\tfrac{1}{2}\sum_p\frac{-\mathrm{Tr}\left[\mathbf{R}^\Sigma\mathbf{B}^\Sigma(p)\begin{pmatrix}0&0\\0&2\end{pmatrix}\right]}{\mathrm{Tr}[\mathbf{R}^\Sigma\mathbf{R}^\Sigma]}\left(\frac{[D^\Sigma(W)]^2}{[D^{\Sigma\prime}(M^\Sigma)]^2(W-M^\Sigma)}\right)'\Bigg|_{W=2M^\Sigma-M^p},$$

$$A_{1e}{}^{Y\Sigma}=\tfrac{1}{2}\sum_p\frac{-\mathrm{Tr}\left[R^YB^Y(p)\begin{pmatrix}0&0\\0&2\end{pmatrix}\right]}{\mathrm{Tr}[\mathbf{R}^Y\mathbf{R}^Y]}\left(\frac{[D^Y(W)]^2}{[D^{Y\prime}(M^Y)]^2(W-M^Y)}\right)'\Bigg|_{W=2M^\Sigma-M^p},$$

$$A_{2e}{}^{\Sigma\Sigma}=-A_{1e}{}^{\Sigma\Sigma},\quad A_{2e}{}^{Y\Sigma}=-A_{1e}{}^{Y\Sigma}. \tag{34}$$

In a few cases where we can compare the results of a calculation of the A_e's by the method given here with those obtained by using the scaling invariance of the bootstrap equations, such as in the N-N^* reciprocal bootstrap, we find that the A_e's from the two different methods are in fairly close agreement.

Now that we have explicit expressions for the elements of the A matrix in terms of $\mathbf{R}^\Sigma$ and $\mathbf{R}^Y$, we are in a position to use a combination of theory and experiment to numerically evaluate $\mathbf{R}^\Sigma$ and $\mathbf{R}^Y$, and

[35] F. J. Gilman, Ph.D. thesis, Princeton University, 1965 (unpublished).

thence $\mathbf{A}_1$ and $\mathbf{A}_2$. Direct experimental evidence for the numerical values of the elements of $\mathbf{R}^\Sigma$ is, however, rather poor. There is some evidence from the study of hypernuclei that $g_{\pi\Lambda\Sigma}^2/4\pi \approx 10$,[36] but this result is inconclusive owing to uncertainties in the theoretical analysis of the experimental data. We shall instead use the values of $g_{\pi\Lambda\Sigma}$, $g_{\pi\Sigma\Sigma}$, and $g_{KN\Sigma}$ given by unitary symmetry (see Appendix A):

$$g_{\pi\Lambda\Sigma} = (2/\sqrt{3})[D/(F+D)]g_{\pi NN},$$
$$g_{\pi\Sigma\Sigma} = [2F/(F+D)]g_{\pi NN},$$
$$g_{KN\Sigma} = [(D-F)/(F+D)]g_{\pi NN},$$

where F/D is the ratio of F to D type pseudoscalar meson-baryon coupling in unitary symmetry.[37] The value of $g_{\pi\Lambda\Sigma}^2$ given above from studies of hypernuclei would indicate that $F/D \approx \frac{1}{3}$. Bootstrap calculations of the baryon octet and decuplet also indicate a value of F/D of $\frac{1}{3}$ to $\frac{1}{2}$.[38] Perhaps the best evidence on the F/D ratio follows from the Cabibbo theory of weak leptonic decays[39] together with the Goldberger-Treiman relation.[40] Again the values of F/D obtained are in the range $\frac{1}{3}$ to $\frac{1}{2}$.[41] More recently, there has been much theoretical interest in the strong interaction symmetry $SU(6)$ and its variants, which predicts $F/D = \frac{2}{3}$.[42] Note that for F/D in the range $\frac{1}{3}$ to $\frac{2}{3}$ we have that $R_{12}^\Sigma = \sqrt{2}g_{\pi\Lambda\Sigma}g_{\pi\Sigma\Sigma}$ is positive and that $g_{KN\Sigma}^2$ is small compared to $g_{\pi\Lambda\Sigma}^2$.[43] It is for this reason together with its higher threshold that we neglect the $\bar{K}N$ channel in the Σ bootstrap. In what follows, we shall vary F/D between $\frac{1}{3}$ and $\frac{2}{3}$ and observe the effect on the A matrix.

TABLE I. Values of the A matrix elements, $\det(1-\mathbf{A}_1)$, and $\det(1-\mathbf{A}_2)$ for $\Gamma(Y_1^*\to\pi\Lambda)=50$ MeV, $\Gamma(Y_1^*\to\pi\Sigma)/\Gamma(Y_1^*\to\pi\Lambda)=9\%$, $W_1=W_2=M^\Sigma+3M^Y$, and $F/D=\frac{1}{3}$, $\frac{1}{2}$, and $\frac{2}{3}$.

$F/D=$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$
$R_{11}^\Sigma = -g_{\pi\Lambda\Sigma}^2/4\pi =$	−11.1	−8.8	−7.1
$R_{12}^\Sigma = \sqrt{2}g_{\pi\Lambda\Sigma}g_{\pi\Sigma\Sigma}/4\pi =$	+9.0	+10.8	+11.6
$R_{22}^\Sigma = -2g_{\pi\Sigma\Sigma}^2/4\pi =$	−7.4	−13.2	−19.0
$R_{11}^Y =$	−17.2	−17.2	−17.2
$R_{12}^Y =$	−12.1	−12.1	−12.1
$R_{22}^Y =$	−8.5	−8.5	−8.5
$A_{1x}^{\Sigma\Sigma} =$	−0.04	+0.03	+0.04
$A_{1x}^{\Sigma Y} =$	+0.99	+0.66	+0.43
$A_{1x}^{Y\Sigma} =$	+0.25	+0.23	+0.22
$A_{1x}^{YY} =$	+0.06	+0.06	+0.06
$A_{1e}^{\Sigma\Sigma} =$	+0.59	+0.53	+0.41
$A_{1e}^{Y\Sigma} =$	+0.00	+0.05	+0.10
$\det(1-\mathbf{A}_1) =$	+0.17	+0.23	+0.38
$A_{2x}^{\Sigma\Sigma} =$	+0.22	+0.19	+0.18
$A_{2x}^{\Sigma Y} =$	−0.37	−0.18	−0.10
$A_{2x}^{Y\Sigma} =$	−0.09	−0.07	−0.05
$A_{2x}^{YY} =$	−0.15	−0.15	−0.15
$A_{2e}^{\Sigma\Sigma} =$	−0.59	−0.53	−0.41
$A_{2e}^{Y\Sigma} =$	−0.00	−0.05	−0.10
$\det(1-\mathbf{A}_2) =$	+1.53	+1.52	+1.39

TABLE II. Values of the A matrix elements, $\det(1-\mathbf{A}_1)$, and $\det(1-\mathbf{A}_2)$ for $\Gamma(Y_1^*\to\pi\Lambda)=50$ MeV, $\Gamma(Y_1^*\to\pi\Sigma)/\Gamma(Y_1^*\to\pi\Lambda)=9\%$, $F/D=\frac{1}{2}$, and $W_1=W_2=M^\Sigma+M^Y$, $M^\Sigma+3M^Y$, and infinity.

