Systematic Study of the Balázs Bootstrap Method*

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The Balázs approach to $\pi\pi$ scattering has been systematically studied numerically. On increasing the number of poles representing the unphysical cut, the solution changes drastically and disappears. We conclude that the single-channel calculations are so unstable as to be meaningless. We claim that such resonances as occur using the Balázs matching process to pole forms do so by accident for small numbers of poles, and are not related to any exact solution. When several partial waves were included, it was very difficult to find solutions because of false minima; however, the P_1-D_0 calculation had at least two solutions. Although agreeing with Mehta and Srivastava in the sensitivity to symmetry-point position, we do not agree with their finding no solutions for $\nu_F<-1.5$.

I. INTRODUCTION

THE purpose of this paper is to report the results of a systematic investigation of the properties of the Balázs bootstrap method[1] and to attempt to draw conclusions. It differs from a recent note[2] in being much more extensive, affording some interpretation and also in actually disagreeing with the numerics of Mehta and Srivastava. Although agreeing with them in the sensitivity to symmetry-point position, we do not agree with their finding no solutions for $\nu_F<-1.5$. The Balázs method was initiated in BI and BII, BI explaining the kernel approximation and BII carrying out the calculation for the $I=1$, $J=1$ $\pi\pi$ state. BIII extended it to the $I=0$ $J=2$ and $I=0$, 2 S waves. The method has also been used by several other authors.[3,4]

We see the task of solution of the Mandelstam representation as one of continuing from the physical region of the t and u channels to the physical region of the s channel, subject to the singularity structure implied by the Mandelstam representation. We do this in three stages. First, we write a fixed s dispersion relation for $s<4$, to give the value of the amplitude on the left of the s channel, with $|\cos\theta|<1$, in terms of the absorptive part on the right of the t channel.

$$A^I(s,t)=\frac{1}{\pi}\int_4^\infty dt' A_t{}^I(\nu',\cos\theta')\times\left(\frac{1}{t'-t}+\frac{(-1)^I}{t'-(4-s-t)}\right). \quad (1a)$$

Our notation is, $\nu=q^2$, $\cos\theta$: center-of-mass variables in the s channel; ν', $\cos\theta'$: those in the t channel,

$$\cos\theta'=1+2[(\nu+1)/\nu'],\qquad \cos\theta=1+2[(\nu'+1)/\nu]. \quad (1b)$$

$A^I(s,t)$ is the amplitude in the isospin eigenchannel I of the t channel, $A_t{}^I(\nu',\cos\theta')$ is the absorptive part in the t channel.

For ν in $[0,-9]$ we can take

$$A_t{}^I(\nu',\cos\theta')=\sum_{l'=0}^{\infty}(2l'+1)P_{l'}(\cos\theta')\,\mathrm{Im}A_{l'}{}^I(\nu'), \quad (1c)$$

which gives this function along the line of integration in (1a), some of which lies outside the physical region of the t channel, and is given by (1c) as a continuation along a line of constant t or ν' from the t-channel physical region.

Next, we project out the partial-wave amplitude in the s channel and take correct linear combinations to give the amplitudes in the isospin eigenchannels in the s channel.

$$A_t{}^I(\nu)=\frac{1}{2}\int_{-1}^{+1}d(\cos\theta)P_l(\cos\theta)\sum_{I'}\beta_{II'}A^{I'}(s,t)$$
$$=\frac{4}{\pi\nu}\int_0^\infty d\nu' Q_l(\cos\theta')\sum_{J'}\beta_{II'}\sum_{l'=0}^{\infty}(2l'+1)\times P_{l'}(\cos\theta')\,\mathrm{Im}A_{l'}{}^{I'}(\nu'). \quad (2)$$

Finally, having got the amplitude on the left-hand side of the s channel, we need to continue to the physical region of the s channel, and it is at this step that there are a variety of methods available.

II. THE BALÁZS BOOTSTRAP METHOD

Following Balázs, we take one term in the sum over l' in (1c), viz.

$$(2l'+1)P_{l'}(\cos\theta')\,\mathrm{Im}A_{l'}{}^{I'}(\nu').$$

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[1] L. A. P. Balázs, Phys. Rev. **128**, 1935 (1962) (hereafter called BI); **128**, 1939 (1962) (BII); **129**, 872 (1963) (BIII); Phys. Rev. Letters **10**, 171 (1963) (BIV); Phys. Rev. **132**, 867 (1963) (BV). BII and BIII have been reprinted in *Complex Angular Momenta and Particle Physics*, edited by E. J. Squires (W. A. Benjamin, Inc., New York, 1963).

[2] M. L. Mehta and P. K. Srivastava, Phys. Rev. **137**, 423 (1965).

[3] S. K. Bose and M. Dersarkissian, Nuovo Cimento **30**, 878 (1963) (D in NN); **30**, 894 (1963) ($\pi\Lambda$ scattering). J. C. Pati, Phys. Rev. **134**, 387 (1964) (Ξ^* in $\bar{K}\Lambda$). G. L. Kane, *ibid.* **135**, 843 (1964) (Ω^- in $\bar{K}\Xi$). S. R. Choudhury and L. K. Pande, *ibid.* **135**, 1027 (1964) (Ω^- in $\bar{K}\Xi$). V. Singh and B. M. Udgaonkar, *ibid.* **128**, 1820 (1962) (nuclear form factor); **130**, 1177, (1963) (N^* in πN).

[4] T. Kanki and A. Tubis, Phys. Rev. **136**, 723 (1964).

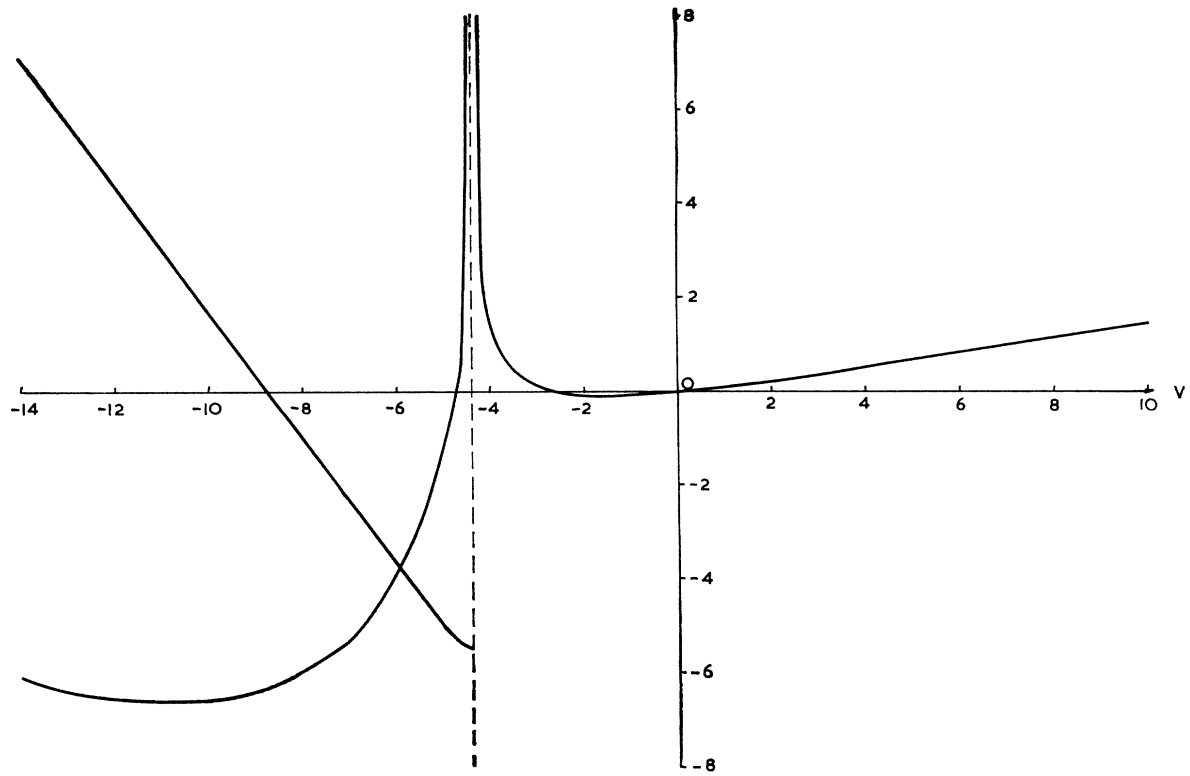


FIG. 1. The Born term. The real and imaginary parts of the function $C(\nu)$.

