Perturbations in the Schwarzschild Metric*

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Exact solutions of the general-relativistic field equations are known only for very simple physical systems. In order to obtain solutions of the field equations for more realistic systems, we consider an expansion of the field equations about some known exact solution, keeping only terms linear in the perturbation. We then choose the unperturbed metric to be the Schwarzschild metric, corresponding to the exterior of any spherically symmetric mass distribution, and consider the case where the perturbing matter is not located close to the Schwarzschild radius. A discussion of the solution for a scalar Green's function in curved space leads directly to an explicit expression for the perturbed metric in terms of the perturbing matter.

I. INTRODUCTION

RECENT astronomical observations have generated considerable interest in the broad category of relativistic astrophysics.[1] In particular, there is increasing probability that astronomical bodies may be found whose description requires the full framework of general relativity. Since the field equations of general relativity yield exact solutions only for oversimplified physical systems, one is forced to consider approximate solutions of the field equations if one wishes to solve physically realistic models. Most approximation methods which have been formulated take the lowest order metric to be that of flat space-time.[2] This is, of course, a reasonable approximation for most gravitational phenomena. However, for the case of systems such as gravitationally collapsed stars, this is a poor first approximation. One approach to this problem is to keep higher powers of the perturbations in the problem and solve the resulting nonlinear equations.[3] The approach we will follow here is to choose the lowest order metric to be some known exact solution of the field equations which approximates the physical system as closely as possible, and then consider the perturbations about this metric which arise when we replace the oversimplified physical model with a more realistic one.[4]

In Sec. II, we consider the general problem of expanding the field equations about a given metric. Equations relating the perturbed metric to the perturbing matter are derived. In Sec. III we specialize the equations to the case where the unperturbed metric is the first-order Schwarzschild metric. Section IV treats the solution of a scalar wave equation in curved space. The results are then applied in Sec. V to give an explicit expression for the perturbed metric.

II. EXPANSION OF THE FIELD EQUATIONS

We wish to solve the field equations of general relativity[5]

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-8\pi GT_{\mu\nu}\,, \tag{2.1}$$

with the boundary condition that $g_{\mu\nu}\rightarrow\delta_{\mu\nu}$ as $r\rightarrow\infty$. We assume that $T_{\mu\nu}$ can be written as the sum of two terms

$$T_{\mu\nu}=T_{\mu\nu}{}^{(0)}+\delta T_{\mu\nu}\,, \tag{2.2}$$

where $T_{\mu\nu}{}^{(0)}$ is chosen so that the field equations for $g_{\mu\nu}$ coupled only to $T_{\mu\nu}{}^{(0)}$ have an exact explicit solution, $g_{\mu\nu}=g_{\mu\nu}{}^{(0)}$. Therefore

$$R_{\mu\nu}{}^{(0)}-\tfrac{1}{2}g_{\mu\nu}{}^{(0)}R^{(0)}=-8\pi GT_{\mu\nu}{}^{(0)}\,, \tag{2.3}$$

where $R_{\mu\nu}{}^{(0)}$ is the Ricci tensor constructed only out of the $g_{\mu\nu}{}^{(0)}$. $\delta T_{\mu\nu}$ is then taken to be a correction to the $T_{\mu\nu}{}^{(0)}$ which we assume gives rise to a small change in $g_{\mu\nu}$; i.e., $T_{\mu\nu}=T_{\mu\nu}{}^{(0)}+\delta T_{\mu\nu}\Rightarrow g_{\mu\nu}=g_{\mu\nu}{}^{(0)}+\delta g_{\mu\nu}$, where $\delta g_{\mu\nu}\ll g_{\mu\nu}{}^{(0)}$. The equation for $\delta g_{\mu\nu}$ in terms of $\delta T_{\mu\nu}$ and the unperturbed metric $g_{\mu\nu}{}^{(0)}$ is found from considering a variation of Eq. (2.1):

$$\delta R_{\mu\nu}-\tfrac{1}{2}\delta g_{\mu\nu}R^{(0)}-\tfrac{1}{2}g_{\mu\nu}{}^{(0)}\delta g^{\alpha\beta}R_{\alpha\beta}{}^{(0)}-\tfrac{1}{2}g_{\mu\nu}{}^{(0)}g^{\alpha\beta(0)}\delta R_{\alpha\beta}=-8\pi G\delta T_{\mu\nu}\,. \tag{2.4}$$

