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Nonrelativistic Motion of Particles in Strongly Bound S States*

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We consider, for some different kinds of potentials, the mathematical question: Can particles in the ground state be strongly bound and still move nonrelativistically? As is well known, this is possible for a properly chosen square well. We show, however, that this is not possible for a Yukawa potential, nor for a purely attractive superposition of Yukawa potentials, nor for a Coulomb potential. For an exponential potential this is possible; however, the criterion for nonrelativistic motion of two particles of mass M in an exponential potential of range m^{-1} is $(mM^{-1})^{1/3}\ll 1$, rather than $mM^{-1}\ll 1$ as might be expected naively. The arguments used are elementary, and rely on exact solutions to soluble problems.

1. INTRODUCTION

THE possibility that the strongly interacting particles are composite objects whose constituents are quarks or other triplets is worth considering as a basis of the $SU(3)$ and $SU(6)$ symmetries which have been helpful in correlating information about low-lying states for baryons and mesons. If triplets exist they are probably sufficiently massive so that they must be strongly bound in baryons and mesons. For a two-body bound state with constituents of mass M, we will take strong binding to be a situation where the binding energy $B\sim 2M$, so that the mass of the bound state is much smaller than the mass of the constituents; however, the value $2M$ is not essential; $B\sim M$ would lead to the same conclusions. The $SU(6)$ symmetry seems to be most helpful for static properties such as masses and magnetic moments, and, on the theoretical side, it seems to be difficult to combine the $SU(6)$ symmetry with Lorentz invariance, both of which may indicate that the triplets move nonrelativistically.

These remarks lead us to the question: Is strong binding compatible with nonrelativistic triplet motion?[1] There seems to be some misunderstanding in the recent literature[2] concerning this question, in particular for a Yukawa potential, which is the type of potential which one expects from the nonrelativistic limit of quantum field theory. Because of this misunderstanding, and because the question is important, even though elementary, we will make some elementary remarks concerning the mathematical question: Given certain specific potentials, can particles in the ground state be strongly bound and still move nonrelativistically? At the end, we will say something about the relevance of this mathematical question to possible strongly bound states occurring in nature. For simplicity, we consider a two-body bound state in the usual reduced one-body form in the center-of-mass frame of reference, and assume the particles are spinless. Because it is the case of most interest, we let both particles have masses M and consider only the ground state, which is an S state. We take $\hbar=c=1$. The radial Schrödinger equation for an S-wave bound state is

$$u''(r)-MV(r)u(r)=MBu(r),$$

where the wave function $\psi(\mathbf{x})=|\mathbf{x}|^{-1}u(|\mathbf{x}|)$ for an S state, $V(r)$ is the (spherically symmetric) potential, and $B>0$ is the binding energy.

2. SQUARE-WELL POTENTIAL

For a properly chosen attractive square-well potential

$$V(r)=-V_0\theta(R-r),\quad V_0>0,$$
$$\theta(x)=1,\quad x>0,$$
$$\theta(x)=0,\quad x<0,$$

the bound particles will move nonrelativistically even for strong binding,[3] because if the binding is strong enough, the wave function of the lowest state can be taken, to a good approximation, as a sine wave,

$$u(r)=(\mathrm{const})\sin(\pi r/R),$$

and the momentum will be approximately $p=\pi R^{-1}$ which can be made to satisfy the nonrelativistic condition $M^{-1}p\ll 1$ provided R is chosen so that $MR\gg 1$. For the square-well potential, the uncertainty principle leads to the same conclusion: For strong binding the particle is known to be in a sphere of radius R, *and with*

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[1] If there are strongly bound heavy triplets of mass $M\gg m_\rho$, say, which move relativistically, then these triplets will be confined within a radius $M^{-1}\ll$the size of strongly interacting particles, $\sim m_\rho^{-1}$, and thus would constitute only the core or nucleus of the baryons and mesons. In any case, one can expect a "cloud" of the known mesons to play some role at distances of $\sim m_\rho^{-1}$. The way in which the triplet and meson cloud structures co-exist in baryons and mesons is an important question which, however, we will not consider in this article.