For $\mathrm{Im}A_{l'}{}^{I'}(\nu')$, we take a δ-function approximation for a Breit-Wigner form.

$$\frac{\Gamma^2\nu'^{2l'}[\nu'/(\nu'+1)]^{l/2}}{(\nu_R-\nu')^2+\Gamma^2\nu'^{2l'}[\nu'/(\nu'+1)]}\to\pi\Gamma\nu_R{}^{l'}\delta(\nu'-\nu_R),$$
$$\mathrm{Im}A_{l'}{}^{I'}(\nu')\approx\pi\Gamma\nu_R{}^{l'}\delta(\nu'-\nu_R). \tag{3}$$

Putting these approximations into (2) we obtain the function

$$C_{ll}{}^{l'I'}(\nu)=\frac{4}{\pi\nu}(2l'+1)\beta_{II'}(\pi\Gamma\nu_R{}^{l'})P_{l'}\left[1+2\left(\frac{\nu+1}{\nu_R}\right)\right]\times Q_l\left[1+2\left(\frac{\nu_R+1}{\nu}\right)\right], \tag{4}$$

which is an estimate for $A_l(\nu)$ in $[0,-9]$, $A_l(\nu)\sim C_l(\nu)$, $\nu\in[0,-9]$. The real and imaginary parts of $C(\nu)$ are plotted in Fig. 1 for $l=l=I=I'=1$, taking the values of ν_R, Γ as the Balázs self-consistent values $\nu_R=3.4$, $\Gamma=0.7$, i.e., $\nu_R\sim585$ MeV, full width $\Gamma\sim150$ MeV. Note that the real part $\sim\nu^l$ near threshold and that it has a logarithmic singularity at $-\nu_R-1$. The cut starts at $\nu=-\nu_R-1$ but would be up to $\nu=-1$ for a continuous Im $A(\nu')$ function in (1). Note that although the function has no right-hand cut it is an estimate for the whole amplitude, contributed to from left and right.

$$C(\nu)\sim A(\nu)=\int_0^\infty+\int_{-\infty}^{-1}\frac{\mathrm{Im}A(\nu')}{\nu'-\nu}d\nu'.$$

It follows that if we take this real and imaginary part in near left, then the correct contribution from the far left for the whole amplitude must be less than that for the Born term, by an amount equal to the contribution from the right-hand cut.

Some authors use

$$C(\nu)\cong B(\nu)\equiv\int_{-\infty}^{-1}\frac{\mathrm{Im}A(\nu')}{\nu'-\nu}d\nu'$$

and the Zachariasen method[5] uses

$$C(\nu)\cong N(\nu)\equiv\int_{-\infty}^{-1}\frac{\mathrm{Im}A(\nu')D(\nu')}{\nu'-\nu}d\nu'.$$

Returning to the Balázs method, we continue to the right-hand side, i.e. to the s-channel physical region, by matching to pole form, which has a right-hand cut:

$$N(\nu)=A(\nu_0)+(\nu-\nu_0)\sum_{i=1}^{n}F^i/(\nu-\nu_i), \tag{5}$$

$$D(\nu)=1-\left(\frac{\nu-\nu_0}{\pi}\right)\sum_{i=1}^{n}F^iK(\nu,\nu_i), \tag{6}$$

$$K(a,b)=\int_0^\infty R(\nu')d\nu'\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{1}{(\nu'-a)(\nu'-b)}$$
$$=K(b,a). \tag{7}$$

[5] F. Zachariasen, Phys. Rev. Letters 7, 112 (1961).

TABLE I. Balázs pole positions.

(1) Basic result		
$n=2$, $\nu=0.5$, $\nu_L=-5$, $P=1$		
$\nu_i=-6.0, -25.0$		
(2) Variation with ν_L		
$\nu_L=-10$,	$\nu_i=-12.3, -42.2$	
$\nu_L=-2$,	$\nu_i=-2.48, -8.696$	
$\nu_L=-1$,	$\nu_i=-1.25, -4.46$	
(3) Variation with p		
$p=0.5$,	$\nu_i=-5.78, -12.3$	
$p=2.0$,	$\nu_i=-6.45, -39.8$	
(4) Variation with n		
$n=3$,	$\nu_i=-5.49, -9.62, -36.1$	
$n=4$,	$\nu_i=-5.316, -7.194, -13.44, -50.8$	
$n=5$,	$\nu_i=-5.13, -6.33, -9.62, -19.3, -64.5$	
$n=6(1)$,	$\nu_i=-5.07, -7.87, -8.13, -11.55, -47.6, +14.6$	
$n=6(2)$,	$\nu_i=-5.29, -6.94, -9.52, -13.89, -14.28, -19.23$	
(5) Variation with ν		
$\nu=$ 1.0,	$\nu_i=-6.135, -22.5$	
$\nu=$ 3.0,	$\nu_i=-6.25, -22.6$	
$\nu=$ 5.0,	$\nu_i=-6.29, -23.4$	
$\nu=$ 10.0,	$\nu_i=-6.37, -25.5$	
$\nu=$ 25.0,	$\nu_i=-6.49, -28.41$	
$\nu=$ 50.0,	$\nu_i=-6.58, -30.68$	
$\nu=$ 100.0,	$\nu_i=-6.67, -32.90$	
$\nu=$ 500.0,	$\nu_i=-6.85, -36.90$	
$\nu=$1000.0,	$\nu_i=-6.80, -37.59$	

(6) χ^2 optimization

n	Initial χ^2	Final χ^2
3	0.9×10^{-8}	0.3×10^{-8}
4	0.8×10^{-11}	0.1×10^{-11}
5	0.5×10^{-14}	0.8×10^{-15}
6(1)	0.3×10^{-14}	0.2×10^{-16}
6(2)	0.3×10^{-15}	0.3×10^{-17}

If $R(\nu')=1$, for all ν' in $[0,\infty]$, then

$$K(a,b)=[2/(a-b)]\{[b/(b+1)]^{1/2}\times\ln[(-b)^{1/2}+(-b-1)^{1/2}]-[a/(a+1)]^{1/2}\times\ln[(-a)^{1/2}+(-a-1)^{1/2}]\}. \quad (8)$$

The ν_i are the Balázs fixed pole positions obtained in an auxiliary calculation (see Sec. III) and the residues F^i are to be determined. The form is, of course, unitary. We match the N/D form to $A(\nu)\sim C(\nu)$ at a finite set of "symmetry points" on the left-hand side by demanding

$$C(\nu_{Fj})=N(\nu_{Fj})/D(\nu_{Fj}),\quad j=1,\cdots n+1,$$

which gives linear algebraic equations for the $\{F^i\}$. The ν_{Fj} are arbitrary, but must lie in the region where $C(\nu)$ is known, viz. $[0,-9]$. In practice, since $C(\nu)$ has a singularity at $\nu=-\nu_R-1$, we must avoid this point, and as the behavior of $C(\nu)$ in $[0,-\nu_R-1]$ is thought to be more relevant to the right-hand side than that in $[-\nu_R-1,-9]$, we take the $\{\nu_{Fj}\}$ in $[-\nu_R-1,0]$, but not too close to the singularity. Putting

$$N(\nu)=(\nu-\nu_0)\sum_{i=0}^{n}F^i/(\nu-\nu_i),$$

with $F^0=A(\nu_0)$, then

$$\frac{(\nu_{Fj}-\nu_0)\sum_{i=0}^{n}F^i/(\nu_{Fj}-\nu_i)}{1-[(\nu_{Fj}-\nu_0)/\pi]\sum_{i=0}^{n}F^iK(\nu_i,\nu_{Fj})}=C(\nu_{Fj}),\quad j=0,\cdots n,$$

i.e.,

$$\sum_i E^{ji}F^i=C^j,\quad C^j\equiv C(\nu_{Fj}).$$

$$E^{ji}=(\nu_{Fj}-\nu_0)\left[\frac{1}{\nu_{Fj}-\nu_i}+\frac{C^jK(\nu_i,\nu_{Fj})}{\pi}\right].$$

The function

$$\mathrm{Im}A_l{}^I(\nu)=-[\nu/(\nu+1)]^{\frac{1}{2}}|N/D|^2$$

in the s-channel physical region constitutes the output to be compared with the $\mathrm{Im}A_{l'}{}^{I'}(\nu')$ input from the t channel, and one demands rough self-consistency by

$$\nu_{R\,\mathrm{out}}=\nu_{R\,\mathrm{in}}, \quad (9)$$

$$\Gamma_{\mathrm{out}}=\Gamma_{\mathrm{in}}, \quad (10)$$

Γ_{out} being defined by

$$(\Gamma\nu_R{}^l)_{\mathrm{out}}=N_{\mathrm{out}}(\nu_{R\,\mathrm{in}})(\nu_{R\,\mathrm{in}}-\nu_0), \quad (11)$$

this definition being more convenient than and very close to

$$\Gamma_{\mathrm{out}}\sim -N(\nu_{R\,\mathrm{in}})/\mathrm{Re}D'(\nu_{R\,\mathrm{in}})$$

and obtained from it by the linear approximation for D.

