PHYSICAL REVIEW VOLUME 147, NUMBER 4 29 JULY 1966

Electromagnetic Mass Splittings of the N and N^*(1238 MeV)

GORDON L. SHAW*
University of California, Irvine, California

AND

DAVID Y. WONG†‡
University of California, San Diego, La Jolla, California

(Received 6 December 1965)

We examine the Dashen-Frautschi calculation of the neutron-proton mass difference $\delta_{n,p}$. Their $SU(2)$ calculation considers the nucleon to be a πN bound state with the dominant forces due to nucleon and N^* (1238-MeV) exchange. $\delta_{n,p}$ then depends linearly on $\delta_{-,++}$ (the mass difference between the N^*'s with charges $-$ and $++$) and the one-photon-exchange driving term Γ. [We note that this $SU(2)$ model predicts $\delta_{0,+}=\frac{1}{3}\delta_{-,++}$.] The N^* is calculated as a πN resonance with N and N^* exchange as the forces. This gives another relation among $\delta_{-,++}$, $\delta_{n,p}$, and Γ. Now in the static Chew-Low theory with a linear D function, the N-N^* reciprocal bootstrap conditions on the residues are exactly satisfied. In this case we show that $\delta_{n,p}$ (and $\delta_{-,++}$) is infinite. (Following Gerstein and Whippman, this divergence is seen to be a general consequence of the static, linear-D, reciprocal bootstrap conditions.) Thus it is only the deviations from the static Chew-Low theory with linear D which give a finite $\delta_{n,p}$. Dashen and Frautschi consider two such effects: (a) They show that the N^* exchange force is suppressed (by a factor of 0.6) because of the detailed shape of the resonance. (b) The physical D function must approach a constant at high energy, and they choose the simple rational form $D\propto(W-M)/(W-7M/3)$ for the P_{11} partial wave which simulates the D function calculated by Balázs. This choice for D leads to an additional suppression of the N^* exchange force. We concentrate our criticism on the nature of the D function. We note that the Balázs D function corresponds to a P_{11} partial wave with a negative definite phase shift, in contradiction to experiment. Using results of πN phase-shift analyses, we calculate the D functions and find that the N^* exchange contribution to the binding of the nucleon is enhanced relative to the linear form for D. Depending on the high-energy behavior of these phase shifts, not only can the calculated $\delta_{n,p}$ have the wrong magnitude, but also the wrong sign. We conclude that the calculation of $\delta_{n,p}$ depends critically on the details of the strong interactions. On the other hand, the ratio $\delta_{-,++}/\delta_{n,p}$ is insensitive to these details and is predicted to be $\sim$3. Thus a less ambitious point of view is to use the experimental value of $\delta_{-,++}(=7.9\pm6.8$ MeV) to get a rough value of $\delta_{n,p}$ (or vice versa).

THERE have been many theoretical attempts following that of Feynman and Speisman[1] to calculate the mass splitting $\delta_{n,p}$ between the neutron and the proton which experimentally is 1.3 MeV. This electromagnetic splitting is calculated from self-energy diagrams (usually keeping only the nucleon-photon intermediate state) with the form factors providing the high-energy cutoff to the integrals. However, the integrals are sensitive to the high-energy behavior of the form factors,[2] i.e., they are sensitive to the details of the strong-interaction dynamics.

Recently, Dashen and Frautschi[3–6] (DF) have introduced a new approach to the problem of determining $\delta_{n,p}$ from the one-photon-exchange diagrams and the shift in the position of the strong-interaction poles due to the electromagnetic splitting of the exchanged and external masses. The N/D equations are used to describe the partial-wave amplitudes. They suggested that their result is insensitive to the exact details of the strong-interaction dynamics. The purpose of this paper is to examine this statement. Following the work of DF very closely, we will make the same approximations to their general equations that they employ except for the nature of the D functions. Here we make use of experimental values of the phase shifts to construct D. We find that $\delta_{n,p}$ is sensitive to the details of the strong interactions: not only is the magnitude uncertain, but also the sign. The result, as we shall see, is not surprising but is due primarily to the fact that the "lowest order approximation" to the problem yields a divergent $\delta_{n,p}$.