Since

$$R_{\mu\nu}=-\left\{\begin{matrix}\alpha\\ \mu\ \nu\end{matrix}\right\}_{,\alpha}+\left\{\begin{matrix}\alpha\\ \mu\ \alpha\end{matrix}\right\}_{,\nu}-\left\{\begin{matrix}\gamma\\ \mu\ \nu\end{matrix}\right\}\left\{\begin{matrix}\alpha\\ \gamma\ \alpha\end{matrix}\right\}+\left\{\begin{matrix}\beta\\ \mu\ \alpha\end{matrix}\right\}\left\{\begin{matrix}\alpha\\ \beta\ \nu\end{matrix}\right\}, \tag{2.5}$$

where $\{{}_{\beta}{}^{\alpha}{}_{\gamma}\}$ is the Christoffel symbol

$$\left\{\begin{matrix}\alpha\\ \beta\ \gamma\end{matrix}\right\}\equiv g^{\alpha\sigma}[\beta\gamma,\sigma]\equiv\tfrac{1}{2}g^{\alpha\sigma}(g_{\beta\sigma,\gamma}+g_{\gamma\sigma,\beta}-g_{\beta\gamma,\sigma})\,, \tag{2.6}$$

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[1] In particular, see *Quasi-Stellar Sources and Gravitational Collapse*, edited by I. Robinson, A. Schild, and E. L. Schucking (The University of Chicago Press, Chicago, 1965), and in Proceedings of the Second Texas Conference on Relativistic Astrophysics (to be published).

[2] See, for example, L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962), Chap. 11.

[3] In situations such as gravitational radiation from nonrelativistic systems, one can obtain explicit solutions of these nonlinear equations. See P. C. Peters, Phys. Rev. **136**, B1224 (1964).

[4] C. Lanczos, Z. Physik **31**, 112 (1925).

[5] Greek indices take values from 0 to 3; Latin indices are restricted to spatial components 1 to 3. At any space-time point we may choose $g_{\mu\nu}=\delta_{\mu\nu}$, where $\delta_{\mu\nu}$ is defined by $\delta_{00}=1$, $\delta_{11}=\delta_{22}=\delta_{33}=-1$, and $\delta_{\mu\nu}=0$ for $\mu\neq\nu$. We shall choose a system of units in which $c=1$. Ordinary differentiation will be denoted by a comma (,), covariant differentiation by a semicolon (;).

then

$$\delta R_{\mu\nu}=-\left[\delta\left\{\begin{matrix}\alpha\\ \mu\ \ \nu\end{matrix}\right\}\right]_{,\alpha}+\left[\delta\left\{\begin{matrix}\alpha\\ \mu\ \ \alpha\end{matrix}\right\}\right]_{;\nu} \tag{2.7}$$

in a system of coordinates where the Christoffel symbols vanish. However $\delta\{{}_{\beta}{}^{\alpha}{}_{\gamma}\}$ is a tensor and therefore we can write (2.7) in covariant form immediately, valid now for any coordinate system,

$$\delta R_{\mu\nu}=-\left[\delta\left\{\begin{matrix}\alpha\\ \mu\ \ \nu\end{matrix}\right\}\right]_{;\alpha}+\left[\delta\left\{\begin{matrix}\alpha\\ \mu\ \ \alpha\end{matrix}\right\}\right]_{;\nu}. \tag{2.8}$$

The covariant differentiation, of course, is taken with respect to the unperturbed metric $g_{\mu\nu}{}^{(0)}$. The expression for $\delta\{{}_{\beta}{}^{\alpha}{}_{\gamma}\}$ in terms of $\delta g_{\mu\nu}$ can be found in a similar manner. In a system of coordinates where the first derivatives of the $g_{\mu\nu}$ vanish, we have

$$\delta\left\{\begin{matrix}\alpha\\ \beta\ \ \gamma\end{matrix}\right\}=\tfrac{1}{2}g^{\alpha\sigma(0)}[(\delta g_{\beta\sigma})_{,\gamma}+(\delta g_{\gamma\sigma})_{,\beta}-(\delta g_{\beta\gamma})_{,\sigma}], \tag{2.9}$$

which can be written in covariant form as

$$\delta\left\{\begin{matrix}\alpha\\ \beta\ \ \gamma\end{matrix}\right\}=\tfrac{1}{2}g^{\alpha\sigma(0)}[(\delta g_{\beta\sigma})_{;\gamma}+(\delta g_{\gamma\sigma})_{;\beta}-(\delta g_{\beta\gamma})_{;\sigma}]. \tag{2.10}$$