$$\mathrm{Re}D'(\nu)=-1/(\nu_{R\,\mathrm{in}}-\nu_0).$$

(9) is specified by

$$\mathrm{Re}D_{\mathrm{out}}(\nu_{R\,\mathrm{in}})=0. \quad (12)$$

III. THE KERNEL APPROXIMATION

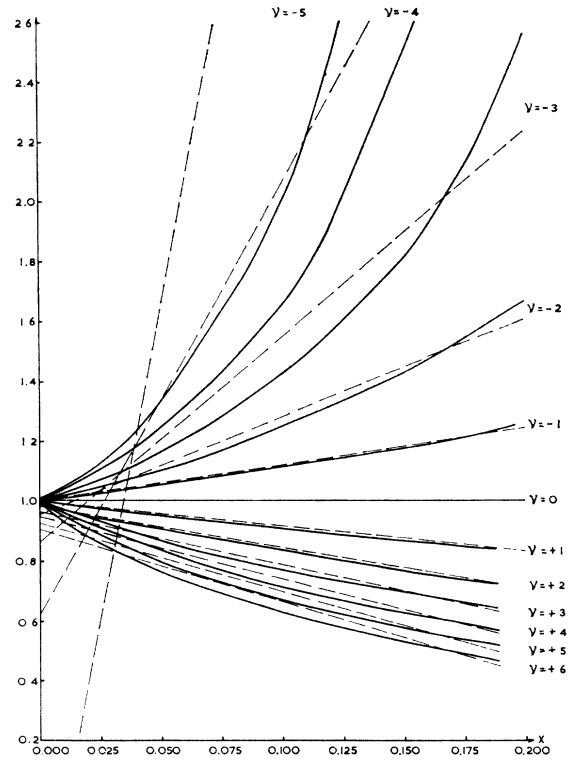
The significance of the Balázs method is in the pole approximation which finds optimum positions for the poles depending only upon the kernel in

$$N(\nu)=A(\nu_0)+\left(\frac{\nu-\nu_0}{\pi}\right)\int_{-\infty}^{-1}\frac{\mathrm{Im}A(\nu')D(\nu')d\nu'}{(\nu'-\nu)(\nu'-\nu_0)}. \quad (13)$$

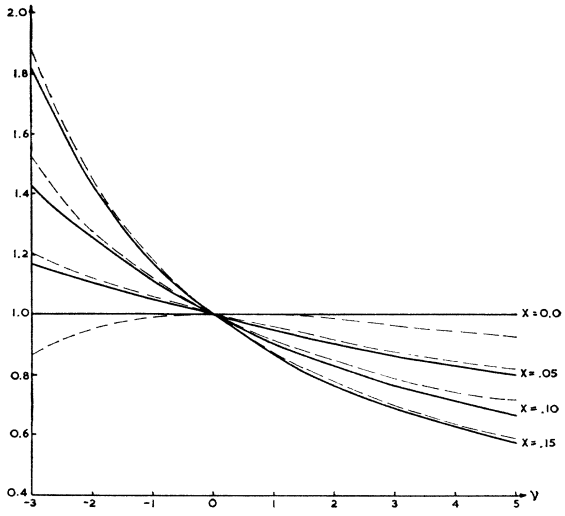
The kernel is approximated by a pole form which simultaneously expresses it as a sum of degenerate kernels and gives $N(\nu)$ a form which is easily integrable to give $D(\nu)$. Writing $x=-1/\nu'$, we approximate

$$K_{\mathrm{exact}}(x,\nu)=\frac{1}{1+x\nu}\approx\sum_{i=1}^{n}\frac{G_i(x)}{1+x_i\nu}=K_{\mathrm{app}}(x,\nu), \quad (14)$$

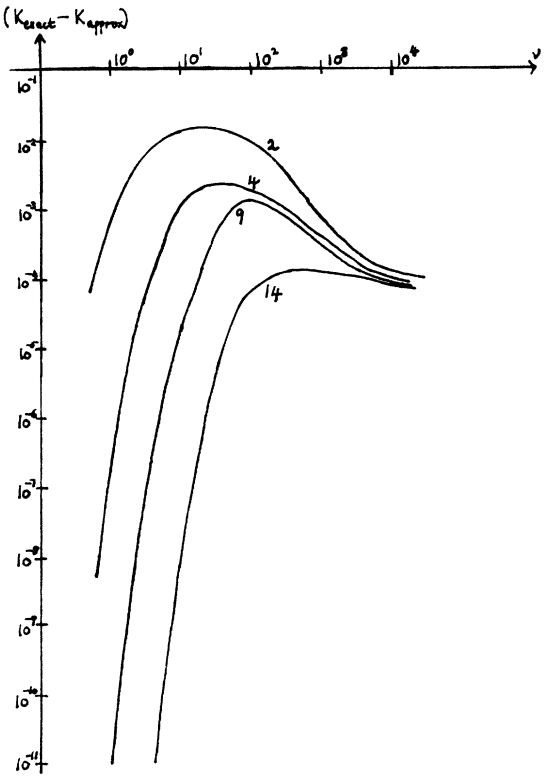
where the $\{G_i(x)\}$ are Lagrange interpolatory rational functions having values unity at $\{x_i\}$. One varies $\{x_i\}$ to give the best agreement for ν fixed at some point in the range over which the approximation is to be used. One also chooses a cutoff ν_L for generality, taking the integral from ν_L to $-\infty$.



(a)



(b)



(c)

FIG. 2. Two-pole fit to the kernel. (a) The exact and approximate kernels plotted against x, for a series of values of ν. Broken line is K_{approx}, continuous line is K_{exact}. (b) Plotted against ν, for a series of values of x. (c) F_1–F_2, typical values over $\{\nu'\}$, $\{\nu_i\}$ are optimum values for $\nu=0.5$, $n=2$, 4, 9, and 14.

There is some freedom of interpretation of the requirement of best fit, and we minimized over a set of points in the range $[0,\nu_L]$. The distribution of points was determined by a power law

$$V^j=\nu_L(j/m)^p, \quad j=1,$$

m being the number of points, usually kept at value 20, and p an index which could be varied. The results of a typical fit are shown in Fig. 2. Values of the pole positions and an indication of dependence on ν_L, ν, and p are shown in Table I, also the change in χ^2 during optimization. The solution 6(1) has a pole on the right-hand side and may be permissible, if we take the N function to include the inelastic cut as well as the left-hand cut, as explained in BII. The effects on the Balázs calculation will be seen later. Note that the difference in χ^2 produced by optimization is only of the same order as that from increasing n by 1.

We wish to make three points about the kernel approximation.

(1) It is valid if and only if the dispersion integral converges.

For the exact integral

$$I_1(\nu)=\int_{-\infty}^{\nu_L}\frac{f(\nu')d\nu'}{\nu'-\nu}=\int_{-\infty}^{\nu_L}f(\nu')\frac{K_{\text{exact}}(\nu',\nu)d\nu'}{\nu'} \quad (15)$$

is replaced by

$$I_1'(\nu)=\int_{-\infty}^{\nu_L}\frac{d\nu' K_{\text{app}}(\nu',\nu)d\nu'}{\nu'}=\frac{C_1 I_2(\nu_2)}{\nu_1-\nu}+\frac{C_2 I_2(\nu_1)}{\nu_2-\nu}, \quad (16)$$

where

$$I_2(a)=\int_{-\infty}^{\nu_L}\frac{f(\nu')(\nu'-a)d\nu'}{\nu'^2} \quad (17)$$

for a two-pole approximation

$$C_1=\frac{\nu_1}{\nu_2}\frac{1}{x_1-x_2}; \quad C_2=\frac{\nu_2}{\nu_1}\frac{1}{x_2-x_1}. \quad (18)$$

Now the conditions for convergence of $I_1(\nu)$ and $I_2(a)$ are identical and, in practice, reduce to either (a) $f(\nu')\to 0$, $\nu'\to\infty$, or (b) $f(\nu')=\sin u\nu'.\, g(\nu')$, u real >0,

$$g(\nu')/\nu'\to 0 \text{ steadily}, \quad \nu'\to\infty,$$

for all finite ν, a.

(2) The asymptotic behavior of the approximated integral is not necessarily $O(1/\nu)$.

For what the kernel approximation asserts is that

$$\int\frac{f(\nu')}{\nu'-\nu}d\nu'\approx\sum_i\frac{F^i}{\nu_i-\nu}$$

in a small range of ν, where ν is sufficiently far from the cut. It does not attempt to be valid on the cut as $\nu\to-\infty$, $\nu_i\to-\infty$. (1) and (2) indicate that it is not a usual pole approximation and not equivalent to a cut off.