DF consider the exchange of the nucleon and the N^*(1238 MeV) to give the dominant forces producing the nucleon as a bound state in the $J=\frac{1}{2}$, $I=\frac{1}{2}$ partial wave and the N^*(1238 MeV) as a resonance with $J=\frac{3}{2}$, $I=\frac{3}{2}$. With this $SU(2)$ model[7] for the N (and N^*), DF treat the electromagnetic splittings of the external and exchanged particles as perturbations with the elastic πN one-photon-exchange diagram (infrared contributions appear in the form of a nonzero-mass photon in the propagator) providing the driving term Γ. Let M^* be the mass of the N^* (and M the mass of the nucleon) and define

$$\delta_{0,+}=M^{*0}-M^{*+},\qquad \delta_{-,++}=M^{*-}-M^{*++}. \tag{1}$$

* Supported in part by the National Science Foundation.
† Alfred P. Sloan Foundation Fellow.
‡ Supported in part by the U. S. Atomic Energy Commission.
[1] R. Feynman and G. Speisman, Phys. Rev. **94**, 500 (1954).
[2] M. Cini, E. Ferrari, and R. Gato, Phys. Rev. Letters **2**, 7 (1959); S. Sunakawa and K. Tanaka, Phys. Rev. **115**, 754 (1959); H. Katsumori and M. Shimada, *ibid.* **124**, 1203 (1961); A. Solomon, Nuovo Cimento **27**, 748 (1963).
[3] R. Dashen and S. Frautschi, Phys. Rev. **135**, B1190 (1964).
[4] R. Dashen, Phys. Rev. **135**, B1196 (1964).
[5] R. Dashen and S. Frautschi, Phys. Rev. **137**, B1318 (1965).
[6] R. Dashen and S. Frautschi, Phys. Rev. **137**, B1331 (1965).
[7] The neglect of K-hyperon channels seems reasonable.

Then the $SU(2)$ model predicts

$$\delta_{-,++}=3\delta_{0,+}. \tag{2}$$

Now following DF[6] we evaluate their Eqs. (21) of Ref. 5 by approximating the short N and N^* exchange cuts by poles, evaluating the mass shifts due to the external nucleon using mass scale invariance of the unperturbed (by electromagnetic effects) solution,[8] ignoring the effects due to the electromagnetic splittings of the coupling constants, and inelastic channels. We obtain

$$\delta_{n,p}=-(1/27)(5+8\beta_{13})\delta_{n,p}+(40/81)\beta_{13}\delta_{-,++}+\Gamma \tag{3}$$

for the shift in the nucleon pole in the $J=\frac{1}{2}$ amplitude due to electromagnetic effects. Similarly from the shift in the N^* position in the $J=\frac{3}{2}$ amplitude,

$$\delta_{-,++}=\tfrac{1}{9}(9+16\beta_{31}+\beta_{33})\delta_{n,p}+(1/27)\beta_{33}\delta_{-,++}+\Gamma^*, \tag{4}$$

where

$$\beta_{13}=C\left(2\frac{\gamma_{33}}{\gamma_{11}}\right)\left(\frac{D_{11}^2(W)}{W-M}\right)'\bigg|_{2M-M^*}(D_{11}'(W)|_M)^{-2},$$

$$\beta_{31}=\left(\frac{\gamma_{11}}{2\gamma_{33}}\right)\left(\frac{D_{33}^2(W)}{W-M^*}\right)'\bigg|_M(D_{33}'(W)|_{M^*})^{-2}, \tag{5}$$

$$\beta_{33}=C\left(\frac{D_{33}^2(W)}{W-M^*}\right)'\bigg|_{2M-M^*}(D_{33}'(W)|_{M^*})^{-2}$$

with D_{11} and D_{33} the D functions for the unperturbed $J=\frac{1}{2}, I=\frac{1}{2}$, and $J=\frac{3}{2}, I=\frac{3}{2}$ partial waves, γ_{11} the residue of the nucleon pole, and γ_{33} the residue of the N^* resonance. DF introduce the factor C because the detailed shape of the N^* resonance reduces the effective N^* exchange force. They evaluate the cancellation of the positive and negative portions of the N^* exchange cut and quote a value of

$$C=0.6. \tag{6}$$

More compactly we can write (4) and (5) as

$$A\begin{pmatrix}\delta_{n,p}\\ \delta_{-,++}\end{pmatrix}=\begin{pmatrix}\Gamma\\ \Gamma^*\end{pmatrix}, \tag{7}$$

where the matrix A depends only on the strong-interaction dynamics.