For notational convenience, we let $h_{\mu\nu}\equiv\delta g_{\mu\nu}$, $h_{\mu}{}^{\alpha}\equiv g^{\alpha\nu(0)}\times\delta g_{\mu\nu}$, etc., and also drop the superscript (0) on the unperturbed metric. Then $\delta R_{\mu\nu}$ becomes

$$\delta R_{\mu\nu}=\tfrac{1}{2}[h_{\mu\nu;\alpha}{}^{;\alpha}-h_{\mu\alpha;\nu}{}^{;\alpha}-h_{\nu\alpha;\mu}{}^{;\alpha}+h_{\alpha}{}^{\alpha}{}_{;\mu;\nu}]. \tag{2.11}$$

If (2.11) is now substituted in Eq. (2.4) and one uses the relation

$$\delta g^{\alpha\beta}=-\delta g_{\gamma\delta}g^{\gamma\alpha}g^{\delta\beta},$$

one finds the equation for $h_{\mu\nu}$, the perturbed metric, in terms of $\delta T_{\mu\nu}$, the perturbing stress-energy tensor,[4]

$$h_{\mu\nu;\alpha}{}^{;\alpha}-h_{\mu\alpha;\nu}{}^{;\alpha}-h_{\nu\alpha;\mu}{}^{;\alpha}+h_{\alpha}{}^{\alpha}{}_{;\mu;\nu}+g_{\mu\nu}[h_{\alpha\lambda}{}^{;\alpha;\lambda}-h_{\alpha}{}^{\alpha}{}_{;\lambda}{}^{;\lambda}]$$
$$-h_{\mu\nu}R+g_{\mu\nu}h_{\alpha\beta}R^{\alpha\beta}=-16\pi G\delta T_{\mu\nu}. \tag{2.12}$$

If the unperturbed stress-energy tensor $T_{\mu\nu}{}^{(0)}$ is taken to be zero, then $g_{\mu\nu}=\delta_{\mu\nu}$ solves Eq. (2.3) and Eq. (2.12) reduces to the usual linearized form of the field equations.[6] The linearized equations are a good approximation to the full field equations only when the space-time curvature is small and when nonlinear gravitational effects can be ignored. If we wish to consider gravitational perturbation effects near a massive body, the nonlinear effects are important and one cannot use the linearized equations. Rather than adding to the linearized field equations the nonlinear terms which arise in an expansion of Eq. (2.1) about flat space, we prefer to expand the field equations about the metric corresponding to the massive body alone. This allows us, in a simple case, to solve (2.12) for the perturbation $h_{\mu\nu}$ in terms of the arbitrary perturbing stress-energy tensor of matter[7] $\delta T_{\mu\nu}$.

The system of coordinates which are used in the expansion depends largely on the system of coordinates one used in giving an explicit expression for the unperturbed metric $g_{\mu\nu}{}^{(0)}$. Thus a finite change of coordinates cannot be made without changing the form of the $g_{\mu\nu}{}^{(0)}$. However, an infinitesimal change of coordinates can still be made which keeps the form of $g_{\mu\nu}{}^{(0)}$ the same, but which changes the $h_{\mu\nu}$. Therefore, under a coordinate transformation

$$x'^{\mu}=x^{\mu}+\eta^{\mu}, \tag{2.13}$$

where η^{μ} is infinitesimal, the metric becomes

$$g'_{\mu\nu}(x')=g_{\mu\nu}(x)-g_{\mu\alpha}\eta^{\alpha}{}_{,\nu}-g_{\nu\alpha}\eta^{\alpha}{}_{,\mu}. \tag{2.14}$$