$W_1=W_2=$	$M^\Sigma+M^Y$	$M^\Sigma+3M^Y$	∞
$R_{11}^\Sigma =$	−8.8	−8.8	−8.8
$R_{12}^\Sigma =$	+10.8	+10.8	+10.8
$R_{22}^\Sigma =$	−13.2	−13.2	−13.2
$R_{11}^Y =$	−17.2	−17.2	−17.2
$R_{12}^Y =$	−12.1	−12.1	−12.1
$R_{22}^Y =$	−8.5	−8.5	−8.5
$A_{1x}^{\Sigma\Sigma} =$	+0.03	+0.03	+0.03
$A_{1x}^{\Sigma Y} =$	+0.46	+0.66	+0.78
$A_{1x}^{Y\Sigma} =$	+0.15	+0.23	+0.28
$A_{1x}^{YY} =$	+0.02	+0.06	+0.08
$A_{1e}^{\Sigma\Sigma} =$	+0.41	+0.53	+0.62
$A_{1e}^{Y\Sigma} =$	+0.04	+0.05	+0.06
$\det(1-\mathbf{A}_1) =$	+0.46	+0.23	+0.06
$A_{2x}^{\Sigma\Sigma} =$	+0.19	+0.19	+0.19
$A_{2x}^{\Sigma Y} =$	−0.12	−0.18	−0.21
$A_{2x}^{Y\Sigma} =$	−0.04	−0.07	−0.08
$A_{2x}^{YY} =$	−0.07	−0.15	−0.22
$A_{2e}^{\Sigma\Sigma} =$	−0.41	−0.53	−0.62
$A_{2e}^{Y\Sigma} =$	−0.04	−0.05	−0.06
$\det(1-\mathbf{A}_2) =$	+1.29	+1.52	+1.71

In contrast to $\mathbf{R}^\Sigma$, we have relatively good experimental results from which to compute $\mathbf{R}^Y$. From $\Gamma(Y_1^* \to \pi\Lambda) = 51.4\pm4$ MeV,[20] and Eq. (17), we calculate $R_{11}^Y = -17.7\pm1.4$. Values for $\Gamma(Y_1^* \to \pi\Sigma)$ are somewhat more uncertain. Early measurements gave $\Gamma(Y_1^* \to \pi\Sigma)/\Gamma(Y_1^* \to \pi\Lambda) = 2$ to 4%,[44] in disagreement with the predictions of (unbroken) $SU(3)$: $\Gamma(Y_1^* \to \pi\Sigma)/\Gamma(Y_1^* \to \pi\Lambda) = 16\%$.[45] In a more recent experiment, Huwe[23] obtains $\Gamma(Y_1^* \to \pi\Sigma)/\Gamma(Y_1^* \to \pi\Lambda) = 9\pm4\%$, which with Eq. (17) yields $R_{22}^Y = -8.5\pm3.8$. $SU(3)$ gives the sign of R_{12}^Y as negative.

In Table I, the values of the A matrix elements for F/D ratios of $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{2}{3}$ are given with $R_{11}^Y = -17.2$ and $R_{22}^Y = -8.5$ corresponding to $\Gamma(Y_1^* \to \pi\Lambda) = 50$ MeV and $\Gamma(Y_1^* \to \pi\Sigma)/\Gamma(Y_1^* \to \pi\Lambda) = 9\%$. We have used $W_1 = W_2 = M^\Sigma + 3M^Y$, which gives D functions with moderate curvature and which go to a constant at infinity. In order to test the sensitivity of our calculation to varying W_1 and W_2, we have in addition calculated in Table II, for a fixed F/D ratio of $\frac{1}{2}$, the two extreme cases of using D functions with no curvature at

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TABLE III. Values of the A matrix elements, $\det(\mathbf{1}-\mathbf{A}_1)$, and $\det(\mathbf{1}-\mathbf{A}_2)$ for $\Gamma(Y_1^*\to\pi\Lambda)=50$ MeV, $F/D=\frac{1}{2}$, $W_1=W_2=M^\Sigma+3M^Y$, and $\Gamma(Y_1^*\to\pi\Sigma)/\Gamma(Y_1^*\to\pi\Lambda)=4$ and 9%.

$(\Gamma Y_1^*\to\pi\Sigma)/\Gamma(Y_1^*\to\pi\Lambda)=$	4%	9%
$R_{11}^\Sigma=$	−8.8	−8.8
$R_{12}^\Sigma=$	+10.8	+10.8
$R_{22}^\Sigma=$	−13.2	−13.2
$R_{11}^Y=$	−17.2	−17.2
$R_{12}^Y=$	−8.1	−12.1
$R_{22}^Y=$	−3.8	−8.5
$A_{1x}^{\Sigma\Sigma}=$	+0.03	+0.03
$A_{1x}^{\Sigma Y}=$	+0.55	+0.66
$A_{1x}^{Y\Sigma}=$	+0.30	+0.23
$A_{1x}^{YY}=$	+0.12	+0.06
$A_{1e}^{\Sigma\Sigma}=$	+0.41	+0.53
$A_{1e}^{Y\Sigma}=$	+0.07	+0.05
$\det(\mathbf{1}-\mathbf{A}_1)=$	+0.28	+0.23
$A_{2x}^{\Sigma\Sigma}=$	+0.19	+0.19
$A_{2x}^{\Sigma Y}=$	−0.20	−0.18
$A_{2x}^{Y\Sigma}=$	−0.11	−0.07
$A_{2x}^{YY}=$	−0.19	−0.15
$A_{2e}^{\Sigma\Sigma}=$	−0.41	−0.53
$A_{2e}^{Y\Sigma}=$	−0.07	−0.05
$\det(\mathbf{1}-\mathbf{A}_2)=$	1.41	1.52

all, corresponding to $W_1=W_2=\infty$, or strongly curved D functions, corresponding to $W_1=W_2=M^\Sigma+M^Y$. To check the sensitivity to the value of $\Gamma(Y_1^*\to\pi\Sigma)/\Gamma(Y_1^*\to\pi\Lambda)$, we have calculated in Table III the A matrix for $F/D=\frac{1}{2}$ and $\Gamma(Y_1^*\to\pi\Sigma)/\Gamma(Y_1^*\to\pi\Lambda)=4\%$ and 9%, corresponding to the earlier and later experiments for this branching ratio.

The first thing to be noticed about Tables I, II, and III is that in all cases[46] the quantity $|\mathbf{1}-\mathbf{A}_1|$ is small. i.e., typically in the range 0.1 to 0.4, whereas $|\mathbf{1}-\mathbf{A}_2|$ is typically in the range 1.3 to 1.7. Recalling that these quantities enter the expressions for the corresponding mass differences in the denominator, we see that the $\Delta I=1$ mass differences will be enhanced over the $\Delta I=2$ mass differences by a factor of three or more if we assume that the driving terms for the two cases are of approximately the same magnitude. Secondly, explicit calculation of the enhanced eigenvector gives its direction as $\approx -0.4\delta M_1^\Sigma+\delta M_1^Y=0$ rather independently of the particular values of the parameters chosen. We then expect $\delta M_1^Y\approx 0.4\delta M_1^\Sigma$ to within about 20%. Finally, since we know from other calculations that typical driving terms are 1 to 3 MeV in magnitude,[9,35] we roughly expect $|\delta M_1^\Sigma|\approx 5$ to 20 MeV, $|\delta M_1^Y|\approx 2$ to 8 MeV, while without enhancement $|\delta M_2^\Sigma|$ and $|\delta M_2^Y|$ are of the same magnitude as the driving terms, 1 to 3 MeV.