[2] G. Morpurgo, Physics **2**, 95 (1965), and unpublished reports by other authors.

[3] Y. Nambu, in *Symmetry Principles at High Energy, II*, edited by B. Kursunoğlu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Company, San Francisco, California, 1965).

good probability can appear anywhere inside this sphere, and thus, by the uncertainty principle has momentum $p \sim R^{-1}$.

The arguments just given for the possibility of strongly bound nonrelativistic particles depend on the shape of the potential and cannot be applied for an attractive Yukawa potential, $-V_0 r^{-1}e^{-mr}$, $V_0>0$, of range m^{-1}, nor for any purely attractive superposition of Yukawa potentials, nor for an attractive Coulomb potential. Further, although for an exponential potential, $-V_0 e^{-mr}$, $V_0>0$, the range m^{-1} can be chosen so that strongly bound particles still move nonrelativistically; the condition for this is $(mM^{-1})^{1/3}\ll 1$, rather than $mM^{-1}\ll 1$ as would be surmised from a naive application of the arguments of the last paragraph.

3. COULOMB AND YUKAWA POTENTIALS

Since, according to the notion of range used for Yukawa potentials, the Coulomb potential has infinite range then if strongly bound particles must be relativistic for a Coulomb potential, the same will be true for the Yukawa potential and for attractive superpositions of Yukawa potentials, as they have shorter range and will bind particles in a smaller region than the Coulomb potential.[4] Let the Coulomb potential energy be

$$V(r)=-G/r.$$

The ground-state wave function for equal-mass particles is

$$u(r)=(\mathrm{const})r\exp[-\tfrac{1}{2}GMr]=(\mathrm{const})r\exp[-(MB)^{1/2}r],$$

where $B=\frac{1}{4}G^2M$. For strong binding, $B\sim 2M$, $(MB)^{1/2}\sim\sqrt{2}M$, and

$$\langle r\rangle=\left[\int_0^\infty u(r)^2\,dr\right]^{-1}\left[\int_0^\infty u(r)^2 r\,dr\right]=\tfrac{3}{2}(MB)^{-1/2}\sim 3/2\sqrt{2}M.$$

Neglecting numerical factors of order 1, $p\sim\langle r\rangle^{-1}\sim M$, so that the particles are relativistic. The virial theorem, which states that the kinetic energy equals one-half the binding energy also leads to $M^{-1}p^2=\frac{1}{2}B$, or $p\sim M$ for strong binding.

Why is the range m^{-1} of a Yukawa or exponential potential not relevant to the size of a strongly bound state? The answer is that although the bound particle is surely inside a sphere of radius m^{-1}, for strong binding it will remain, with good probability, inside a smaller sphere of radius M^{-1} (for $M>m$). The uncertainty-principle estimate then leads to $p\sim M$ rather than $p\sim m$. A rule of thumb for monotonically increasing attractive potentials is that the strongly bound particle in its ground state remains inside the distance r_0 where the binding energy and potential energy are equal in absolute magnitude, $|V(r_0)|=B$, since that is where the radial wave function has its inflection and changes from oscillatory to exponentially damped behavior. For the Coulomb potential, strong binding, $B\sim 2M$, requires $G^2\sim 8$, and gives $r_0\sim\sqrt{2}M^{-1}$, which agrees qualitatively with the estimates found above. Another rule of thumb for such potentials is that the particle is located approximately at the position $r_{\max}$ where $u(r)$ has its maximum. For the Coulomb potential, this rule leads to $r_{\max}=2(MB)^{-1/2}\sim\sqrt{2}M^{-1}$, for strong binding, in qualitative agreement with the other estimates.