The crucial point is that the simplest model for the strong interactions leads to an A which has zero determinant and hence divergent δ's: The static Chew-Low theory[9] in the narrow resonance approximation ($C=1$) and the *linear approximation for the D functions* yields a solution to the N-N^* reciprocal bootstrap equations (and predicts $\gamma_{33}/\gamma_{11}=\frac{1}{2}$, in agreement with experiment). In this model (setting $C=1$) all the B_{ij} in (5) are 1 which immediately leads to an A with zero determinant.[10] The same result for A follows in the static Chew-Low model with linear D and $C=1$ regardless of the spin, parity, and isospin of the N, N^*, and π, as long as the reciprocal bootstrap equations have a solution. Following Gerstein and Whippman,[11] we show this in the Appendix.

Thus it is the deviations from the static Chew-Low N-N^* reciprocal bootstrap theory with linear D function which give a finite $\delta_{n,p}$. DF consider two such deviations: (1) Their factor C due to the detailed shape of the N^* reduces the N^* exchange contribution. (The N-N^* reciprocal bootstrap does not then have a solution. Contributions from other forces could presumably remedy this.) (2) The second feature they consider is the nonlinearity of the unperturbed D function. DF use a D function similar to one determined by Balázs.[12] This has the effect of further diminishing the contribution from N^* exchange factor β_{13} to 0.23. (β_{31} and β_{33} are also reduced by a "Balázs" D function.)

We will concentrate our criticism on the nature of D functions used by DF: We will discuss the reasons why the parametrized form they chose is inadequate. Then we use experimental values for the phase shifts to calculate the D functions and solve (7) for the mass splittings. The calculated splittings are very sensitive to the high-energy behavior of the phase shifts.

DF chose the simplest phenomenological form for the D function which had the desired characteristics that it approaches a constant as $W\to\infty$ and goes through zero at the bound state or resonance. For the unperturbed $I=\frac{1}{2}$, $J=\frac{1}{2}$ D function they used

$$D_{11}=(W-M)(M-M')/(W-M'), \tag{8}$$

with M' taken equal to $7M/3$ to simulate the D function calculated by Balázs. D_{11} as given by (8) has the feature that its slope continually decreases for $W<M$ which leads to the suppression of the N^* exchange factor β_{13}. Thus the form (8) completely prejudices the issue of the nonlinearity of the D function. To discuss this quantitatively, we consider the form in which D is written as the exponential of an integral over the phase shift. We write the $J=\frac{1}{2}$, $I=\frac{1}{2}$ S matrix as

$$S_{11}=\eta_{11}e^{2i\alpha_{11}}, \tag{9}$$

where η_{11} is the inelastic factor and α_{11} is the real-part phase shift. The results of the extensive energy-dependent complex phase-shift analyses[13–15] determine

[8] See, e.g., Eqs. (43′) and (44) of Ref. 5. Note that terms of order $(M^*-M)/M$ are neglected. Strictly speaking, mass scale invariance is valid only if all allowed channels are explicitly included.

[9] G. Chew, Phys. Rev. Letters **9**, 23 (1962); F. Low, *ibid.* **9**, 279 (1962).

[10] We note that in Ref. 6, p. B1346, a statement was made to the contrary. It follows, however, from Tables XI–XX of Ref. 6 that this statement is erroneous.

[11] I. Gerstein and M. Whippman, Ann. Phys. (N. Y.) **34**, 488 (1965).

[12] L. Balázs, Phys. Rev. **128**, 1935 (1962).

[13] L. Roper, Phys. Rev. Letters **12**, 340 (1964); L. Roper and R. Wright, Phys. Rev. **138**, B921 (1965).