These theoretical results are in rather good agreement with experiment. One set of experimental values for the Σ mass differences[47] is $\delta M_1^\Sigma=M^{\Sigma^+}-M^{\Sigma^-}=-7.6\pm0.4$ MeV and $\delta M_2^\Sigma=M^{\Sigma^+}+M^{\Sigma^-}-2M^{\Sigma^0}=1.9\pm0.4$ MeV. More recent values are $\delta M_1^\Sigma=-7.89\pm0.12$ and $\delta M_2^\Sigma=2.09\pm0.24$.[48] If we assume approximately equal driving terms, then we may take $(\delta M_1^\Sigma/\delta M_2^\Sigma)_{\rm exp}$ as an experimental measure of the enhancement of the $\Delta I=1$ mass differences over the $\Delta I=2$ mass differences. For either of the experimental results above, we then have an "experimentally measured" enhancement of about a factor of 4 to 5, while the theoretical prediction is an enhancement by a factor of 3 or more.

Experimental results for the Y_1^* electromagnetic mass differences are somewhat unsettled at present. One experiment[23] gives $\delta M_1^Y=M^{Y^+}-M^{Y^-}=-17\pm7$ MeV, while others[49] give -4.3 ± 2.2 and -2.0 ± 1.5 MeV. Taking $\delta M_1^\Sigma\approx-8$ MeV from experiment, the direction of the enhanced eigenvector would predict $\delta M_1^Y\approx-3$ MeV, agreeing in sign with all the experiments and in fairly good quantitative agreement with the latter ones.

Looking at some of the finer details of the calculation, we see from Table I that smaller F/D ratios, which are favored by reciprocal bootstrap calculations of the octet and decuplet, result in small values of $|\mathbf{1}-\mathbf{A}_1|$, and thus more enhancement of the $\Delta I=1$ mass differences. Similarly, from Table II we see that curved D functions lead to less enhancement. This could have been easily predicted from Eqs. (28) and (34) where each A matrix element is multiplied by a factor which is one for a linear D function and decreases as D acquires curvature.

Although Table III would seem to indicate that the calculation of the A matrix is not sensitive to changing $\Gamma(Y_1^*\to\pi\Sigma)/\Gamma(Y_1^*\to\pi\Lambda)$, it should be emphasized that the $\pi\Sigma$ channel is important, especially to the strong-interaction Σ bootstrap. With only a $\pi\Lambda$ channel for both the Σ and Y_1^*, we would have found that $|\mathbf{1}-\mathbf{A}_1|=|\mathbf{1}-\mathbf{A}_2|$ so that there would be no enhancement of the mass differences transforming as $\Delta I=1$ over those transforming as $\Delta I=2$.

Now let us note the effect of additional channels which have been neglected. The most important of these is likely to be the $\bar{K}N$ channel for the Y_1^*. It is clear that in our calculation the effect of such an additional channel will be to decrease the enhancement, for neither the Σ nor Y_1^* can be an exchanged or external particle in the $\bar{K}N$ channel, so that changes in the Σ and Y_1^* masses do not affect the $\bar{K}N$ component of the Y_1^*. This can be seen more directly from the expressions for the A matrix elements in Eq. (28). The addition of the $\bar{K}N$ channel will not change the numerators of the expressions for the A matrix elements, but the sum of the squares of the coupling constants to the various channels which appears in the denominator will increase.

[46] A strong enhancement of the $\Delta I=1$ mass differences is also found using the broken $SU(3)$ values for $g_{\pi\Lambda\Sigma}$ and $g_{\pi\Sigma\Sigma}$ given in Ref. 6.

[47] H. C. Dosch *et al.*, Phys. Letters **14**, 239 (1965).

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To check on the effect of the $\bar{K}N$ channel, let us assume for the moment that the residue at the Y_1^* pole in the $J^P=(\frac{3}{2})^+$ partial wave is the same for the $\bar{K}N$ and $\pi\Sigma$ channels, as it is in unbroken $SU(3)$.[45] If we take $F/D=\frac{1}{2}$, $\Gamma(Y_1^*\to\pi\Sigma)/\Gamma(Y_1^*\to\pi\Lambda)=9\%$, and D functions with $W_1=W_2=M^\Sigma+3M^Y$, we find that $|\mathbf{1}-\mathbf{A}_1|=0.43$ with the $\bar{K}N$ channel present, as contrasted with $|\mathbf{1}-\mathbf{A}_1|=0.23$ with the $\bar{K}N$ channel absent. There is thus still a strong enhancement effect, and in actuality we expect the $\bar{K}N$ channel to have less effect than even this.

Since it is the reciprocals of $|\mathbf{1}-\mathbf{A}_1|$ and $|\mathbf{1}-\mathbf{A}_2|$ which enter the expression for the mass differences, a calculation of the $\Delta I=1$ mass differences will necessarily be rather sensitive to small changes in the small quantity $|\mathbf{1}-\mathbf{A}_1|$, but not to small changes in $|\mathbf{1}-\mathbf{A}_2|$, which is of order one. A better determination of $\mathbf{A}_1$ and $|\mathbf{1}-\mathbf{A}_1|$ in order to eliminate the uncertainties in our A matrix calculation requires much better bootstrap calculations to start with than have so far been performed.

IV. THE Ξ AND Ξ* SYSTEM

In this section we attempt a calculation of the self-consistent terms contributing to the electromagnetic mass differences of the Ξ^0-Ξ^- and Ξ^{*0}-Ξ^{*-} baryons in a manner very similar to that of the previous section. We consider the Ξ and Ξ^* to be strong interaction bound states and resonances in pseudoscalar meson-baryon scattering amplitudes. Both the Ξ and Ξ^* have been obtained as bound states (resonances) in bootstrap calculations of the whole baryon octet and decuplet.[38,50] However, detailed bootstrap calculations of the Ξ and Ξ^* which take into account the breaking of $SU(3)$ symmetry are in a much poorer state than those for the N and N^* or Σ and Y_1^*. The work of Dashen *et al.*[6] on coupling constants in broken $SU(3)$ indicates that one must take account of at least the two-body $\pi\Xi$ and $\bar{K}\Sigma$ channels for a Ξ bootstrap, and the $\pi\Xi$, $\bar{K}\Lambda$, and $\bar{K}\Sigma$ channels for a Ξ^* bootstrap in broken $SU(3)$. In our calculation, we shall rely heavily on the $SU(3)$ symmetric bootstraps of the octet and decuplet, but shall break the $SU(3)$ by noting the changes in masses and coupling constants of the baryons and mesons and in particular by nelgecting the $\eta\Xi$ channel present in the $SU(3)$ symmetric bootstrap of the Ξ and Ξ^*. We then consider the Ξ and Ξ^* to be bound states or resonances in the three pseudoscalar meson-baryon channels with the lowest thresholds, namely, the $\pi\Xi$ channel (threshold 1460 MeV), the $\bar{K}\Lambda$ channel (threshold 1610 MeV), and the $\bar{K}\Sigma$ channel (threshold 1690 MeV). We shall thus assume that the Ξ can be obtained as a $J^P=(\frac{1}{2})^+$, $I=\frac{1}{2}$, $\pi\Xi$, $\bar{K}\Lambda$, and $\bar{K}\Sigma$ bound state with a mass of 1320 MeV, and similarly the Ξ^* as a $J^P=(\frac{3}{2})^+$, $I=\frac{1}{2}$, $\pi\Xi$, $\bar{K}\Lambda$, and $\bar{K}\Sigma$ resonance with a mass of 1530 MeV.