If the form of the unperturbed metric is to remain the same, we must have that $g'_{\mu\nu}(x')=g'_{\mu\nu}{}^{(0)}(x')+h_{\mu\nu}(x')$ and $g_{\mu\nu}(x)=g'_{\mu\nu}{}^{(0)}(x)+h_{\mu\nu}(x)$, where $g'_{\mu\nu}{}^{(0)}(x)$ is the same function of x as $g'_{\mu\nu}{}^{(0)}(x')$ is of x'. But since $g'_{\mu\nu}{}^{(0)}(x)=g'_{\mu\nu}{}^{(0)}(x'-\eta)\cong g'_{\mu\nu}{}^{(0)}(x')-g'_{\mu\nu,\alpha}{}^{(0)}\eta^{\alpha}$, we have from (2.14) that the $h_{\mu\nu}$ transform under the infinitesimal coordinate change (2.13) like

$$h_{\mu\nu}(x')=h_{\mu\nu}(x)-g_{\mu\alpha}\eta^{\alpha}{}_{,\nu}-g_{\nu\alpha}\eta^{\alpha}{}_{,\mu}-g_{\mu\nu,\alpha}\eta^{\alpha} \tag{2.15}$$

or

$$h_{\mu\nu}(x')=h_{\mu\nu}(x)-\eta_{\mu;\nu}-\eta_{\nu;\mu}. \tag{2.16}$$

By choosing the η_{μ} appropriately, we can impose four constraints on the $h_{\mu\nu}$. In analogy with the linearized theory, we define

$$\bar{h}_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}h_{\alpha}{}^{\alpha}. \tag{2.17}$$

Under the transformation (2.13), the quantity $\bar{h}_{\mu\nu}{}^{;\nu}$ undergoes a transformation

$$\bar{h}'_{\mu\nu}{}^{;\nu}=\bar{h}_{\mu\nu}{}^{;\nu}-\eta_{\mu;\nu}{}^{;\nu}. \tag{2.18}$$

Since ${}_{;\nu}{}^{;\nu}$ is the generalized D'Alembertian operator, we can choose η_{μ} so that in the new system $\bar{h}_{\mu\nu}{}^{;\nu}=0$ (De Donder's gauge) or, more generally,

$$\bar{h}_{\mu\nu}{}^{;\nu}=f_{\mu}, \tag{2.19}$$

where f_{μ} is an arbitrary infinitesimal vector function.[8]

Equation (2.12) can be written, using (2.17), in the form

$$\bar{h}_{\mu\nu;\alpha}{}^{;\alpha}-\bar{h}_{\mu\alpha;\nu}{}^{;\alpha}-\bar{h}_{\nu\alpha;\mu}{}^{;\alpha}+g_{\mu\nu}\bar{h}_{\alpha\beta}{}^{;\alpha;\beta}-h_{\mu\nu}R$$
$$+g_{\mu\nu}h_{\alpha\beta}R^{\alpha\beta}=-16\pi G\delta T_{\mu\nu}. \tag{2.20}$$

Since $\bar{h}_{\mu\alpha;\nu}{}^{;\alpha}=\bar{h}_{\mu\alpha}{}^{;\alpha}{}_{;\nu}+R^{\sigma}{}_{\mu\nu}{}^{\alpha}\bar{h}_{\sigma\alpha}+R^{\sigma}{}_{\alpha\nu}{}^{\alpha}\bar{h}_{\mu\sigma}$, we have, from (2.19), that

$$\bar{h}_{\mu\alpha;\nu}{}^{;\alpha}=f_{\mu;\nu}+R^{\sigma}{}_{\mu\nu}{}^{\alpha}\bar{h}_{\sigma\alpha}-R^{\sigma}{}_{\nu}\bar{h}_{\mu\sigma}. \tag{2.21}$$

[6] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill Publishing Company, Inc., New York, 1965), Chap. 8.

[7] An analogous expansion of the field equations about a uniform, isotropic, cosmological model has been carried out by W. M. Irvine, Ann. Phys. (N. Y.) 32, 322 (1965).

[8] A choice of f_{μ} must be made in any problem in order to define the coordinate system one is using. We prefer to keep f_{μ} arbitrary at this point so that we may simplify our equations later by a convenient choice of f_{μ}.