4. EXPONENTIAL POTENTIALS

The situation as concerns strong binding and nonrelativistic motion of constituents for an attractive exponential potential differs from that for the Coulomb and Yukawa potentials. Here, again, since we know the exact wave function, it is simple to study what happens. For

$$V(r)=-V_0e^{-mr},\quad V_0>0,$$

the ground-state wave function[5] is

$$u(r)=(\mathrm{const})J_{(4MB/m^2)^{1/2}}\left(\left(\frac{4MV_0}{m^2}\right)^{1/2}e^{-mr/2}\right),$$

where the eigenvalue condition relating B and V_0 is

$$J_{(4MB/m)^{1/2}}((4MV_0/m^2)^{1/2})=0.$$

For the ground state, we want the first zero of the Bessel function. For strong binding and a long-range potential, $(4MBm^{-2})^{1/2}\gg 1$, and we can use the formula[6]

$$x_0\cong\nu+(1.86)\nu^{1/3}+O(\nu^{-1/3})$$

for the first zero of $J_\nu(x)$ for large ν. We use $r_{\max}$ as our estimate of the position of the particle. This is found using the formula [7]

$$x_{\max}\cong\nu+(0.81)\nu^{1/3}+O(\nu^{-1/3})$$

for the first maximum of $J_\nu(x)$ for large ν. Then, for $(4MBm^{-2})^{1/2}\gg 1$,

$$\left(\frac{4MV_0}{m^2}\right)^{1/2}\cong\left(\frac{4MB}{m^2}\right)^{1/2}+(1.86)\left(\frac{4MB}{m^2}\right)^{1/6},$$

$$\left(\frac{4MV_0}{m^2}\right)^{1/2}e^{-mr_{\max}/2}\cong\left(\frac{4MB}{m^2}\right)^{1/2}+(0.81)\left(\frac{4MB}{m^2}\right)^{1/6},$$

[4] See the Appendix for a demonstration of this.

[5] E. Kamke, *Differentialgleichungen, Band 1: Gewohnliche Differentialgleichungen* (Chelsea Publishing Company, New York, 1959), pp. 403, 422.

[6] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1944), 2nd ed., p. 516.

[7] Reference 6, p. 521.

and

$$r_{\max}\cong(1.05)2(4mMB)^{-1/3},$$

or

$$r_{\max}\sim(mM^2)^{-1/3},\quad \frac{p}{M}\sim\left(\frac{m}{M}\right)^{1/3}$$

for strong binding. The estimate based on the inflection r_0 is qualitatively the same, $r_0=(1.86)2(4mMB)^{-1/3}$. These considerations hold for any purely attractive superposition of exponential functions,[4] with the lowest mass playing the role of m above, for example, the difference of two Yukawa potentials of equal strength,

$$V(r)=-V_0\left[\frac{e^{-m_1r}}{r}-\frac{e^{-m_2r}}{r}\right]=-V_0\int_{m_1}^{m_2}d\kappa\, e^{-\kappa r},$$

$$V_0>0,\quad m_1<m_2.$$

This last example is a boundary situation, since for the difference of two Yukawa potentials where the repulsive one is stronger,

$$V(r)=-V_1\frac{e^{-m_1r}}{r}+V_2\frac{e^{-m_2r}}{r},\quad V_2>V_1>0,\quad m_2>m_1,$$

the potential has a zero at $R_0=(m_2-m_1)^{-1}\ln(V_2V_1^{-1})$. A bound particle will be located at distances greater than this; therefore $pM^{-1}<(MR_0)^{-1}$, and since m_1, m_2, $(V_2V_1^{-1})$ can be chosen to make $(MR_0)^{-1}$ small, the bound particles can move nonrelativistically in this potential, even for strong binding. This last example shows that superpositions of Yukawa potentials which are small or repulsive at the origin and have their minima sufficiently far from the origin can lead to nonrelativistic strong binding.

Finally, we mention that Eden and Goldstone[8] considered a problem in the context of nuclear physics which has some similarity to the problem we discussed above. Their problem was to see how a charge form factor $F(r)$ which is a superposition of exponential or Yukawa functions of range $\leq m^{-1}$ could lead to nuclei whose size $R_0\gg m^{-1}$, as is the case for heavy nuclei. They pointed out that such functions $F(r)$ must have oscillatory Laplace transforms $\tilde{F}(\alpha)$, and gave as examples r^ne^{-mr} which have maxima at $r=nm^{-1}$.