We now proceed to calculate the contribution of self-consistent terms to the Ξ and Ξ^* mass differences arising from the presence of the Ξ and Ξ^* in the reciprocal bootstrap. It should be noticed from the outset that here we are not dealing with a closed Ξ-Ξ^* reciprocal bootstrap subsystem as was the case for the Σ-Y_1^* system. In fact, the most important $\pi\Xi$ channel is $\bar{K}\Sigma$ in which the Ξ and Ξ^* are neither exchanged nor external particles. First, however, let us address ourselves to the question of whether the Ξ and Ξ^* self-consistent terms are sufficient *alone* to provide an enhancement of the Ξ and Ξ^* electromagnetic mass differences.

The partial-wave-scattering matrices we deal with in this section will then be 3×3, with channel one being $\pi\Xi$ scattering, channel two $\bar{K}\Lambda$ scattering, and channel three $\bar{K}\Sigma$ scattering. The Λ is an isotopic spin singlet and the π^+, π^0, π^- and Σ^+, Σ^0, Σ^- make up isospin triplets which we define as in Sec. III. The Ξ, Ξ^*, and $\bar{K}$ are all isodoublets. Writing them in the form of isospinors, we define

$$\Xi=\begin{pmatrix}\Xi^0\\ \Xi^-\end{pmatrix},\quad \Xi^*=\begin{pmatrix}\Xi^{*0}\\ \Xi^{*-}\end{pmatrix},\quad K^c=\begin{pmatrix}-\bar{K}^0\\ \bar{K}^-\end{pmatrix}. \tag{35}$$

With the above definitions, we label the partial-wave amplitudes by the bound state which occurs in that partial wave, e.g., the unperturbed (by electromagnetism) partial-wave amplitudes in the $J^P=(\frac{1}{2})^+$, $I=\frac{1}{2}$, $I_z=+\frac{1}{2}$ and $-\frac{1}{2}$ channels are denoted by $\mathbf{T}^{\Xi^0}(W)$ and $\mathbf{T}^{\Xi^-}(W)$. In the absence of electromagnetism, we assume that both $\mathbf{T}^{\Xi^0}(W)$ and $\mathbf{T}^{\Xi^-}(W)$ have poles at the unperturbed Ξ mass, M^Ξ,

$$\mathbf{T}^\Xi(W)=\mathbf{R}^\Xi/(W-M^\Xi) \tag{36}$$

with R^Ξ related to the coupling constants defined in Appendix A by

$$\mathbf{R}^\Xi=\frac{1}{4\pi}\begin{bmatrix}-3g_{\pi\Xi\Xi}^2 & +\sqrt{3}g_{\pi\Xi\Xi}g_{K\Lambda\Xi} & -3g_{\pi\Xi\Xi}g_{K\Sigma\Xi}\\ +\sqrt{3}g_{\pi\Xi\Xi}g_{K\Lambda\Xi} & -g_{K\Lambda\Xi}^2 & +\sqrt{3}g_{K\Lambda\Xi}g_{K\Sigma\Xi}\\ -3g_{\pi\Xi\Xi}g_{K\Sigma\Xi} & +\sqrt{3}g_{K\Lambda\Xi}g_{K\Sigma\Xi} & -3g_{K\Sigma\Xi}^2\end{bmatrix}. \tag{37}$$

Similarly, in the $J^P=(\frac{3}{2})^+$, $I=\frac{1}{2}$, $I_z=+\frac{1}{2}$ and $-\frac{1}{2}$ channels, the corresponding partial-wave amplitudes, which are denoted by $\mathbf{T}^{\Xi^{*0}}(W)$ and $\mathbf{T}^{\Xi^{*-}}(W)$ have poles at the unperturbed Ξ^* mass, M^{Ξ^*}, of the form

$$\mathbf{T}^{\Xi^*}(W)=\frac{\mathbf{R}^{\Xi^*}}{W-M^{\Xi^*}}=\frac{1}{W-M^{\Xi^*}}\begin{bmatrix}R_{11}^{\Xi^*} & R_{12}^{\Xi^*} & R_{13}^{\Xi^*}\\ R_{21}^{\Xi^*} & R_{22}^{\Xi^*} & R_{23}^{\Xi^*}\\ R_{31}^{\Xi^*} & R_{32}^{\Xi^*} & R_{33}^{\Xi^*}\end{bmatrix}. \tag{38}$$

The Ξ^* can only decay strongly into $\pi\Xi$. The width for $\Xi^*\to\pi\Xi$ is related to $R_{11}^{\Xi^*}$ by[25]

$$\Gamma(\Xi^*\to\pi\Xi)=-2\rho_{11}(M^{\Xi^*})R_{11}^{\Xi^*}. \tag{39}$$

[50] Y. Hara, Phys. Rev. 135, B1079 (1964).

The magnitude of off-diagonal residues is fixed by $|R_{ij}|=(R_{ii}R_{jj})^{1/2}$, and the relative signs of the Ξ and Ξ^* coupling constants needed to determine the signs of off-diagonal residues will be taken from $SU(3)$ predictions.

When the electromagnetic interaction is turned on, $\mathbf{T}^{\Xi^*}(W)$ and $\mathbf{T}^{\Xi^-}(W)$ will no longer have a common pole at $W=M^{\Xi}$, but will have poles at the Ξ^0 and Ξ^- masses, M^{Ξ^0} and M^{Ξ^-}, respectively. We write $\delta M^{\Xi^0}=M^{\Xi^0}-M^{\Xi}$ and $\delta M^{\Xi^-}=M^{\Xi^-}-M^{\Xi}$ as the Ξ^0 and Ξ^- mass shifts, respectively. Similarly, $\mathbf{T}^{\Xi^{*0}}(W)$ and $\mathbf{T}^{\Xi^{*-}}(W)$ will have poles at $M^{\Xi^{*0}}$ and $M^{\Xi^{*-}}$ when the electromagnetic interaction is turned on, and we define the Ξ^* mass shifts: $\delta M^{\Xi^{*0}}=M^{\Xi^{*0}}-M^{\Xi^*}$, $\delta M^{\Xi^{*-}}=M^{\Xi^{*-}}-M^{\Xi^*}$.

Since both the Ξ and Ξ^* are isodoublets, there is only one mass difference to be measured in each multiplet. We define

$$\delta M^{\Xi}=M^{\Xi^0}-M^{\Xi^-}=\delta M^{\Xi^0}-\delta M^{\Xi^-},$$
$$\delta M^{\Xi^*}=M^{\Xi^{*0}}-M^{\Xi^{*-}}=\delta M^{\Xi^{*0}}-\delta M^{\Xi^{*-}}. \quad (40)$$

If we wrote the Ξ and Ξ^* mass operators in isospin space as sums of parts transforming according to irreducible representations of the group of isospin rotations, then we would find that δM^{Ξ} and δM^{Ξ^*} are the coefficients of the $\Delta I=1$ parts of the mass operators.