Substituting (2.21) into the second, third, and fourth terms of (2.20) we find

$$\bar{h}_{\mu\nu;\alpha}{}^{;\alpha}-f_{\mu;\nu}-f_{\nu;\mu}+g_{\mu\nu}f_{\alpha}{}^{;\alpha}-2\bar{h}_{\alpha\beta}R^{\alpha}{}_{\mu\nu}{}^{\beta}+\bar{h}_{\mu\alpha}R^{\alpha}{}_{\nu}$$
$$+\bar{h}_{\nu\alpha}R^{\alpha}{}_{\mu}-h_{\mu\nu}R+g_{\mu\nu}h_{\alpha\beta}R^{\alpha\beta}=-16\pi G\delta T_{\mu\nu}, \quad (2.22)$$

with f_μ chosen to simplify the equations as much as possible once $g_{\mu\nu}{}^{(0)}$ has been chosen.

Equation (2.22) is consistent only if $\delta T_{\mu\nu}$ obeys a conservation law. Since Eq. (2.1) requires that $T_{\mu\nu}$ satisfy

$$T_{\mu\nu}{}^{;\nu}=0 \quad (2.23)$$

taking the variation of (2.23) yields the constraint on $\delta T_{\mu\nu}$

$$(\delta T_{\mu\nu})^{;\nu}=f_\alpha T_\mu{}^\alpha+\bar{h}^{\alpha\beta}T_{\mu\alpha;\beta}+\tfrac{1}{2}\bar{h}^{\alpha\beta}{}_{;\mu}T_{\alpha\beta}. \quad (2.24)$$

If the matter which composes $\delta T_{\mu\nu}$ is spatially separated from the unperturbed matter $T_{\mu\nu}$, then $\delta T_{\mu\nu}$ satisfies

$$(\delta T_{\mu\nu})^{;\nu}=0. \quad (2.25)$$

By the same arguments which are used to derive the equations of motion of a particle in curved space[9] from Eq. (2.23), one finds from (2.25) that the matter making up $\delta T_{\mu\nu}$ follows a geodesic path in the unperturbed metric $g_{\mu\nu}{}^{(0)}$.

III. EXPANSION ABOUT THE SCHWARZSCHILD METRIC

Equations (2.22) and (2.24) are valid for an arbitrary unperturbed metric. At this point we choose $g_{\mu\nu}{}^{(0)}$ to be the Schwarzschild metric in isotropic coordinates:

$$\begin{aligned} g_{00}{}^{(0)}&=e^{\nu}\equiv[(1+\phi/2)/(1-\phi/2)]^2,\\ g_{0i}{}^{(0)}&=0, \\ g_{ij}{}^{(0)}&=-\delta_{ij}e^{\lambda}\equiv-\delta_{ij}[1-\phi/2]^4, \end{aligned} \quad (3.1)$$

where ϕ is the gravitational potential,

$$\phi=-GM/rc^2, \quad (3.2)$$

and M is the mass of the Schwarzschild body. The Schwarzschild solution is chosen because in the exterior region (where $T_{\mu\nu}=0$), any metric which is the solution of (2.1) for a spherically symmetric mass distribution can be reduced to the Schwarzschild metric.[10] We choose isotropic coordinates so that, at a point, angular orientation does not affect the coordinate length of a measuring rod or the magnitude of the velocity of light.[11]

If the perturbing matter consists of moving particles, then the $h_{\mu\nu}$ will just be a sum over the solutions of (2.22) for each individual perturbing particle, a result of the linearity of the equations in the perturbations. Thus we may consider the perturbing matter to be only one particle of mass m ($m\ll M$) moving in the metric of the background Schwarzschild solution, the case of many particles or a fluid being a simple linear superposition of one-particle solutions. Even though the perturbing particle is assumed to be spatially separated from the matter which gives rise to the $g_{\mu\nu}{}^{(0)}$, we obtain, in general, contributions to the $\delta T_{\mu\nu}$ not only from the particle itself, but also from the reaction of the original matter to the presence of the particle. In our case the unperturbed matter is at rest with respect to our coordinate system. When we introduce the particle of mass m, the T_{00} and T_{0i} components of the original stress-energy tensor pick up terms proportional to m, which therefore contribute to $\delta T_{\mu\nu}$. The T_{ij} components, however, being quadratic in the velocity of the large mass, give terms proportional to m^2, which can be neglected since we are considering only perturbations linear in m. We will later need only the spatial components of $\delta T_{\mu\nu}$, so that the contributions arising from the large mass need not be considered in more detail.