5. CONCLUDING REMARKS

We conclude with some remarks about the relevance of the observations above to possible strongly bound states occurring in nature. In this article we considered a mathematical question: Given a potential of some form, can, and if so under what conditions, parameters be chosen so that particles can be strongly bound and still move nonrelativistically? We did not consider the physical question: What is the form of the potential which acts between strongly bound particles and what values can the parameters in this potential assume? Nor did we ask the more difficult question: Under what conditions, if any, can the interactions between strongly bound particles be represented by a potential? We plan to study these questions, using the N-quantum approximation,[9] in a separate article. We considered the Yukawa and related potentials in the present article because they seem likely to appear in the nonrelativistic limit of field theory.

Note added in proof. Professor S. T. Epstein has pointed out (private communication) that the virial theorem for a central potential gives a simple demonstration that the kinetic energy is greater than the binding energy in a purely attractive superposition of Yukawa potentials. This gives another argument that the kinetic energy in such a potential is greater than the kinetic energy in a Coulomb potential which gives the same binding energy. The argument starts from the virial theorem

$$2\langle(2M)^{-1}p^2\rangle-\left\langle r\frac{\partial V}{\partial r}\right\rangle=0.$$

For a purely attractive superposition of Yukawa potentials,

$$V(r)=\int_0^\infty\rho(\alpha)\frac{e^{-\alpha r}}{r}d\alpha,\quad \rho(\alpha)\leq 0.$$

Then

$$\left\langle r\frac{\partial V}{\partial r}\right\rangle=-\langle V\rangle-\left\langle\int_0^\infty\alpha\rho(\alpha)e^{-\alpha r}d\alpha\right\rangle>-\langle V\rangle,$$

and

$$\langle(2M)^{-1}p^2\rangle>B.$$

We thank Professor Epstein for allowing us to quote his argument.

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APPENDIX

Here we will give a simple qualitative argument to show that for given binding energy $B>0$ the ground state in a Coulomb potential, $V(r)=-Gr^{-1}$, has a larger size than the ground state in a Yukawa potential, $V'(r)=-G'r^{-1}e^{-mr}$, of range m^{-1} whose strength G' is chosen so that the ground state has the same binding energy B. We will use the inflection to characterize the size of the state. Our argument will show that the in-

[8] R. J. Eden and J. Goldstone, Nucl. Phys. **49**, 33 (1963).

[9] O. W. Greenberg, Phys. Rev. **139**, B1038 (1965) and Bull. Am. Phys. Soc. **10**, 484 (1965).

flection r_0 for $V(r)$ is greater than the inflection r_0' for $V'(r)$. Consider a third potential

$$V''(r) = -G[\exp(mG/B)]r^{-1}e^{-mr}$$

which has the same range m^{-1} as $V'(r)$, but a greater strength so that it passes through the point $(r_0, -B)$ where the binding energy plotted negatively intersects $V(r)$. For $r<r_0$, $V''(r)<V(r)$, so that the binding energy B'' of the ground state in $V''(r)$ is greater than B. Therefore the strength G' of $V'(r)$ necessary to give binding energy B is less than $G[\exp(mG/B)]$ and $V'(r_0') = -B$ occurs for $r_0'<r_0$ as was to be shown. This same argument shows that for given binding energy the size of the ground state in a Yukawa potential of range m^{-1} is larger than that in a Yukawa potential of range $(m')^{-1}$ for $m'>m$. A similar statement holds for any family of monotonically increasing (attractive) potentials characterized by a single range parameter.