As we have mass differences transforming according to only one irreducible representation of the isospin group in this case, we may define

$$\delta M^{\Xi}=A_x^{\Xi\Xi}\delta M^{\Xi}+A_x^{\Xi\Xi^*}\delta M^{\Xi^*}+A_e^{\Xi\Xi}\delta M^{\Xi}+d^{\Xi},$$
$$\delta M^{\Xi^*}=A_x^{\Xi^*\Xi}\delta M^{\Xi}+A_x^{\Xi^*\Xi^*}\delta M^{\Xi^*}+A_e^{\Xi^*\Xi}\delta M^{\Xi}+d^{\Xi^*}, \quad (41)$$

where the A's have the same meaning as in Sec. III with a suitable transposition of superscripts, and the d's again represent driving terms.[51] We then define the A matrix for the Ξ-Ξ^* subsystem as

$$\mathbf{A}=\begin{pmatrix}A^{\Xi\Xi} & A^{\Xi\Xi^*}\\ A^{\Xi^*\Xi} & A^{\Xi^*\Xi^*}\end{pmatrix}=\begin{pmatrix}A_x^{\Xi\Xi}+A_e^{\Xi\Xi} & A_x^{\Xi\Xi^*}\\ A_x^{\Xi^*\Xi}+A_e^{\Xi^*\Xi} & A_x^{\Xi^*\Xi^*}\end{pmatrix}. \quad (42)$$

As in Sec. III we shall use the approximation of keeping only the singularities of the u-channel Ξ and Ξ^* exchange amplitudes which are close to $W=M^{\Xi}$. Taking the static limit, we approximate these short cuts near $W=M^{\Xi}$ by poles. For the purpose of calculating the A matrix, we shall also neglect differences in the external baryon masses, setting them all equal to M^{Ξ}. The use of the static approximation is, of course, less justifiable here than in Sec. III since the pseudoscalar mesons involved in some channels are K mesons whose mass is an appreciable fraction of the baryon masses. However, in spite of our rather rough model, we still expect to obtain at least a good qualitative indication of the effect of the self-consistent terms on the mass differences.

We take for the $\mathbf{D}$ functions an approximation similar to that used in the previous section, namely, in a representation where $\mathbf{D}(W)$ is diagonal at the bound state or resonance mass, we assume

$$\mathbf{D}^{\Xi}(W)=\mathbf{D}^{\Xi}(M^{\Xi})+D^{\Xi}(W)\mathbf{1} \quad (43a)$$

and

$$\mathbf{D}^{\Xi^*}(W)=\mathbf{D}^{\Xi^*}(M^{\Xi^*})+D^{\Xi^*}(W)\mathbf{1} \quad (43b)$$

where $\mathbf{1}$ is the 3×3 unit matrix and

$$D^{\Xi}(W)=D^{\Xi\prime}(M^{\Xi})(W-M^{\Xi})\times(M^{\Xi}-W_1)/(W-W_1), \quad (44a)$$

$$D^{\Xi^*}(W)=D^{\Xi^*\prime}(M^{\Xi^*})(W-M^{\Xi^*})\times(M^{\Xi^*}-W_2)/(W-W_2). \quad (44b)$$

Equation (8) written for the Ξ and Ξ^* mass differences then simplifies to

$$\delta M^{\Xi^0,\Xi^-}=\frac{\mathrm{Tr}\left[\mathbf{R}^{\Xi}\frac{1}{\pi}\int_{\mathrm{cuts}}\frac{dW'}{W'-M^{\Xi}}\mathrm{Im}[(D^{\Xi})^2\delta\mathbf{T}^{\Xi^0,\Xi^-}](W')\right]}{[D^{\Xi\prime}(M^{\Xi})]^2\,\mathrm{Tr}[\mathbf{R}^{\Xi}\mathbf{R}^{\Xi}]}, \quad (45a)$$

$$\delta M^{\Xi^{*0},\Xi^{*-}}=\frac{\mathrm{Tr}\left[\mathbf{R}^{\Xi}\frac{1}{\pi}\int_{\mathrm{cuts}}\frac{dW'}{W'-M^{\Xi^*}}\mathrm{Im}[(D^{\Xi^*})^2\delta\mathbf{T}^{\Xi^{*0},\Xi^{*-}}](W')\right]}{[D^{\Xi^*\prime}(M^{\Xi^*})]^2\,\mathrm{Tr}[\mathbf{R}^{\Xi^*}\mathbf{R}^{\Xi^*}]}. \quad (45b)$$

Using the above approximations, exchange of a particle of mass M^x then leads to a pole in the partial-wave amplitude $\mathbf{T}(W)$ at $W=2M^{\Xi}-M^x$. Because of conservation of strangeness, Ξ and Ξ^* exchange can only occur in the channel $\pi\Xi\to\pi\Xi$. Using the Lagrangian density in Appendix A and the static crossing relations to compute the contribution of exchange of the various Ξ and Ξ^* charge states to $\mathbf{T}^{\Xi^0}$, $\mathbf{T}^{\Xi^-}$, $\mathbf{T}^{\Xi^{*0}}$, and $\mathbf{T}^{\Xi^{*-}}$,[35] we find for the A

[51] We have, however, put terms from shifts in the masses of barons other than the Ξ and Ξ^*, e.g., external Σ mass shifts, into the d's.

matrix elements defined in Eq. (41),

$$
\begin{aligned}
A_x^{\Xi\Xi}&=\frac{1}{3}\frac{(5/9)(R_{11}^{\Xi})^2}{\mathrm{Tr}[\mathbf{R}^{\Xi}\mathbf{R}^{\Xi}]}\left(\frac{[D^{\Xi}(W)]^2}{[D^{\Xi\prime}(M^{\Xi})]^2(W-M^{\Xi})}\right)'\Bigg|_{W=2M^{\Xi}-M^{\Xi}},\\
A_x^{\Xi\Xi^*}&=-\frac{4}{3}\frac{(5/9)R_{11}^{\Xi}R_{11}^{\Xi^*}}{\mathrm{Tr}[\mathbf{R}^{\Xi}\mathbf{R}^{\Xi}]}\left(\frac{[D^{\Xi}(W)]^2}{[D^{\Xi\prime}(M^{\Xi})]^2(W-M^{\Xi})}\right)'\Bigg|_{W=2M^{\Xi}-M^{\Xi^*}},\\
A_x^{\Xi^*\Xi}&=-\frac{2}{3}\frac{(5/9)R_{11}^{\Xi^*}R_{11}^{\Xi}}{\mathrm{Tr}[\mathbf{R}^{\Xi^*}\mathbf{R}^{\Xi^*}]}\left(\frac{[D^{\Xi^*}(W)]^2}{[D^{\Xi^*\prime}(M^{\Xi^*})]^2(W-M^{\Xi^*})}\right)'\Bigg|_{W=2M^{\Xi}-M^{\Xi}},\\
A_x^{\Xi^*\Xi^*}&=-\frac{1}{3}\frac{(5/9)(R_{11}^{\Xi^*})^2}{\mathrm{Tr}[\mathbf{R}^{\Xi^*}\mathbf{R}^{\Xi^*}]}\left(\frac{[D^{\Xi^*}(W)]^2}{[D^{\Xi^*\prime}(M^{\Xi^*})]^2(W-M^{\Xi^*})}\right)'\Bigg|_{W=2M^{\Xi}-M^{\Xi^*}}.
\end{aligned}
\tag{46}
$$