(3) It is still a good approximation for oscillating functions, i.e., conditionally convergent integrands. For, from (1) it still converges and, in BI, Balázs gives an estimate for the percent error for an oscillating function. Now when the oscillations exactly cancel the exact integral has a zero so the percent error is infinite; however, this is merely due to the fact that the approximate integral has a zero slightly displaced. Numerical estimates of the Balázs expression from BI for the percent error, using the kernel approximation of Fig. 2,

$$E=\{[\Delta D_1'/D_1-\Delta D_2'/D_2]/[D_1'/D_1-D_2'/D_2]\}\times 100\%, \quad (19)$$

if we take the oscillations to exactly cancel at $\nu=-2$, $E=\infty$, then when

$$\nu=-1, \quad E=0.4\%,$$
$$\nu=+1, \quad E=0.04\%,$$
$$\nu=+5, \quad E=0.3\%,$$
$$\nu=0, \quad E=0\% \text{ always}.$$

Note from Figs. 2 the percent error is zero for $\nu=0$ and 5% as $\nu\to-\infty$.

The assertion is not that any kernel approximation within 5% of the exact is accurate for oscillating functions, for this is clearly untrue, but that this particular approximation is. We have made it plausible that the $N(\nu)$ function can be well approximated by the pole form in a certain finite range. For example, a two-pole fit is accurate to 10%, for ν in $[-3, 6]$, for all ν' in $[\nu_L, -\infty]$, and a six-pole fit is accurate to 1%, for ν in $[-3, 15]$. However, to form the $D(\nu)$ function we need an estimate for $N(\nu)$ for the entire range $[0, \infty]$.[6] To see how the approximation varies in this range, we did two studies. First, we found the optimum ν_i for $n=2$, for a series of values of ν in the range. The results are shown in Table I(5) and show that the optimum ν_i change slowly and monotonically from $\nu=1.0$, $\nu_i=6.17$, 21.6, to $\nu=100.0$, $\nu_i=6.80$, 37.6, but that the value of χ^2 at the optimum value changes rapidly, starting at 10^{-5}, rising to 10^{-1} at $\nu=50$, and then decreasing to 10^{-3} for $\nu=10^3$. Second, we evaluated $K_{\text{exact}}-K_{\text{approx}}$ for a series of values of ν, the ν_i being the optimum values for $\nu=0.5$. $K_{\text{exact}}-K_{\text{approx}}$ was roughly constant for ν' in $[\nu_L, -\infty]$, except that, for large negative ν', it rose by a factor of 10 or so. The values for typical ν' are plotted versus ν in Fig. 2(c). We see the following:

(a) The "asymptotic" value of $K_{\text{exact}}-K_{\text{approx}}$ is roughly the same for all n and is 10^{-4}, so the percent error = 10%.

[6] I am grateful to B. Martin for discussion on this point.

(b) The error rises from very low values for $\nu<10$ to 10^{-3} or so for $\nu=50$. For $n=2$, $K_{\text{exact}}-K_{\text{approx}}=10^{-2}$ and the percent error $<10\%$.

(c) As n increases the error gets very small for ν in [0,100]. Now, because we are evaluating $D(\nu)$, the very high ν values in the integral will be damped by the Cauchy denominators. Also, for large negative ν', the error rises by a factor of 10 for all ν; however, the contributions from this region are damped by the $(\nu'-\nu_0)$ denominator in the $N(\nu)$ integral.

Thus, our conclusion is that, for a sufficient number of poles, one can achieve sufficiently accurate values of $N(\nu)$, to achieve an accurate value of $D(\nu)$, provided one has accurate values of F^i.

We should finally like to compare the pole positions obtained by the Balázs optimization procedure with those obtained by using a numerical integration formula. If we wish to integrate

$$\int_{-\infty}^{\nu_L} dx\, f(x),$$

then we can first transform to a finite range by

$$z=(x-\nu_L)/x;$$

then z decreases from 0 to -1 as x decreases from $-\nu_L$ to $-\infty$. Further

$$x=\nu_L/(1-z),$$

so

$$dx=\nu_L dz/(1-z)^2,$$

and we have

$$\int_{-1}^{0} dz\frac{\nu_L}{(1-z)^2} f[x(z)].$$

If we do this integration numerically, we take n integration points z^i and weights w^i and take the value of the integral as

$$\sum_{i-1}^{n} w^i\frac{\nu_L}{(1-z_i')^2} f[x(z_i)].$$

Thus for a dispersion integral

$$f(x)=\frac{h(x)}{(x-\nu)(x-\nu_0)},$$

say

$$\int_{-\infty}^{\nu_L}\frac{h(x)dx}{(x-\nu)(x-\nu_0)}\approx\sum_{i=1}^{n} w^i\frac{\nu_L}{(1-z_i)^2}\times\frac{h[x(z_i)]}{[x(z_i)-\nu][x(z_i)-\nu_0]},$$

a pole approximation with poles at

$$x(z_i)=\nu_L/(1-z_i).$$

This is equivalent to approximation of the integrand $f[x(z)]/(1-z)^2$ by polynomials $g^n(z)$, so $f(x)/(1-z)^2$ is approximated by $g^n[(x-\nu_L)/x]$ and $f(x)$ is approximated by $(1/x^2)g^n[(x-\nu_L)/x]$. For Gaussian integration the $g^n(z)$ are Legendre functions $P_n(z)$ and z^i are its roots. Hence $f(x)$ is approximated by poles of various orders at the origin.

TABLE II. Pole distributions.

Gaussian integration pole positions	
For $\nu_L=5$	
$n=2$	$-6, -30$
$n=3$	$-5.6, -10.0, -50.0$
$n=4$	$-5.4, -7.5, -15.0, -70.0$
$n=5$	$-5.15, -6.0, -8.05, -13.1, -29.4, -143.0$
Evenly spaced integration pole positions	
For $\nu_L=5$	
$n=2$	$-7.5, -15.0$
$n=3$	$-6.5, -10.0, -20.0$
$n=4$	$-6.25, -8.33, -12.5, -25.0$
Balázs pole positions	
For $\nu_L=5$	
$n=2$	$-6, -25$
$n=3$	$-5.49, -9.62, -36.1$
$n=4$	$-5.32, -7.19, -13.44, -50.8$
$n=5$	$-5.13, -6.33, -9.62, -19.3, -64.5$

If we put in Gaussian points z^i and calculate the corresponding $x(z_i)$, the pole positions on the left-hand cut, we obtain the results shown in Table II. If one takes equally-spaced points, omitting the end points, then

$$z^i=i/(n+1),\quad i=1,\cdots,n$$

$$\nu^i=\frac{\nu_L}{1-i/(n+1)}=\frac{(n+1)\nu_L}{n+1-1}$$

reordering

$$\nu^j=(n+1)\nu_L/j;\quad j=1,\cdots,n,$$

a harmonic series. The pole distributions are shown in Table II.

The positions for Gaussian integration are remarkably similar to the Balázs positions and agree much better than those from equal spacing integration.

Now, Gaussian points can be generated by a best-fit procedure[7] and one wonders whether there is any connection between this and the Balázs optimization procedure.

Note that there is a simple connection between the pole positions for different ν_L, viz.,

$$x^i\propto\nu_L.$$

If we look at the ν_L variation of the Balázs poles, shown in Table I(a), we see a similar behavior. Of course, in this treatment, there is no dependence on ν. However, one can see that, for ν near the cut, some modification of the ν^i to fit best might be necessary. Further, the Balázs poles become very insensitive to ν, when it is reasonably far from the cut.

[7] See, for example, Z. Kopal, *Numerical Analysis* (Chapman and Hall Ltd., London 1961), 2nd ed., p. 349 ff.