[14] P. Auvil, A. Donnachi, A. Lea, and C. Lovelace, Phys. Letters **12**, 76 (1964).

[15] P. Bareyre, C. Brickman, A. Stirling, and G. Villet, Phys. Letters **18**, 342 (1965).

the behavior of both α_{11} and η_{11}. α_{11} starts off negative and small ($\geq -2°$) but quickly turns over and becomes large and positive, going through $\pi/2$ at incident pion laboratory kinetic energy $E_L \sim 600$ MeV ("Roper resonance"). Now let D_{11} be free of left-hand cuts and D_{11}^*/D_{11} have the phase of S_{11}.[16] Then, assuming that the "Roper resonance" as well as the nucleon bound state are predominately due to forces in the πN channel, we find[17,18]

$$D_{11}=(W-M) \times \exp\left[-\frac{W-M}{\pi}\int_{M+1}^{\infty}\frac{\alpha_{11}(W')dW'}{(W'-W)(W'-M)}\right] \quad (10)$$

with $\alpha_{11}(\infty)=-\pi$. On the other hand, if the "Roper resonance" is mainly due to inelastic channels as suggested by the small value of η_{11}, then a pair of apparent "Castillejo-Dalitz-Dyson" (CDD) zeros in S_{11} at $W=W_R \pm iW_I$ appear on the physical sheet.[19] Then [20] $\alpha_{11}(\infty)=0$ and we replace (10) by the form

$$D_{11}=(W-M)\left[\frac{(M-W_R)^2+W_I^2}{(W-W_R)^2+W_I^2}\right]^{1/2} \times \exp\left[-\frac{W-M}{\pi}\int_{M+1}^{\infty}\frac{\alpha_{11}(W')dW'}{(W'-W)(W'-M)}\right]. \quad (11)$$

Note that this form [as well as (10)] approaches a constant as $W\to\infty$.

The P_{11} phase shift calculated by Balázs[12] is always negative and as a result the exponential factor in (10) would be less than one for $W<M$. Thus we see that the large suppression of the N^* exchange term in (3) found by DF is a result of using a negative definite P_{11} phase shift in contradiction with experiment.

The existing phase-shift analyses[13–15] extend up to $E_L \sim 1$ BeV.[21] We employ these results to determine the D functions: Above this energy we let α_{11} go smoothly to $-\pi$ and 0 as $W\to\infty$ when evaluating forms (10) and (11), respectively. Explicitly, we use

$$\alpha_{11}(W)=-8\pi q^3(W-8.5)(W-W_0)/(W-2.0)^5 \quad (12)$$

for Eq. (10) and

$$\alpha_{11}(W)=-W_1 q^3(W-8.5)/(W-2.0)^5 \quad (13)$$

for Eq. (11). Here, q is the center-of-mass momenta, W_0 and W_1 are adjusted so that $\alpha_{11}=\pi/2$ at $W=10.7$ (Roper resonance) in (12) and (13), respectively. The above expressions give a good fit to the experimental phase shift below 1 BeV. When using (11), we choose as an example a CDD zero near the pole of the DF D_{11} function:

$$W_R=16, \quad W_I=2. \quad (14)$$

A cutoff form of (11) is also considered with α_{11} set equal to zero for $W>15$. In all the above cases, we find that the nonlinearity of D_{11} enhanced the N^* exchange term in (3) instead of reducing it as the DF form (8) did. To calculate the D function for the P_{33} amplitude (in order to calculate β_{31} and β_{33}) we use the form

$$D_{33}=\exp\left[-\frac{1}{\pi}\int_{M+1}^{\infty}\frac{\alpha_{33}(W')dW'}{(W'-W)}\right] \quad (15)$$

with $\alpha_{33}(\infty)=0$. (There seems to be no CDD ambiguity for this state.) A good fit to the experimental phase shift is given by

$$\alpha_{33}(W)=W_2 q^3/(W-3.8)^4, \quad (16)$$

with W_2 adjusted to give $\alpha_{33}=\pi/2$ at $W=M^*=8.8$.