Turning to the calculation of $A_e^{\Xi\Xi}$ and $A_e^{\Xi^*\Xi}$, our method is again very similar to that in Sec. III. We first approximate $\mathbf{T}(W)$ on the left-hand cut by a sum of pseudopoles as in Eq. (29). There is then a change in $\mathbf{T}$ on both the right- and left-hand cuts due to a change in the external baryon masses, but again the contribution of the right-hand cut turns out to be negligible.[35] We then derive the analogs of Eq. (34) for the shift in the Ξ or Ξ^* masses due to a change in the external Ξ masses. These are

$$
\begin{aligned}
A_e^{\Xi\Xi}&=(-\tfrac{1}{3})\sum_p\frac{-\mathrm{Tr}\left[\mathbf{R}^{\Xi}\mathbf{B}^{\Xi}(p)\begin{pmatrix}2&0&0\\0&0&0\\0&0&0\end{pmatrix}\right]}{+\mathrm{Tr}[\mathbf{R}^{\Xi}\mathbf{R}^{\Xi}]}\left(\frac{[D^{\Xi}(W)]^2}{[D^{\Xi\prime}(M^{\Xi})]^2(W-M^{\Xi})}\right)'\Bigg|_{W=2M^{\Xi}-M^{p}},\\
A_e^{\Xi^*\Xi}&=(-\tfrac{1}{3})\sum_p\frac{-\mathrm{Tr}\left[\mathbf{R}^{\Xi^*}\mathbf{B}^{\Xi^*}(p)\begin{pmatrix}2&0&0\\0&0&0\\0&0&0\end{pmatrix}\right]}{\mathrm{Tr}[\mathbf{R}^{\Xi^*}\mathbf{R}^{\Xi^*}]}\left(\frac{[D^{\Xi^*}(W)]^2}{[D^{\Xi^*\prime}(M^{\Xi^*})]^2(W-M^{\Xi^*})}\right)'\Bigg|_{W=2M^{\Xi}-M^{p*}}.
\end{aligned}
\tag{47}
$$

The number of possible Born poles due to exchange of particles in the u channel makes the computation of the $\mathbf{B}(p)$'s and A matrix elements somewhat more complicated then in the previous section. There the Λ, Σ, and Y_1^* were the only members of the octet and decuplet which could be exchanged. Here the Ξ, Ξ^*, Λ, Σ, Y_1^*, N, and N^* may be exchanged. The various Born residue matrices are tabulated in Appendix C.

We have numerically evaluated our expressions for the A matrix elements first in Table IV using the $SU(3)$ symmetric coupling constants with $F/D=\frac{1}{2}$, $g_{\pi NN}^2/4\pi$, and $\Gamma(\Xi^*\to\pi\Xi)=7.5\pm1.7$ MeV assumed given, and secondly in Table V using the broken $SU(3)$ coupling constants of Dashen *et al.*[6] In both cases, we have used linear D functions.

There is clearly no enhancement of the mass differ-

TABLE IV. Values of the residue and A matrix elements for the Ξ and Ξ^* mass differences using a linear D function, $F/D=\frac{1}{2}$, and $SU(3)$ symmetric coupling constants.

$R_{11}^{\Xi}=-4.8$	$R_{11}^{\Xi^*}=-7.4$
$R_{22}^{\Xi}=-0.5$	$R_{22}^{\Xi^*}=-7.4$
$R_{33}^{\Xi}=-44.4$	$R_{33}^{\Xi^*}=-7.4$
$R_{12}^{\Xi}=-1.5$	$R_{12}^{\Xi^*}=-7.4$
$R_{13}^{\Xi}=-14.5$	$R_{13}^{\Xi}=+7.4$
$R_{23}^{\Xi}=-4.7$	$R_{23}^{\Xi^*}=+7.4$
$A_x^{\Xi\Xi}=+0.00$	$A_e^{\Xi\Xi}=+0.09$
$A_x^{\Xi\Xi^*}=-0.01$	$A_e^{\Xi^*\Xi}=+0.05$
$A_x^{\Xi^*\Xi}=-0.03$	
$A_x^{\Xi^*\Xi^*}=-0.02$	$\det(\mathbf{1}-\mathbf{A})=+0.93$

TABLE V. Values of the residue and A matrix elements for the Ξ and Ξ^* mass differences using a linear D function and the broken $SU(3)$ coupling constants of Dashen *et al.* (Ref. 6).

$R_{11}^{\Xi}=-3.4$	$R_{11}^{\Xi^*}=-7.4$
$R_{22}^{\Xi}=-0.1$	$R_{22}^{\Xi^*}=-11.4$
$R_{33}^{\Xi}=-9.6$	$R_{33}^{\Xi^*}=-3.7$
$R_{12}^{\Xi}=-0.6$	$R_{12}^{\Xi^*}=-9.2$
$R_{13}^{\Xi}=-5.7$	$R_{13}^{\Xi^*}=+5.2$
$R_{23}^{\Xi}=-1.0$	$R_{23}^{\Xi^*}=+6.5$
$A_x^{\Xi\Xi}=+0.02$	$A_e^{\Xi\Xi}=+0.22$
$A_x^{\Xi\Xi^*}=-0.16$	$A_e^{\Xi^*\Xi^*}=+0.01$
$A_x^{\Xi^*\Xi}=-0.05$	
$A_x^{\Xi^*\Xi^*}=-0.05$	$\det(\mathbf{1}-\mathbf{A})=+0.80$

ences due to self-consistent terms from a Ξ-Ξ^* reciprocal bootstrap. Changing the F/D ratio has only a small effect on the A matrix elements and on $\det(\mathbf{1}-\mathbf{A})$, and using a curved D function only makes the elements of the A matrix even smaller than they are in Tables IV and V.

A glance at the residue matrices $\mathbf{R}^{\Xi}$ and $\mathbf{R}^{\Xi^*}$ in Tables IV and V shows immediately why there is no enhancement. If the values of the coupling constants given there are at all correct, then the $\pi\Xi$ component of the Ξ and Ξ^* is not the largest component. For the Ξ in particular, it is the $\bar{K}\Sigma$ channel which overshadows all the rest of the two-body channels. Thus, since changes in exchanged or external Ξ or Ξ^* masses effect only the small $\pi\Xi$ component of the Ξ or Ξ^*, the self-consistent terms have only a small effect on the Ξ or Ξ^* masses. This conclusion is very general and could only be changed by a very drastic departure from the predictions of $SU(3)$ or the results of Ref. 6 in broken $SU(3)$ for the coupling constants.

However, as already noted in Sec. II, we find that the Ξ^0-Ξ^- mass difference can be quite large even though it is not enhanced by self-consistent terms from a Ξ-Ξ^* reciprocal bootstrap. This is precisely because of the large $\bar{K}\Sigma$ component of the Ξ which both makes the possible Ξ-Ξ^* self-consistent terms small, as shown above, and the possible effects of external Σ mass shifts large. Proceeding just as in the calculation above of the effects of external Ξ mass shifts, we find with our approximations that the contribution to δM^{Ξ} of external Σ mass shifts is

$$\delta M^{\Xi}=M^{\Xi^0}-M^{\Xi^-}=\sum_p \frac{-\mathrm{Tr}\left[\mathbf{R}^{\Xi}\mathbf{B}^{\Xi}(p)\begin{pmatrix}0&0&0\\0&0&0\\0&0&2\end{pmatrix}\right]}{\mathrm{Tr}[\mathbf{R}^{\Xi}\mathbf{R}^{\Xi}]}\left(\frac{[D^{\Xi}(W)]^2}{[D^{\Xi\prime}(M^{\Xi})]^2(W-M^{\Xi})}\right)'\Bigg|_{W=2M^{\Xi}-M^p} \tfrac{2}{3}(M^{\Sigma^+}-M^{\Sigma^-}). \quad (48)$$

Using the experimental value for $M^{\Sigma^+}-M^{\Sigma^-}$, linear D functions, and $SU(3)$ symmetric coupling constants, we find a contribution to $M^{\Xi^0}-M^{\Xi^-}$ of ~ -6 MeV. A moderately curved D function reduces this to about -5 MeV. Using the broken $SU(3)$ coupling constants of Dashen *et al.*,[6] slightly increases this contribution to -6 to -8 MeV, depending on the D functions used.