This argument assumes, of course, that the large mass is described by a stress-energy tensor appropriate for a point mass. The assumption that the velocity of the large mass be zero in the absence of a perturbing mass is actually more restrictive then we need for the following analysis. If the velocity of the large mass is negligible compared to the velocity of the perturbing mass, then we may again ignore contributions arising from the reaction of the large mass to the presence of the perturbing mass. In the same manner, if the large mass is described by an energy density ϵ and pressure p, then the condition that the reaction of the large mass to the perturbing particle be negligible is that p satisfy $p\ll\epsilon v^2/c^2$, where v is the velocity of the perturbing particle. In the particular case where we take the lowest order metric to be given by the expansion of Eq. (3.1) to order ϕ, we are assuming that the large mass is a point mass initially at rest.

The part of $\delta T_{\mu\nu}$ arising directly from the particle, denoted by $\delta T_{\mu\nu}{}^{(m)}$, can be written as

$$\delta T_{\mu\nu}{}^{(m)}=g_{\mu\alpha}g_{\nu\beta}\delta T^{\alpha\beta(m)}, \quad (3.3)$$

where $\delta T^{\alpha\beta(m)}$ is given by

$$\delta T^{\alpha\beta(m)}=m\int ds\,\delta^4(x,z(s))\frac{dz^\alpha}{ds}\frac{dz^\beta}{ds}. \quad (3.4)$$

Here $\delta^4(x,z(s))$ is a two-point scalar function[12] defined to be 0 except when $x^\mu=z^\mu(s)$ and to have the integral property

$$\int\delta^4(x,z)[\sqrt{(-g)}]\,d^4\tau=1.$$

Equation (3.4) satisfies the condition (2.25) providing the space-time position of the particle, $z^\mu(s)$, satisfies the

[9] R. Adler, M. Bazin, and M. Schiffer, Ref. 6, p. 296.
[10] G. Birkhoff, *Relativity and Modern Physics* (Harvard University Press, Cambridge, Massachusetts, 1923), p. 253.
[11] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, London, 1960), p. 93.
[12] J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Company, Amsterdam, 1960), Chap. II.

geodesic equation

$$\frac{d^2z^\mu}{ds^2}+\left\{\begin{matrix}\mu\\ \alpha\ \beta\end{matrix}\right\}\frac{dz^\alpha}{ds}\frac{dz^\beta}{ds}=0, \tag{3.5}$$

with

$$g_{\mu\nu}\frac{dz^\mu}{ds}\frac{dz^\nu}{ds}=1. \tag{3.6}$$

The equations which we have written are linear in the perturbations. By our choice of the unperturbed metric (3.1), the coefficients of the perturbations are independent of time. Therefore, if $h_{\mu\nu}$ and $\delta T_{\mu\nu}$ are Fourier-decomposed in time,

$$h_{\mu\nu}(\mathbf{r},t)=\int h_{\mu\nu}(\mathbf{r},\omega)e^{-i\omega t}\,d\omega,$$
$$\delta T_{\mu\nu}(\mathbf{r},t)=\int \delta T_{\mu\nu}(\mathbf{r},\omega)e^{-i\omega t}\,d\omega, \tag{3.7}$$

then Eqs. (2.19), (2.22), and (2.25) remain the same for the Fourier components $h_{\mu\nu}(\mathbf{r},\omega)$ and $\delta T_{\mu\nu}(\mathbf{r},\omega)$, except that time derivatives $\partial/\partial t$ are replaced by the factor $-i\omega$. With a given choice of f_μ, Eq. (2.19) allows us to express $\bar{h}_{00}$ and $\bar{h}_{0i}$ in terms of the spatial components $\bar{h}_{ij}$. Thus we need consider Eq. (2.22) only for μ, $\nu=i$, j. As noted before, this has the additional advantage that the stress contributions δT_{ij} are easily given in terms of the motion of the perturbing particle.