Thus we calculate D_{11} and D_{33} using (10)–(16) and hence determine the A matrix, Eq. (7), from (3)–(5).

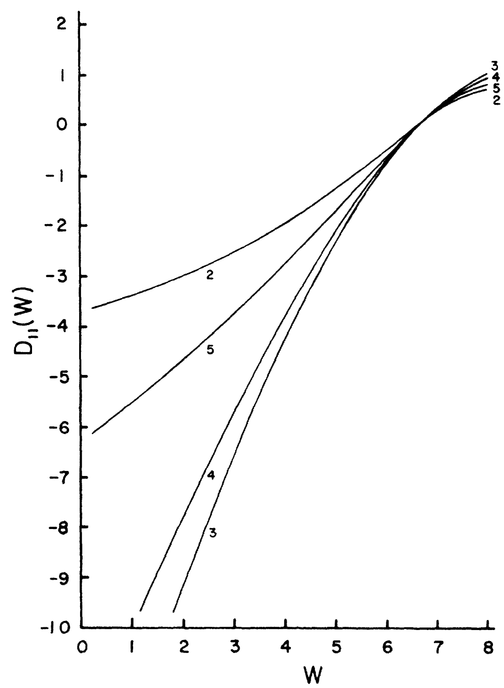

FIG. 1. Plots of the D_{11} function versus W corresponding to the cases 2–5 described in Table I.

[16] This leads to the Frye-Warnock [Phys. Rev. 130, 478 (1963)] formalism for the N/D equations.

[17] We ignore the $-W$ contribution from the S_{11} phase shift. Similarly the $-W$ contribution from the D_{33} phase shift is neglected in (16).

[18] We use units $\hbar=c=m_\pi=1$.

[19] M. Bander, P. Coulter, and G. Shaw, Phys. Rev. Letters 14, 270 (1965).

[20] R. Warnock, Phys. Rev. 131, 1320 (1963); J. Hartle and C. Jones, *ibid.* 140, B90 (1965); Ann. Phys. (N. Y.) (to be published). The presence of the CDD zeros in S_{11} implies that inelastic channels are important. However, the present treatment, following that of DF, neglects explicit inelastic contributions to the right-hand branch cut. Note that expression (11) contains a pair of complex CDD poles in D_{11}^2 which could be considered as an extra driving term. This is also neglected. A multichannel calculation would eliminate these poles.

[21] They are well determined up to ~700 MeV.

TABLE I. Calculated values of mass shifts using different approximations for the D functions.

Case	Type of D function	C	$\delta_{n,p}$ (MeV)	$\delta_{-,++}$ (MeV)	$\delta_{-,++}/\delta_{n,p}$
1	$D_{11}=(W-M)$, $D_{33}=(W-M^*)$	1	∞	∞	3
2	$\begin{Bmatrix} D_{11}=(W-M)(M-15.5)/(W-15.5) \\ D_{33}=(W-M^*)(M^*-15.5)/(W-15.5) \end{Bmatrix}$	0.6	+1.8	+7.0	3.9
3	Form (10) of present paper for D_{11} and (15) for D_{33}. Expressions (12) for P_{11} and (16) for P_{33} phase shifts with no cutoff.	0.6	−1.9	−4.7	2.5
4	From (11) for D_{11} and (15) for D_{33}. Expressions (13) P_{11} and (16) for P_{33} phase shifts with no cutoff.	0.6	−2.3	−6.5	2.9
5	Same as case 4 but with cutoff at $W=15$.	0.6	+6.4	+22.9	3.6