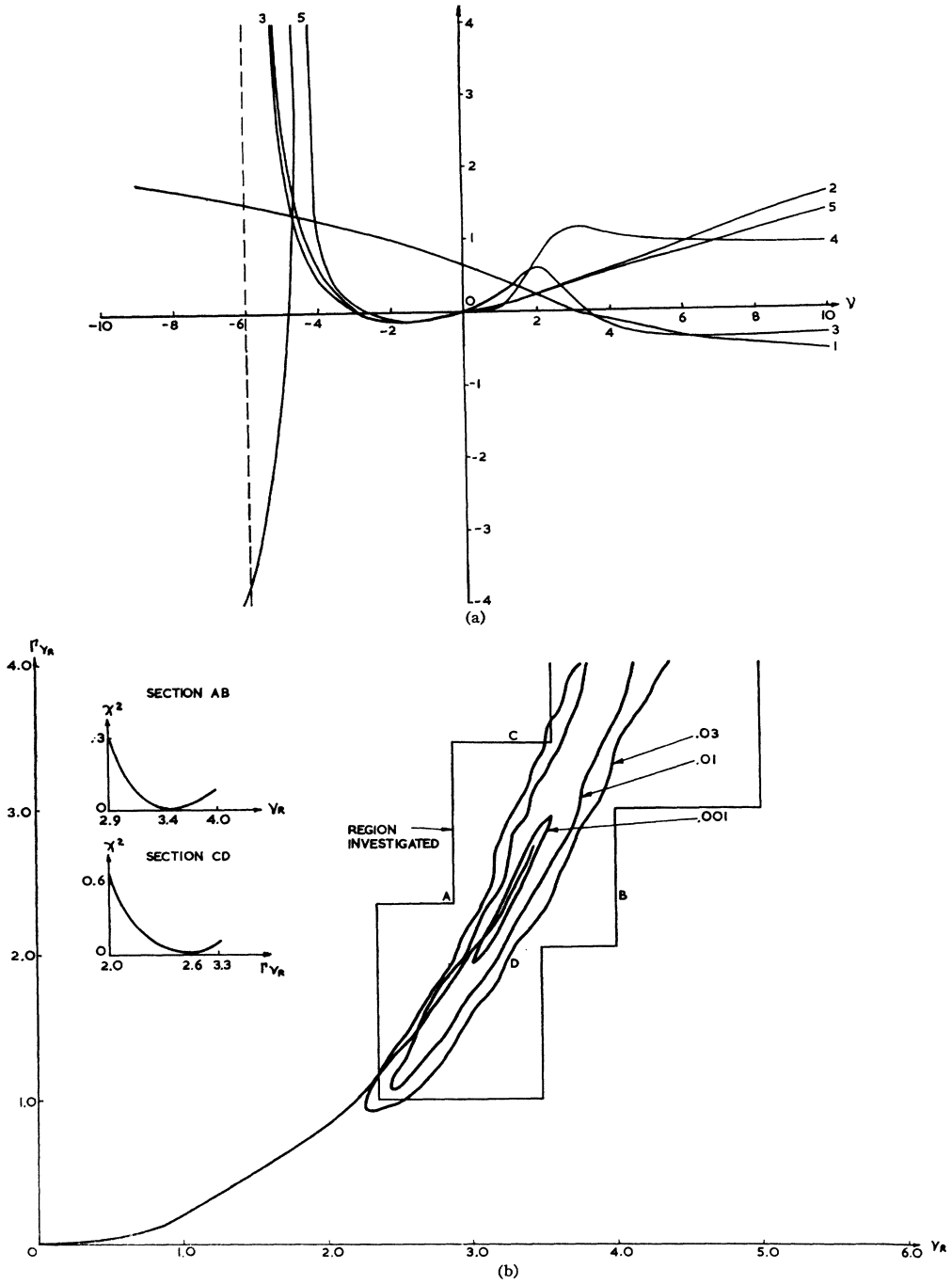


FIG. 3. The mechanism of the Balázs approximation. (a) Graphs of (1) Re$D(\nu)$, (2) Re$N(\nu)$, (3) Re$A(\nu)$, (4) Im$A(\nu)$, and (5) Re$C(\nu)$, showing how matching between Re$A(\nu)$ and Re$C(\nu)$ occurs and the functions it produces. (b) χ^2 map for the two-pole P_1 calculation.

Since the integral is

$$\frac{\nu_L}{\nu\nu_0}\int_{-1}^{0}\frac{h[x(z)]}{(a-z)(a_0-z)(1+z)^2},$$

$$a=\nu_L/\nu-1;\quad a_0=\nu_L/\nu_0-1;\quad a,\ a_0 \text{ real}>0,$$

then presumably, the best positions are given by the roots of polynomials $g^n(z)$,

$$g^n(z_i)=0,$$

where $g^n(z)$ are defined by the weight function

$$w(z)=[(a-z)(a_0-z)(1+z)^2]^{-1}$$

and interval $[0,-1]$.

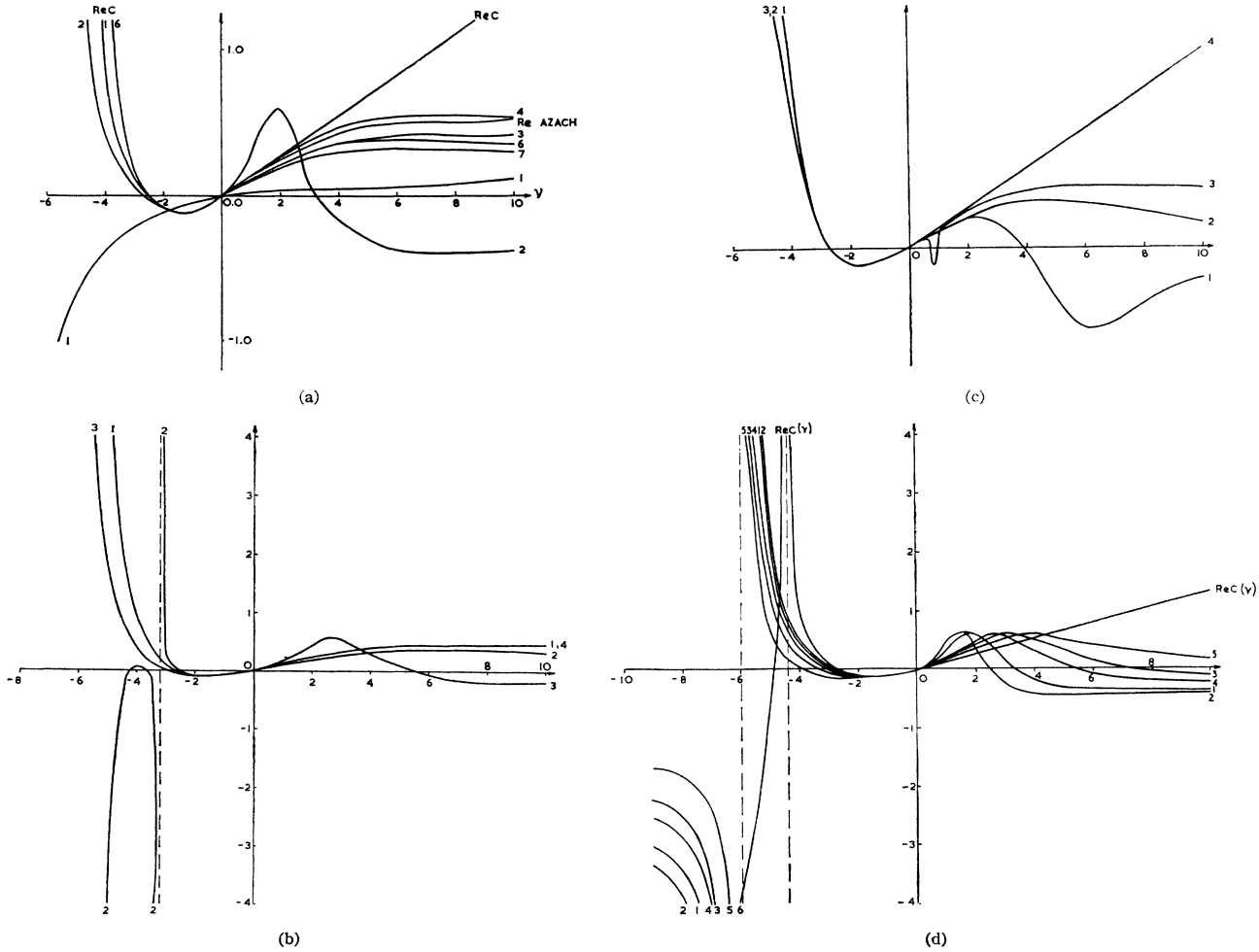


FIG. 4. Static studies. $\nu_R=3.4$, $\Gamma\nu_R=2.6$. (a) Derivative matching. $n=1, 2, 3, 4, 6, 7$. (b) Distributed matching. $n=2, 3, 4, 5$; curve 1 is Re$C(\nu)$. (c) Distributed matching $n=6(1), 6(2), 7$; corresponding curves 1, 2, 3; curve 4 is Re$C(\nu)$. (d) Different sets of symmetry points. (1) -2.1, -1.9, 0.0. (2) -2.5, -2.3, 0.0. (3) -1.5, -1.0, 0.0. (4) -2.1, -1.0, 0.0. (5) -0.8, -0.5, 0.0.

Now there is an essential difference between the numerical integration procedure and the Balázs procedure, namely that the Balázs F^i are determined by matching the approximate form to the exact integral, but the numerical integration procedure uses the $f[x(z_i)]$ to get them. The numerical integration procedure is only valid for integrands which can be approximated by polynomials whereas the Balázs procedure is valid for a much wider range of integrands.

The similarity between the two lies in the fact that if we tried to approximate a well-behaved integral by the Balázs procedure, then we should get the same answer as by numerical integration, hence the pole positions are the same.