It is straightforward, but tedious, to write out explicitly the equation for $\bar{h}_{ij}$ using the full Schwarzschild metric. However, this appears to be too general a procedure if one wishes to obtain explicit solutions for the perturbed metric given an arbitrary motion of the source.[13] Therefore we restrict ourselves to the case where the perturbing particle is not close to the Schwarzschild singularity ($\phi=-2$). Neglecting powers of ϕ^2 in our expression for the metric, we approximate (3.1) by

$$g_{00}{}^{(0)}=(1+2\phi),\quad g_{0i}{}^{(0)}=0,\quad g_{ij}{}^{(0)}=-\delta_{ij}(1-2\phi). \tag{3.8}$$

The only nonvanishing Christoffel symbols are

$$\left\{\begin{matrix}k\\ 0\ 0\end{matrix}\right\}=\left\{\begin{matrix}0\\ k\ 0\end{matrix}\right\}=\phi_{,k},$$
$$\left\{\begin{matrix}k\\ i\ j\end{matrix}\right\}=\delta_{ij}\phi_{,k}-\delta_{ik}\phi_{,j}-\delta_{jk}\phi_{,i}, \tag{3.9}$$

[13] The analysis of the perturbed metric into spherical harmonics has been carried out for this case by T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957). This analysis was, however, applied only to the case in which the perturbing matter vanishes. In the presence of a perturbing mass one would expect that the perturbed metric would be given as a sum over all harmonics with coefficients determined by the matter distribution, in analogy with the similar flat-space decomposition of $h_{\mu\nu}$ into spherical harmonics given by J. Mathews, J. Soc. Indian Appl. Math. **10**, 768 (1962).

and the only nonvanishing components of the Riemann tensor are

$$R_{0ij0}=\phi_{,ij};\quad R_{ijkl}=\delta_{jk}\phi_{,il}-\delta_{jl}\phi_{,ik}-\delta_{ik}\phi_{,jl}+\delta_{il}\phi_{,jk},$$
$$R_{ij}=-\delta_{ij}\phi_{,kk};\quad R_{00}=-\phi_{,kk};\quad R=2\phi_{,kk}, \tag{3.10}$$

together with those components which can be obtained from the above by a trivial permutation of indices.

Using Eqs. (3.8), (3.9), and (3.10), the field equation for $\bar{h}_{ij}$ can then be written, to order ϕ, as

$$\Box\bar{h}_{ij}+2\Box\phi\bar{h}_{ij}-4[\phi\bar{h}_{ij,00}+\phi_{,i}\bar{h}_{0j,0}+\phi_{,j}\bar{h}_{0i,0}+\phi_{,ij}h_{00}]$$
$$+2(\phi_{,l}\bar{h}_{lj})_{,i}+2(\phi_{,l}\bar{h}_{li})_{,j}-2\delta_{ij}[\bar{h}_{kl}\phi_{,kl}-h_{00}\phi_{,kk}]$$
$$-f_{i,j}-f_{j,i}-\delta_{ij}f_{0,0}+\delta_{ij}f_{k,k}=-16\pi G\delta T_{ij}, \tag{3.11}$$

where $\Box\equiv\partial^2/\partial t^2-\nabla^2$. We now choose our coordinate system so that

$$f_\mu=2\phi_{,l}\bar{h}_{l\mu}. \tag{3.12}$$

Equation (3.11), with (3.12), then simplifies to

$$\Box[(1+2\phi)\bar{h}_{ij}]-4[\phi\bar{h}_{ij,00}+\phi_{,i}\bar{h}_{0j,0}+\phi_{,j}\bar{h}_{0i,0}$$
$$+(\phi_{,ij}-\tfrac{1}{2}\delta_{ij}\phi_{,kk})h_{00}]=-16\pi G\delta T_{ij}. \tag{3.13}$$

The coordinate conditions (2.19) with (3.12) become

$$(1-2\phi)\bar{h}_{00,0}-[(1+2\phi)\bar{h}_{0k}]_{,k}=0,$$
$$(1-2\phi)\bar{h}_{0i,0}-[(1+2\phi)\bar{h}_{ik}]_{,k}=2\phi_{,i}h_{00}. \tag{3.14}$$

Therefore, through the use of Eqs. (3.7) and (3.14), we may express (3.13) in a form where only the spatial components of $\bar{h}_{\mu\nu}$ appear. From Eqs. (3.3) and (3.4), we also have that

$$\delta T_{ij}=m(1-4\phi)\int ds\,\delta^4(x,z(s))\frac{dz^i}{ds}\frac{dz^j}{ds}, \tag{3.15}$$

where $z^\mu(s)$ is the equation of the path of the perturbing particle. The perturbation in the metric, $h_{\mu\nu}$, caused by the perturbing stress-energy tensor $\delta T_{\mu\nu}$ is therefore determined by Eqs. (3.13), (3.14), and (3.15).