The photon exchange driving terms Γ and Γ^* depend not only on the $\pi\pi$ isovector form factor and the $N\bar{N}$ isoscalar form factor,[4] but on D_{11} and D_{33}, respectively. Since the major variation of the mass shifts with respect to the nonlinearity of D originates from the A matrix rather than the driving term, we evaluate Γ and Γ^* using linear D functions. Following Dashen[4] we have

$$\tfrac{1}{3}\Gamma^*=\Gamma=1.4\text{ MeV}. \tag{17}$$

The results of all the cases discussed above are presented in Table I. In order to illustrate the sensitivity of these results, we plot the D_{11} functions and the quantity $\beta(W)=[D_{11}^2(W)/(W-M)]'$ in Figs. 1 and 2, respectively. Although $\beta(\infty)$ vanishes in all the cases 2–5, solutions 3 and 4 will enhance higher mass exchange contributions. If these solutions correspond to physical D functions, then higher mass exchange effects are necessarily important. We note that there can also be cases like solution (5) where $\beta(W)$ peaks in the neighborhood of the N^* exchange pole ($W=2M-M^*=4.6$). In this case the mass shift results are also very different from those given by DF. Although the magnitude and the sign of the mass differences are strongly model-dependent, their ratio

$$\delta_{-,++}/\delta_{n,p}\approx 3 \tag{18}$$

for all cases considered.

A less ambitious point of view (as considered by DF) is to abandon the attempt to calculate both $\delta_{n,p}$ and $\delta_{-,++}$: Experimentally[22]

$$\delta_{-,++}=7.9\pm 6.8\text{ MeV}. \tag{19}$$

Using this determination of $\delta_{-,++}$ to calculate $\delta_{n,p}$ from (3), our models for D give the correct sign for $\delta_{n,p}$ but an uncertainty of a factor ~ 3 in the magnitude.

One of the interesting features of the DF formalism is that not only does it allow one to calculate electromagnetic mass splittings but also splittings of the coupling constants. Although their effects on the mass splittings appear to be small if the determinant of A is not near zero, the effect of the mass splittings on the coupling-constant shifts will again be sensitive to the nonlinearity of the D functions.

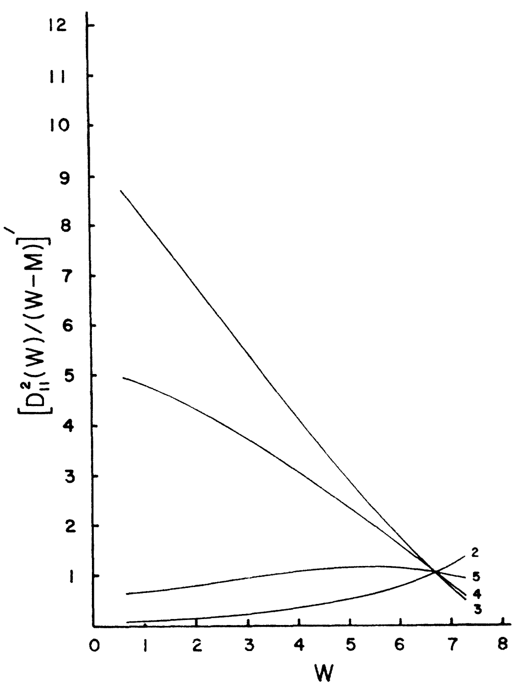

FIG. 2. Plots of the quantity $[D_{11}^2/(W-M)]'$ versus W corresponding to the cases 2–5 described in Table I.

APPENDIX

Consider the Chew-Low static model with a linear D function and the narrow-resonance approximation for the reciprocal bootstrap of N and N^*. Here we demonstrate that a generalization of this situation in which the meson M, baryon B, and resonance B^* have any particular spin, parity, and isospin leads to electromagnetic mass splittings[23] δ_3 for B and δ_3^* for B^* which

[22] G. Gidal, A. Kernan, and S. Kim, Phys. Rev. 141, 1261 (1966).

[23] Following Ref. 6 we use the subscript 3 to denote mass-splitting terms which are proportional to the charge and the subscript 1 to denote mass-splitting terms which are the same for each member of the multiplet.

are infinite. Let γ be the residue of the baryon pole and γ^* the residue of B^*. Then the static model which considers B and B^* exchange forces to produce the B and B^* gives

$$\begin{aligned}\gamma&=a\gamma+b\gamma^*,\\ \gamma^*&=c\gamma+d\gamma^*.\end{aligned}\tag{A1}$$