IV. THE P_1 CHANNEL

The N/D form is matched to the $C(\nu)$ function at $(n+1)$ symmetry points to give $\{F_i\}$ and $A(\nu_0)$. The properties of the pole form and the mechanism of the process are shown in Fig. 3. Note the near linearity of Re$D(\nu)$ although of course $D \xrightarrow[\nu\to\infty]{}$ constant, justifying the self-consistency conditions used by Balázs and supporting the linear approximation of Pati and also Kane.[3] Note that $C(\nu)$ has a logarithmic singularity at $-\nu_R-1$ but that N/D has a pole at -6.

Symmetry points close together are equivalent to matching derivatives. Because of the sensitivity to symmetry points, it was found clearest to adopt two classes of distribution:

(1) "derivative," i.e., l at the origin to give ν^l behavior, and $(n-l)$ clustered round one point usually $\nu=-2$.

(2) "distributed," i.e., spread out in $[0, -2.5]$.

We did static studies, i.e., fix t channel, match and find s channel, and dynamic studies, i.e., self-consistent values of ν_R, $\Gamma\nu_R{}^l$. The result of a static study is thus a tabulation of functions $N(\nu)$ and $D(\nu)$, and we shall,

TABLE III. Dynamic studies. Self-consistent values of $(\nu_R,\Gamma\nu_R)$.

(i) Variation with $\{\nu_{Fi}\}$				
$\{\nu_{Fi}\}$			ν_R	$\Gamma\nu_R$
−2.1	−1.9	0.0	3.308	2.525
−2.1	−1.9	−0.1	3.436	2.721
−1.5	−1.0	0.0	1.88	1.193
−2.5	−2.3	0.0	3.82	3.129
−2.1	−1.0	0.0	2.568	1.760
−2.8	−2.2	−1.6	5.02	4.71
(ii) Variation with ν_0				
ν_0				
−10			4.34	4.96
− 3			3.465	2.849
− 2			3.308	2.525
− 1			3.22	2.35
− 0.001			3.23	2.37
(iii) Variation with n				
n				
2			3.4	2.6
4			2.08	0.15
3				
5				
7			no solution	
1				
6(1)				
6(2)				

in general, display the function $\mathrm{Re}A(\nu)$ derived from them. The result of a dynamic study is whether a self-consistent acceptable solution was obtained and, if so, the self-consistent values of ν_R and $\Gamma\nu_R$[1].

We obtained self-consistency by minimizing the function

$$F=[\mathrm{Re}D_{\mathrm{out}}(\nu_{R\ \mathrm{in}})]^2+[\Gamma_{\mathrm{in}}\nu_{R\ \mathrm{in}}{}^{l}-N_{\mathrm{out}}(\nu_{R\ \mathrm{in}})\times(\nu_{R\ \mathrm{in}}-\nu_0)]^2.$$

Figure 3(b) is a contour plot of it. This contour plot illustrates an important point in bootstrap calculations. We see a long shallow valley roughly along the line

$$\Gamma\nu_R=\nu_R.$$

Instead of finding the absolute minimum and taking those values of $(\Gamma\nu_R,\nu_R)$ as the solution, one ought to deduce that only Γ is fixed by the minimization process to be ≈ 1 and that ν_R is a free parameter. In this way, one can hope to determine the number of free parameters that the model predicts.[8]

TABLE IV. Self-consistent values. Variation with $\{\nu_{ij}\}$, the pole positions.

$\{\nu_{ij}\}$		$\sim\nu_L$	ν_R	$\Gamma\nu_R$
−6.0	−50.0		3.33	2.525
−6.0	−25.0	−5	3.06	2.788
−10.0	−50.0		4.17	3.00
−6.25	−100.0		3.55	2.29
−12.3	−42.2	−10	4.41	3.22
−1.25	−4.46	−1	0.08	0.3
−2.480	−8.696	−2	0.93	1.17

[8] I am very grateful to Dr. R. F. Streater for information on this point.

Looking first at the $I=I$, $J=1$ channel, the static results for variation of $\{\nu_{Fi}\}$ and n are shown in Figs. 4. The variation of ν_0 from −3 to 0 produced no detectable change in $\mathrm{Re}A(\nu)$ to 1% over the entire range $[-9,+10]$. The dynamic studies show a similar insensitivity to ν_0. The dynamic results are shown in Table III. Sensitivity to $\{\nu_{Fi}\}$ is very marked and makes predictions by the Balázs method so unreliable as to be meaningless. The significance of the $\{\nu_{Fi}\}$ −2.8, −2.2, −1.6 is that these are the positions used by Kanki and Tubis[4] in a Balázs $\pi\pi$ calculation using a different crossing continuation. Their result was (3.2, 3.0).

The pole positions were also varied to find what effect they would have. The results are shown in Table IV. This enabled the cutoff ν_L in the kernel approximation to be varied and the effect seen. From Table IV, the values of $(\nu_R,\Gamma\nu_R)$ are fairly sensitive to the pole positions but not as much as to the symmetry-point positions. For the last two entries, the symmetry points had also to be moved in, in proportion, to satisfy the condition that $\{\nu_{Fi}\}$ not too close to ν_L or $-\nu_R-1$.

Possibly more striking than the $\{\nu_{Fi}\}$ dependence is the n dependence where the position and width change drastically and the solution even disappears for some n. In Fig. 4(b) the first six-pole fit has $\mathrm{Re}A(\nu)=0$ when $\nu\sim 4$, because of a zero of $\mathrm{Re}N$, so the phase shift is zero, indicating that 6(1) pole positions do not belong to the same approximation as the others. The seven-pole fit has $\mathrm{Re}A(\nu)=0$ at two points, the first being due to a zero of $\mathrm{Re}N$ and the second to a zero of $\mathrm{Re}D$, which are very close together for both derivative and distributed matching and are presumably striving to cancel each other out. Whether it comes back for larger n as one might expect is not clear. From the n dependence it is suggested that the occurrence of resonances is an accident of the details of approximation. In the P_1 case, resonances only occur in $n=2$, $n=4$ because this form is a ratio of two cubics (quintics) the denominator having its zeros to the left and the numerator thus dominating the shape.

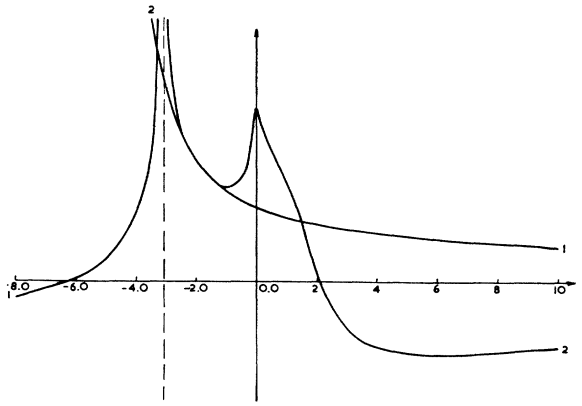


FIG. 5. S-wave self-consistent solution. Two-pole S_0 single channel.