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Meson Masses and Decays*

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We consider here the masses, decay rates, and decay spectra of the octet of 0^- mesons, the nonet of 1^- mesons, the $X(960)$ resonance, and a proposed $\eta^*(1620)$ resonance having the quantum numbers of $\eta(550)$. We use spin–unitary-spin symmetry and nonet symmetry, so that the mesons fall into 36 states, combining an $SU(6)\mathbf{1}$, with an $SU(6)\mathbf{35}$. Further, we fix the strength of the couplings of the $SU(3)$ octets in the **36** with respect to the $SU(3)$ singlets by assuming that the form of the coupling remains unchanged when we include the singlet states. We are then able—after making a particular choice of Hamiltonion—to predict the masses $(M_{\eta^*}, M_{\rho^\pm}, M_{\rho^0}, M_{K^{*\pm}}, M_{K^{*0}})$ and the decay rates $\Gamma_{\omega\to\pi^+\pi^-}$, $\Gamma_{\phi\to\pi^+\pi^-}$, $\Gamma_{\phi\to\pi^+\pi^-\pi^0}$, using as input the masses of the octet of 0^- mesons and (M_ϕ, M_ω). These predictions are in reasonable agreement with presently existing experimental data. We further discuss related meson decays using the above scheme to eliminate the arbitrariness in the relative coupling strength of the singlet states. Specifically we consider $V\to P+P'$, $V\to 3P$, $V\to P+\gamma$, $V\to P+P'+\gamma$ [discussing C noninvariance and S-wave pion-pion resonances and their effect in $(\omega,\phi)\to\pi^+\pi^-\gamma$], $V\to l^+ + l^-$, $P\to 2\gamma$, $P\to P'+P''+\gamma$, and $P\to l^+ + l^- + \gamma$. We use the model of Gell-Mann, Sharp, and Wagner and give invariant-mass spectra for the three-body decays.

INTRODUCTION

In the present work, we consider a theory in which the eight 0^- mesons, the nine 1^- mesons, and a possible ninth 0^- meson, are all equivalent in the absence of symmetry-breaking forces. We thus begin with 36 equivalent states. The particular form of the symmetry breaking and the symmetry itself, are made plausible by the assumption that the known particles are built up out of three very heavy fractionally charged objects,[1–4] schematically named (n_0, p_0, Λ_0). Following Zweig,[1] we call these objects "aces." It is to be noted that the symmetry and the form of its breaking are in no sense rigorously derivable from the ace assumption. In fact, the aces could be considered merely a mathematical convenience, and thus at this level are a purely phenomenological construction. In order to decide why nature seems to reflect certain properties of our symmetry a more extensive theoretical investigation is required, dealing with the underlying dynamics.[5]

In Sec. I we discuss the consequences of the assumption that the mesons are bound states of ace-antiace pairs. We assume that in the limit of perfect symmetry, the ace-ace forces are unitary-spin, and spin, independent. This will lead us to a symmetry between 36 meson states: $3\times 9=27$ vector-meson states, and 9 pseudoscalar-meson states. It should be emphasized that we assume a coupling of the form $\mathrm{Tr}(VVP)$ and thus fix the couplings of the $SU(3)$ singlets with respect to the $SU(3)$ octets. There are not then two arbitrary amplitudes for the VVP couplings,[6,7] and we shall see the consequences of this assumption in detail

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[1] G. Zweig, CERN Report TH. 412, 1964 (unpublished); lectures given at the Majorana Summer School, Erice, Sicily, August 1964 (unpublished).

[2] M. Gell-Mann, Phys. Letters **8**, 214 (1964).

[3] T. D. Lee, M. Nauenberg, and F. Gürsey, Phys. Rev. **135**, B467 (1964).

[4] J. Schwinger, Phys. Rev. **135**, B816 (1964).

[5] Experimentally, there are lower limits to the ace mass of a few tens of BeV. For a detailed discussion of the implications of the existence of real aces, fundamental triplets, etc., see: Y. Nambu, University of Chicago report, 1965 (unpublished); in *Proceedings of the 1965 Coral Gables Conference* (W. H. Freeman and Company, San Francisco, California, 1965), p. 274; Y. Nambu and S. Fenster, Progr. Theoret. Phys. (Kyoto) (to be published).

[6] S. Okubo, Phys. Letters **5**, 165 (1965).

[7] S. Glashow, Phys. Rev. Letters **11**, 48 (1963).