In order that a solution exist, it is necessary that

$$\begin{vmatrix}1-a & -b\\ -c & 1-d\end{vmatrix}=0$$

and thus

$$R=\gamma^*/\gamma=(1-a)/b=c/(1-d).\tag{A2}$$

Now from the Dashen-Frautschi formalism, using the approximations described in the sentence preceding Eq. (3) of the present paper, we have[23,24]

$$\begin{aligned}\delta_1&=(E_1^{BB\text{ ext}}+E_1^{BB\text{ exch}})\delta_1+E_1^{BB^*\text{ exch}}\delta_1^*+\Gamma,\\ \delta_1^*&=(E_1^{B^*B\text{ ext}}+E_1^{B^*B\text{ exch}})\delta_1+E_1^{B^*B^*\text{ exch}}\delta_1^*+\Gamma^*,\end{aligned}\tag{A3}$$

with

$$\begin{aligned}&E_1^{BB\text{ exch}}=-a, && E_1^{BB^*\text{ exch}}=-bR,\\ &E_1^{BB\text{ ext}}=(1+b+bR), && E_1^{B^*B\text{ exch}}=-c/R,\\ &E_1^{B^*B^*\text{ exch}}=-d, && E_1^{B^*B\text{ ext}}=(1+c/R+d).\end{aligned}\tag{A4}$$

Writing (A3) in form

$$A_1\begin{pmatrix}\delta\\ \delta^*\end{pmatrix}=\begin{pmatrix}\Gamma\\ \Gamma^*\end{pmatrix}\tag{A5}$$

we immediately have, using the bootstrap conditions (A2) that A_1 has zero determinant. This means that there is a uniform over-all mass shift of the B (and B^* multiplet) which is infinite. More important, however, are the δ_3 and δ_3^* splittings.[23] Let I_B, I_{B^*}, and I_M denote the isospins of the particles and define

$$B\equiv I_B(I_B+1),\quad B^*\equiv I_{B^*}(I_{B^*}+1),$$

and

$$M\equiv I_M(I_M+1).$$

The ratios of the E_3 to the E_1 coefficients are given by Gerstein and Whippman [Eq. (A7) of Ref. 11] as

$$\begin{aligned}&E_3^{BB\text{ exch}}/E_1^{BB\text{ exch}}=(B-M)/B,\\ &E_3^{BB^*\text{ exch}}/E_1^{BB^*\text{ exch}}\\ &\quad=E_3^{B^*B\text{ exch}}/E_1^{B^*B\text{ exch}}=(B-M)/[BB^*]^{1/2},\\ &E_3^{BB\text{ ext}}/E_1^{BB\text{ ext}}=1-M/2B,\\ &E_3^{B^*B^*\text{ exch}}/E_1^{B^*B^*\text{ exch}}=(B-M)/B^*,\\ &E_3^{B^*B\text{ ext}}/E_1^{B^*B\text{ ext}}=(B+B^*-M)/2(BB^*)^{1/2}.\end{aligned}\tag{A6}$$

Then from (A2), (A4), and (A6) we obtain the A matrix:

$$\begin{aligned}A_3&=\begin{bmatrix}1+a\left(\dfrac{B-M}{B}\right)-2\left(1-\dfrac{M}{2B}\right) & (1-a)\dfrac{B-M}{(BB^*)^{1/2}}\\ (1-d)\dfrac{B-M}{(BB^*)^{1/2}}-2\left(\dfrac{B+B^*-M}{2(BB^*)^{1/2}}\right) & 1+d\left(\dfrac{B-M}{B^*}\right)\end{bmatrix}\\ &=\begin{bmatrix}\dfrac{(a-1)(B-M)}{B} & \dfrac{(1-a)(B-M)}{(BB^*)^{1/2}}\\ -\dfrac{d(B-M)+B^*}{(BB^*)^{1/2}} & \dfrac{d(B-M)+B^*}{B^*}\end{bmatrix}.\end{aligned}\tag{A7}$$

Thus A has zero determinant and the mass splittings δ_3 and δ_3^* are infinite.

[24] The superscript notation ext and exch denote external and exchange so that, e.g., $E^{B^*B\text{ exch}}\delta$ is the contribution to the shift in position of the B^* due to a shift in the exchanged B mass.