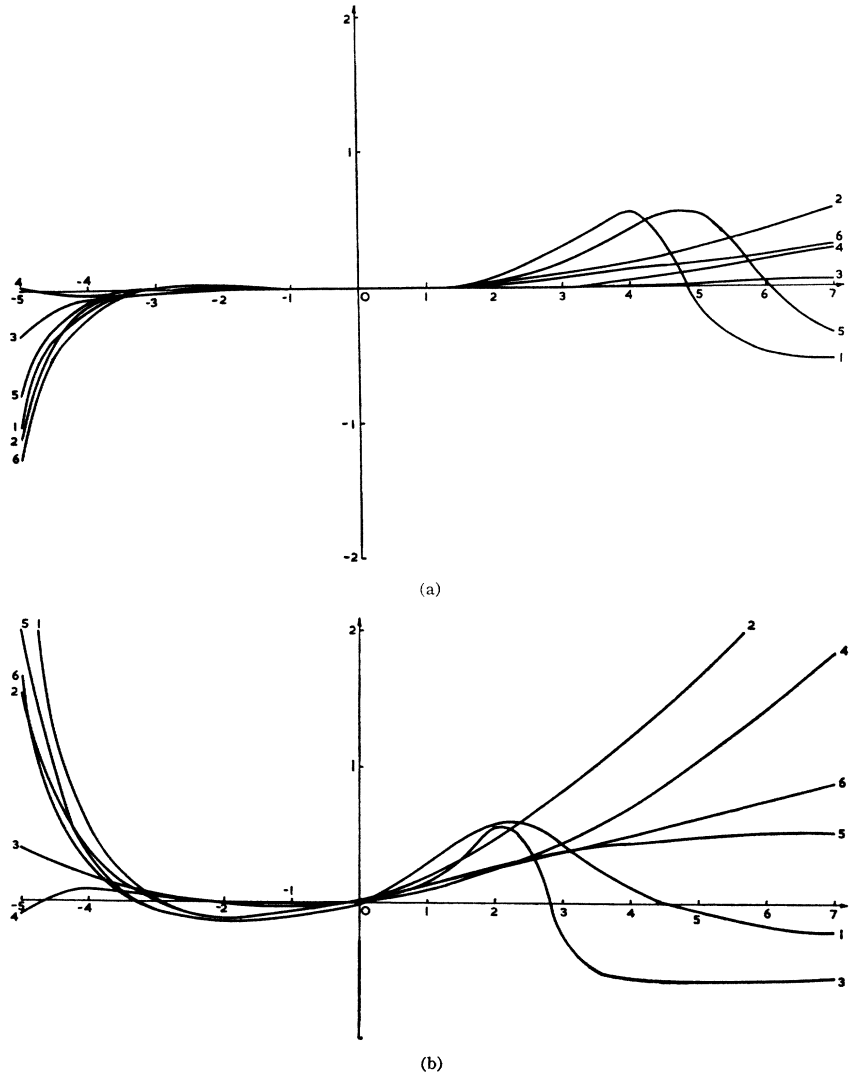


FIG. 6. The P_1–D_0 calculation. For P_1, $\nu_R=5.76$, $\Gamma\nu_R=5.0$; for D_0, $\nu_R=9.37$, $\Gamma\nu_R{}^2=4.58$. (a) Reflections into the P wave. (1) ReA, $P+D\to P$. (2) ReC, $P+D\to P$. (3) ReA, $D\to P$. (4) ReC, $D\to P$. (5) ReA, $P\to P$. (6) ReC, $P\to P$. (b) Reflections into the D wave. (1) ReA, $P+D\to D$. (2) ReC, $P+D\to D$. (3) ReA, $D\to D$. (4) ReC, $D\to D$. (5) ReA, $P\to D$. (6) ReC, $P\to D$.

The dependence of the solution upon the parameters of the kernel approximation is shown in Table IV. The most suitable choice of ν_L is presumably $-\nu_R-1$, however there would then be a big jump in changing to a distributed Im$A(\nu')$ with cut starting at -1.

Mehta and Srivastava conclude that no solutions exist for $\nu_F<-1.5$, our results show this is not the case.

V. OTHER SINGLE-CHANNEL BOOTSTRAPS

Table V shows the results of some searches for resonances in other channels, taken singly. The 2-pole S-wave is not really sensible as the N/D form has a cusp at threshold, see Fig. 5.

The calculation gives values of ν_R and $G=\Gamma\nu_R{}^l$ and

TABLE V. Single-channel searches. Values of $(\nu_R,\Gamma\nu_R)$.[a]

n $\{\nu_{Fi}\}$ Channel	2 distributed	2 derivative	3 distributed	3 derivative	4 distributed	4 derivative
S_0	...	2.048, 2.166	...	...	...	...
S_2	...	1.80, 1.79	...	...	...	...
P_1	2.57, 1.76	3.4, 2.6	...	...	1.86, 0.06	2.6, 0.26
D_0	...	...	...	...	2.0, 0.1	...
D_2	...	...	...	...	1.7, 0.06	...
F_1			...	...	...	...

[a] Dots indicate search done but no solution found.

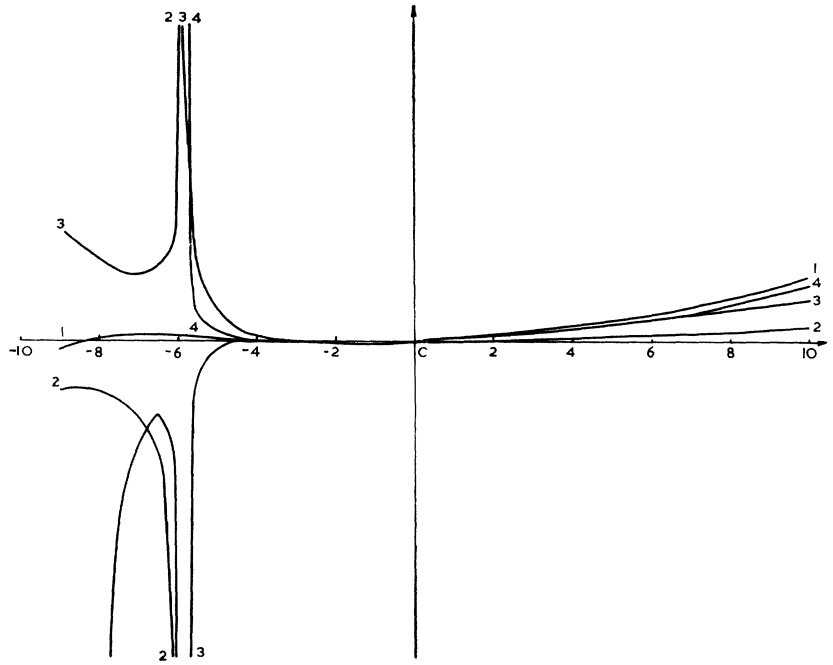


FIG. 7. Variation of n for $D_0 \rightarrow P_1$. (1) ReC. (2) $n=2$. (3) $n=3$. (4) $n=4$.

the subsequent interpretation of the width Γ as $G/\nu_R{}^l$ is a matter of conjecture.

The comparison of a δ-function input with a continuous output must lead to a lot of uncertainty as to the predicted value of the width. We did some dynamic studies taking the input width to be multiplied by a factor f, so, in the function $C(\nu)$, $\Gamma \rightarrow f\Gamma$. In the self-consistency condition (10), $\Gamma_{\text{out}}=\Gamma_{\text{in}}$, of course, Γ_{in} was taken as Γ. Self-consistent values were obtained for several values of f, in the 2-pole P_1 single-channel calculations, and it was found that $\nu_R \propto f$, $\Gamma\nu_R \propto f$.

TABLE VI. P_1–D_0 self-consistent values.

(1) For different $\{\nu_{Fi}\}$				P_1		D_0	
$\{\nu_{Fi}\}$				ν_R	$\Gamma\nu_R$	ν_R	$\Gamma\nu_R{}^2$
−2.1	−2.05	−1.0	0.0	4.53	4.26	4.817	1.378
−2.1	−2.05	−0.5	0.0	4.26	4.08	3.74	0.84
−2.1	−2.05	−0.2	0.0	4.02	3.82	3.09	0.58
−2.1	−2.05	−0.05	0.0	4.39	4.37	3.40	0.75
−2.1	−2.05	−1.5	0.0	4.74	4.33	5.90	2.007
−2.1	−2.05	−1.95	0.0	4.89	4.37	6.90	2.667

(2) With derivative configuration $\{\nu_{Fi}\}$

P_1	$\nu_R=5.76$	$\Gamma\nu_R=5.0$
D_0	$\nu_R=9.37$	$\Gamma\nu_R{}^2=4.58$

which is near Balázs's result

(3) Possible multiple solution with $\{\nu_{Fi}\}=-2.05, -1.95, -0.05, 0.0$ in both P_1 and D_0 waves

Solution 1

P_1	$\nu_R=6.663$	$\Gamma\nu_R=6.548$
D_0	$\nu_R=6.691$	$\Gamma\nu_R{}^2=2.766$

Solution 2

P_1	$\nu_R=4.378$	$\Gamma\nu_R=4.372$
D_0	$\nu_R=3.228$	$\Gamma\nu_R{}^2=0.693$

both appeared to satisfy the self-consistency requirements

VI. COUPLED CHANNELS

Simultaneous self-consistency of two and three channels was searched for in the cases S_0–S_2, 2 pole; S_0–P, 2 pole; D_0–D_2, 4 pole; P_1–D_0–D_2, 4 pole; S_0–S_2–P, 2 pole. Only one search was done, i.e., minimization of the discrepancy from given initial values, and the only solutions found were for the case found by Balázs in BIII, viz. P_1–D_0, 3 pole. However, the sensitivity of the calculation to the parameters and the pathology of the discrepancy function lead one to believe that the search is a difficult one. We even found a possible multiple solution, i.e. two sets of values both satisfying the self-consistency conditions. See Table VI. A pictorial breakdown of the P_1–D_0 calculation is shown in Figs. 6. Note the smallness of the $D_0 \rightarrow D_0$ and $D_0 \rightarrow P_1$ interaction, so that the D_0 wave comes mainly to P_1 exchange modified by D_0 exchange and the P_1 by self-stability modified numerically by D_0 exchange. The effect of the D_0 upon the P_1 channel for different n is shown in Fig. 7 and P_1 upon D_0 for different n in Fig. 8(c). Again we see the accidental nature of the solutions in the P_1–D_0 case, where only $n=3$ can give a D_0 resonance as seen from Fig. 8(c). Hence only $n=3$ can give P_1–D_0 self-consistent values. One expects on intuitive grounds that the self-consistency will become better resolved, i.e., there will be sharper minima for a larger number of component systems but that, due to complication, solutions will be harder to find.