Stated in terms of the A matrix defined in Sec. II, we have just shown that $A^{\Xi\Sigma}\sim 1$ due to external Σ mass shifts. Calculations of the effects of other external mass shifts and of the effects of shifts in exchanged N, Λ, Σ, N^*, and Y_1^* masses give contributions to $M^{\Xi^0}-M^{\Xi^-}$ of the order of 1 MeV.[35] Thus our rough model calculations indicate that most of the observed Ξ^0-Ξ^- mass difference of -6.5 ± 1.0 MeV is explained by external Σ mass shifts in the $\bar{K}\Sigma$ channel of a Ξ bootstrap. A calculation of the Ξ^{*0}-Ξ^{*-} mass difference where the $\bar{K}\Sigma$ channel is not dominant requires a more extensive analysis of the driving terms, such as that carried out by Kumar.[52]

V. SUMMARY AND CONCLUSION

In this paper we have considered the contributions of the self-consistent terms arising from strong interaction reciprocal bootstraps to the electromagnetic mass differences of the Σ, Y_1^*, Ξ, and Ξ^* baryons. We have particularly considered the effects of the breaking of $SU(3)$ symmetry on the octet enhancement calculations of Dashen and Frautschi.[4]

In Sec. II we saw how the breaking of $SU(3)$ symmetry is reflected in an important asymmetry of the A matrix which summarizes the effects of the self-consistent terms arising from shifts in the masses of members of the octet and decuplet when acting as exchanged or external particles in the reciprocal bootstrap. This asymmetry is directly connected with the splitting off in broken $SU(3)$ of the N and N^* and Σ and Y_1^*, but not the Ξ and Ξ^*, from the rest of the octet and decuplet to form separate, closed reciprocal bootstrap subsystems.

As a particular case, we considered in Sec. III the Σ-Y_1^* reciprocal bootstrap subsystem. In contrast to the N-N^* subsystem where no enhancement is found in broken $SU(3)$,[4] we found that the self-consistent terms arising from the strong interaction reciprocal bootstrap of the Σ and Y_1^* result in an enhancement by a factor of 3 or more of the $\Delta I=1$ over the $\Delta I=2$ electromagnetic mass differences of the Σ and Y_1^*.

If we take the experimental value of $(M^{\Sigma^+}-M^{\Sigma^-})/(M^{\Sigma^+}+M^{\Sigma^-}-2M^{\Sigma^0})$ as a measure of the enhancement, then we find that the Σ mass differences transforming as $\Delta I=1$ are experimentally found to be enhanced by a factor of 4 to 5. Also, from the direction of the enhanced eigenvector, we predict $M^{Y^+}-M^{Y^-}\approx 0.4(M^{\Sigma^+}-M^{\Sigma^-})\approx -3$ MeV. It is, in fact, very difficult to explain theoretically the large magnitude of $M^{\Sigma^+}-M^{\Sigma^-}\approx -8$ MeV without some sort of dynamical enhancement mechanism, since typical driving terms are on the order of 1 or 2 MeV. We regard the enhancement prediction as a major success of the calculation, although we are unable to predict the exact value of the enhancement due to a lack of knowledge of the strong interaction physics of the Σ and Y_1^*.

In Sec. IV we attempted much the same type of calculation for the Ξ and Ξ^* electromagnetic mass differences. Here, however, we were not dealing with a

[52] A. Kumar, Phys. Rev. 140, B202 (1965).

closed subsystem split off from the rest of the octet and decuplet. The $\pi\Xi$ component of either the Ξ or Ξ^* is not the largest component, and we found that the self-consistent contributions to the mass differences arising from changes in the masses of the Ξ and Ξ^* acting as exchanged and external particles did not lead to an enhancement of the mass differences. However, we found that there is an important contribution to the Ξ mass differences from changes in the mass of external Σ's in the $\bar{K}\Sigma$ channel, so that the Ξ^0-Ξ^- mass difference follows the enhanced eigenvector of the Σ^+-Σ^- mass difference. This leads to a contribution of both the right sign and magnitude to explain the observed large Ξ^0-Ξ^- mass difference.

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APPENDIX A

The rationalized, renormalized coupling constants for the interactions of the members of the pseudoscalar meson octet with the baryon octet are defined by the interaction Lagrangian density (the space-time dependence is suppressed),

$$\mathcal{L}_I = g_{\pi NN}(\bar{N}\gamma_5\boldsymbol{\tau}N)\cdot\boldsymbol{\pi} + ig_{\pi\Lambda\Sigma}(\boldsymbol{\Sigma}\gamma_5\Lambda)\cdot\boldsymbol{\pi} + g_{\pi\Sigma\Sigma}(\boldsymbol{\Sigma}\gamma_5\times\boldsymbol{\Sigma})\cdot\boldsymbol{\pi} + ig_{KN\Sigma}(\bar{N}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\Sigma})K + ig_{\pi\Xi\Xi}(\bar{\Xi}\gamma_5\boldsymbol{\tau}\Xi)\cdot\boldsymbol{\pi}$$
$$+ ig_{K\Sigma\Xi}(\bar{\Xi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\Sigma})K^c + ig_{K\Lambda\Xi}(\bar{\Xi}\gamma_5\Lambda)K^c + ig_{KN\Lambda}(\bar{N}\gamma_5\Lambda)K + \text{H.c.}, \quad \text{(A1)}$$

where

$$N=\begin{pmatrix} p \\ n\end{pmatrix},\quad K^c=\begin{pmatrix} -\bar{K}^0 \\ K^-\end{pmatrix},\quad \text{etc.}$$

Within the eightfold way, these coupling constants are related to one another as follows:

$$\begin{aligned}
g_{\pi\Lambda\Sigma} &= \frac{2}{\sqrt{3}}\frac{D}{F+D}g_{\pi NN}, & g_{K\Lambda\Xi} &= -\frac{1}{\sqrt{3}}\frac{D-3F}{F+D}g_{\pi NN},\\
g_{\pi\Sigma\Sigma} &= \frac{2F}{F+D}g_{\pi NN}, & g_{K\Sigma\Xi} &= -g_{\pi NN},\\
g_{KN\Sigma} &= \frac{D-F}{F+D}g_{\pi NN}, & g_{KN\Lambda} &= -\frac{1}{\sqrt{3}}\frac{D+3F}{F+D}g_{\pi NN},
\end{aligned}\quad \text{(A2)}$$
$$g_{\pi\Xi\Xi} = [(F-D)/(F+D)]g_{\pi NN},$$

where $g_{\pi NN}^2/4\pi = 14.8$, and F/D is the ratio of F to D type couplings.[35,53]

APPENDIX B

The residue matrices at the pseudopoles due to Σ and Y_1^* exchange in the Σ and Y_1^* channels are given in terms of the elements of the direct channel residue matrices defined in Eqs. (15) and (16) by

$$\begin{aligned}
\mathbf{B}^\Sigma(\Lambda\text{ exch.}) &= +\frac{1}{3}\begin{pmatrix} 0 & 0 \\ 0 & -R_{11}^\Sigma\end{pmatrix}, & \mathbf{B}^Y(\Lambda\text{ exch.}) &= -\frac{2}{3}\begin{pmatrix} 0 & 0 \\ 0 & -R_{11}^\Sigma\end{pmatrix},\\
\mathbf{B}^\Sigma(\Sigma\text{ exch.}) &= +\frac{1}{3}\begin{pmatrix} R_{11}^\Sigma & -R_{12}^\Sigma \\ -R_{12}^\Sigma & \frac{1}{2}R_{22}^\Sigma\end{pmatrix}, & \mathbf{B}^Y(\Sigma\text{ exch.}) &= \frac{2}{3}\begin{pmatrix} R_{11}^\Sigma & -R_{12}^\Sigma \\ -R_{12}^\Sigma & \frac{1}{2}R_{22}^\Sigma\end{pmatrix},\\
\mathbf{B}^\Sigma(Y_1^*\text{ exch.}) &= -\frac{4}{3}\begin{pmatrix} R_{11}^Y & -R_{12}^Y \\ -R_{12}^Y & \frac{1}{2}R_{22}^Y\end{pmatrix}, & \mathbf{B}^Y(Y_1^*\text{ exch.}) &= -\frac{1}{3}\begin{pmatrix} R_{11}^Y & -R_{12}^Y \\ -R_{12}^Y & \frac{1}{2}R_{22}^Y\end{pmatrix}.
\end{aligned}\quad \text{(B1)}$$

[53] P. McNamee and F. Chilton, Rev. Mod. Phys. 36, 1005 (1964).

APPENDIX C

In this Appendix we write down the residue matrices of the Born pseudopoles needed in the calculations described in Sec. IV. For compactness of the resulting expressions, we shall use the $BB\Pi$ coupling constants defined in Appendix A and define the $\Delta B\Pi$ coupling constants as follows: Writing $R_{\pi N}{}^{N^*}$ to denote the residue at the N^* pole in the $J^P=(\frac{3}{2})^+$ partial-wave amplitude for πN scattering, $R_{\pi\Lambda}{}^{Y}$ to denote the residue at the Y_1^* pole for $\pi\Lambda$ scattering, etc., we define

$$\begin{aligned} &R_{\pi N}{}^{N^*}=-3h_{N^*N\pi}{}^2/4\pi\,, && R_{K\Lambda}{}^{\Xi^*}=-h_{\Xi^*\Lambda K}{}^2/4\pi\,, && R_{\pi\Sigma}{}^{Y^*}=-2h_{Y\Sigma\pi}{}^2/4\pi\,,\\ &R_{K\Sigma}{}^{N^*}=-3h_{N^*\Sigma K}{}^2/4\pi\,, && R_{K\Sigma}{}^{\Xi^*}=-3h_{\Xi^*\Sigma K}{}^2/4\pi\,, && R_{KN}{}^{Y^*}=-2h_{YNK}{}^2/4\pi\,,\\ &R_{\pi\Xi}{}^{\Xi^*}=-3h_{\Xi^*\Xi\pi}{}^2/4\pi\,, && R_{\pi\Lambda}{}^{Y^*}=-h_{Y\Lambda\pi}{}^2/4\pi\,, && R_{K\Xi}{}^{Y^*}=-2h_{Y\Xi K}{}^2/4\pi\,. \end{aligned} \tag{C1}$$

In unbroken $SU(3)$ symmetry,[53]

$$h_{N^*N\pi}:h_{N^*\Sigma K}:h_{Y^*\Lambda\pi}:h_{Y^*\Sigma\pi}:h_{Y^*NK}:h_{Y^*\Xi K}:h_{\Xi^*\Xi\pi}:h_{\Xi^*\Lambda K}:h_{\Xi^*\Sigma K}=\sqrt{2}:-\sqrt{2}:-\sqrt{3}:+1:-1:-1:+1:-\sqrt{3}:-1\,. \tag{C2}$$

Using this notation, our Born pseudopole residue matrices are as follows:

$$\mathbf{B}^{\Xi}(\Xi\text{ exch.})=\frac{1}{3}\frac{1}{4\pi}\begin{bmatrix} g_{\pi\Xi\Xi}{}^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad \mathbf{B}^{\Xi^*}(\Xi\text{ exch.})=-2\mathbf{B}^{\Xi}(\Xi\text{ exch.})\,, \tag{C3a}$$

$$\mathbf{B}^{\Xi}(N\text{ exch.})=\frac{1}{3}\frac{1}{4\pi}\begin{bmatrix} 0 & 0 & 0\\ 0 & -g_{KN\Lambda}{}^2 & -3g_{KN\Sigma}g_{KN\Lambda}\\ 0 & -3g_{KN\Sigma}g_{KN\Lambda} & +g_{KN\Sigma}{}^2\end{bmatrix},\quad \mathbf{B}^{\Xi^*}(N\text{ exch.})=-2\mathbf{B}^{\Xi}(N\text{ exch.})\,, \tag{C3b}$$

$$\mathbf{B}^{\Xi}(\Lambda\text{ exch.})=\frac{1}{3}\frac{1}{4\pi}\begin{bmatrix} 0 & 0 & -g_{\pi\Lambda\Sigma}g_{K\Lambda\Xi}\\ 0 & 0 & 0\\ -g_{\pi\Lambda\Sigma}g_{K\Lambda\Xi} & 0 & 0\end{bmatrix},\quad \mathbf{B}^{\Xi^*}(\Lambda\text{ exch.})=-2\mathbf{B}^{\Xi}(\Lambda\text{ exch.})\,, \tag{C3c}$$

$$\mathbf{B}^{\Xi}(\Sigma\text{ exch.})=\frac{1}{3}\frac{1}{4\pi}\begin{bmatrix} 0 & 3g_{\pi\Lambda\Sigma}g_{K\Sigma\Xi} & -2g_{\pi\Sigma\Sigma}g_{K\Sigma\Xi}\\ 3g_{\pi\Lambda\Sigma}g_{K\Sigma\Xi} & 0 & 0\\ -2g_{\pi\Sigma\Sigma}g_{K\Sigma\Xi} & 0 & 0\end{bmatrix},\quad \mathbf{B}^{\Xi^*}(\Sigma\text{ exch.})=-2\mathbf{B}^{\Xi}(\Sigma\text{ exch.})\,, \tag{C3d}$$

$$\mathbf{B}^{\Xi}(\Xi^*\text{ exch.})=-\frac{4}{3}\frac{1}{4\pi}\begin{bmatrix} h_{\Xi^*\Xi\pi}{}^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad \mathbf{B}^{\Xi^*}(\Xi^*\text{ exch.})=\tfrac{1}{4}\mathbf{B}^{\Xi}(\Xi^*\text{ exch.})\,, \tag{C3e}$$

$$\mathbf{B}^{\Xi}(Y_1^*\text{ exch.})=-\frac{4}{3}\frac{1}{4\pi}\begin{bmatrix} 0 & \sqrt{3}h_{Y\Lambda\pi}h_{Y\Xi K} & -2h_{Y\Sigma\pi}h_{Y\Xi K}\\ \sqrt{3}h_{Y\Lambda\pi}h_{Y\Xi K} & 0 & 0\\ -2h_{Y\Sigma\pi}h_{Y\Xi K} & 0 & 0\end{bmatrix},\quad \mathbf{B}^{\Xi^*}(Y_1^*\text{ exch.})=\tfrac{1}{4}\mathbf{B}^{\Xi}(Y_1^*\text{ exch})\,, \tag{C3f}$$

$$\mathbf{B}^{\Xi}(N^*\text{ exch.})=-\frac{4}{3}\frac{1}{4\pi}\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -4h_{N^*\Sigma K}{}^2\end{bmatrix},\quad \mathbf{B}^{\Xi^*}(N^*\text{ exch.})=\tfrac{1}{4}\mathbf{B}^{\Xi}(N^*\text{ exch.})\,. \tag{C3g}$$