An attempt was made to find the effect of the ρ upon other channels by taking the ρ pole only to give the left-hand side and then solving by matching. The results are shown in Figs. 8.

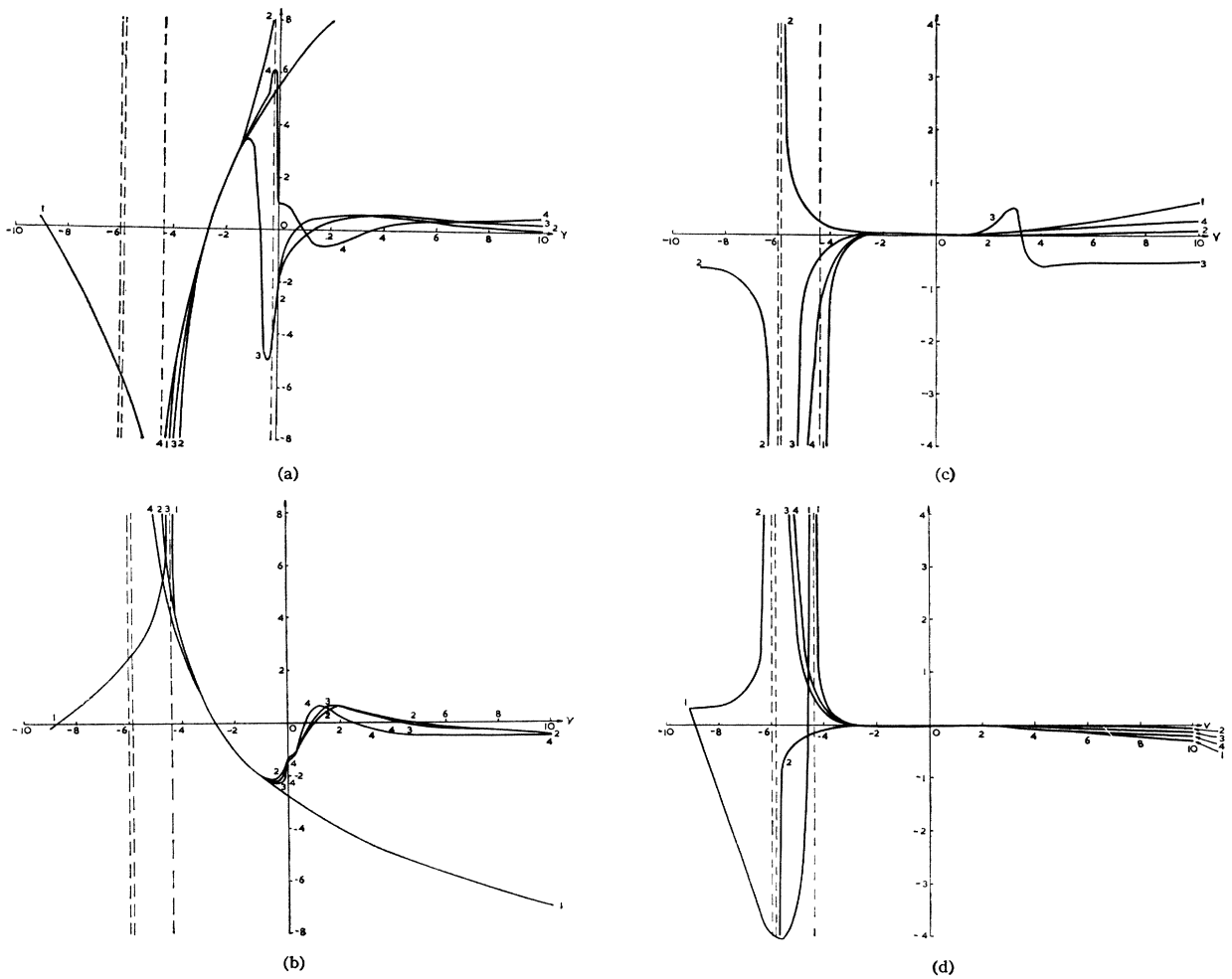


FIG. 8. Effect of the ρ on other channels. $\nu_R=3.4$, $\Gamma\nu_R=2.6$. (a) $P_1 \to S_0$; (b) $P_1 \to S_2$; (c) $P_1 \to D_0$; (d) $P_1 \to D_2$. In (a), (b), (c), (d), (1) ReC; (2) $n=2$; (3) $n=3$; (4) $n=4$.

VII. CONCLUSIONS

Clearly, the method as used by Balázs is very sensitive to $\{\nu_{Fi}\}$ and to n and is thus unreliable for making predictions about the $\pi\pi$ system.

The kernel approximation seems reasonable where ν is sufficiently far from the cut. It is not obviously equivalent to a cutoff. The pole positions are close to those obtained by a Gaussian integration procedure. The determination of the $\{F^i\}$ by matching points $\{\nu_{fj}\}$ is justified provided we have a reasonable estimate of the value of the integral and provided the $\{\nu_{Fj}\}$ are sufficiently far from the $\{\nu_i\}$. Herein almost certainly lies the source of instability and is strongly indicated by the sensitivity of $\{F^i\}$ to $\{\nu_{Fj}\}$. In short, $-\nu_R-1$ is too close to $\{\nu_{Fj}\}$ and $\{\nu_{Fj}\}$ are too close to $\{\nu_i\}$.

As one must take $\nu_{Fj}<0$ and $\nu_L \ll \nu_{Fj}$ one must have a special treatment of the near left, if only in specifically neglecting it, and use the approximation only for the far left.

Other authors have almost without exception used the kernel approximation only as a part of a more detailed treatment of the unphysical cuts. The method is insensitive to variation of the normalization point ν_0. The uncertainty of interpretation of the width can lead to comparable uncertainties in both position and width of the self-consistent solution.

Note added in proof. In approximating (1a) by (4), we have neglected contributions from the high-t' part of the integral. Two different ways of dealing with this have been suggested, by Singh and Udgaonkar[3] and Balázs (BV).[1] The present author has considered the integral (2) with $\mathrm{Im}A_{l'}{}^{I'}(\nu')$ given by a Breit-Wigner formula, so $\mathrm{Im}A_{l'}{}^{I'}(\nu') \to$ constant. The integral converges $\nu' \to \infty$ for $l \geq 1$, since $P_{l'} \to$ constant and $Q_l \to (\nu')^{-l-1}$ and $\nu' \to \infty$. Thus, for a small number of partial waves in the crossed channel, the high-energy region can only make a small contribution to this integral. The author has evaluated the integrals

numerically and found only a small change upon moving a cutoff in the integral from $\nu'=10$ to $\nu'=30$. For small values of $|\nu|$, however, there will be accumulation of partial waves of high l', producing the diffraction peak in the t channel. In this case, one may well get a large contribution from high-l' values. This will only apply in the diffraction region, say $0>\nu>-2$.

It has been pointed out to the author that the extreme insensitivity of the static studies to variation of ν_0 is probably a rigorous result derivable from existing theorems. In fact it does follow easily but not obviously. Here we prove that a once subtracted N/D pole form, with given left-hand pole positions ν_i, whose $n+1$ parameters are determined by matching to values C_j at points ν_{Fj}, is independent of the choice of subtraction point. For, given $A=N/D$,

$$N(\nu)=A_0+(\nu-\nu_0)\sum_{i=1}^{n} F^i/(\nu-\nu_i), \quad D(\nu)=1-\left(\frac{\nu-\nu_0}{\pi}\right)\int_0^\infty d\nu' R(\nu')\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{N(\nu')}{(\nu'-\nu)(\nu'-\nu_0)},$$

one can construct an N', D' with the same form such that $A'\equiv N'/D'=A$, with different subtraction point ν_0'. This can be verified by direct manipulation; N' and D' differ from N and D by a factor $D(\nu_0')$. Then, if A obeys the conditions $A(\nu_{Fj})=C_j$, $j=1, \cdots, n+1$ these will be linear equations for A_0, $\{F^i\}$ and will uniquely determine them, and hence $A(\nu)$ at all points. Since $A=A'$ at all points, in particular $A'(\nu_{Fj})=C_j$, $j=1, \cdots, n+1$. Thus, we can construct explicitly an identical N/D amplitude with different subtraction point and obeying the same matching conditions. This proves the desired result.[9]

The small ν_0 dependence of the dynamic studies is due to the ν_0 dependence of the self-consistency criterion given by Eq. (11).

ACKNOWLEDGMENTS

The author wishes to thank C. Lovelace for suggesting this problem, R. F. Streater for helpful discussions, and R. G. Moorhouse and W. Walkinshaw for their hospitality at the Rutherford Laboratory.

[9] The author is grateful to D. Morgan for help with this proof.