It should be noted that Eq. (3.13) could also have been obtained from an expansion of the field equations of general relativity about the flat-space metric.[3] If the deviation of the metric from that of flat space is given as the sum of $h_{\mu\nu}$ and a term linear in ϕ arising from the large mass, then we find nonlinear terms of order ϕ^2, ϕh, and h^2, ignoring terms of higher than second order in the gravitational constant. The terms of order ϕ^2 can then be eliminated by including in the lowest order metric the ϕ^2 terms of the expansion of (3.1). The terms of order ϕh are identical to those of Eq. (3.13) if one uses the coordinate condition (3.12). The terms of order h^2, being proportional to m^2, are ignored since we are looking only for perturbations in the metric linear in the perturbing mass m.

IV. GREEN'S FUNCTIONS IN CURVED SPACE

The problem of finding the solution to (3.13) is made much easier if we first digress and consider the general problem of Green's functions in curved space. In flat space with Cartesian coordinates, the Green's function for the scalar wave equation is defined by

$$\Box G(x,z)=\delta^4(x-z). \tag{4.1}$$

If one specifies that $G(x,z)\rightarrow 0$ for large spatial distances and that only contributions from the δ^4 function at the earlier time are allowed, then one obtains the retarded Green's function $G_R(x,z)$:

$$G_R(x,z)=\delta(z^0-t+R)/4\pi R, \tag{4.2}$$

where $x^\mu=(t,\mathbf{r})$, $z^\mu=(z^0,\mathbf{z})$, and $R=[(\mathbf{r}-\mathbf{z})\cdot(\mathbf{r}-\mathbf{z})]^{1/2}$. We could, of course, have taken the advanced solution to (4.1), $G_A(x,z)$:

$$G_A(x,z)=\delta(z^0-t-R)/4\pi R, \tag{4.3}$$

except that this can usually be ruled out through causality arguments.[14] Linear combinations of (4.2) and (4.3) are also solutions of (4.1). If we take the sum of (4.2) and (4.3), we obtain[15]

$$\begin{aligned} G_R+G_A&=(1/2\pi)\delta((t-z^0)^2-R^2)\\ &=(1/2\pi)\delta((x_\alpha-z_\alpha)(x^\alpha-z^\alpha)). \end{aligned} \tag{4.4}$$

If we define $\Omega_0\equiv\frac{1}{2}(x_\alpha-z_\alpha)(x^\alpha-z^\alpha)$, then

$$G_R+G_A=(1/4\pi)\delta(\Omega_0), \tag{4.5}$$

where Ω_0 is one-half the square of the proper time between the points x^α and z^α, taken along the "straight" line joining the two points. The Green's function (4.5) has a disadvantage for most problems in that it contains advanced contributions as well as the desired retarded ones. However, if we consider the solution of the scalar wave equation with a point source

$$\Box\psi(x)=\int_{-\infty}^{\infty} f(s)\delta^4(x,z(s))\,ds, \tag{4.6}$$

where $z^\mu(s)$ is the parametric equation of the path of the source and $f(s)$ is its strength, then we may ensure that only retarded contributions are obtained in the solution of (4.6) by letting

$$\psi(x)=\frac{1}{4\pi}\int_{-\infty}^{s_0}\delta(\Omega_0[x,z(s)])f(s)\,ds, \tag{4.7}$$

where s_0 is chosen so that $z^\mu(s_0)$ is outside the light cone centered on x^μ.

Equation (4.6) is written explicitly in Cartesian coordinates. In order to study the solutions to wave equations in curved space, we first write (4.6) in terms of general coordinates

$$\psi_{;\alpha}{}^{;\alpha}=\int_{-\infty}^{\infty} f(s)\delta^4(x,z(s))\,ds, \tag{4.8}$$

where $\delta^4(x,z(s))$ is a two-point scalar function[12] defined as in Eq. (3.4). If we now define[12] $\Omega(x,z)$ to be one-half the square of the proper time along the geodesic joining x^μ and z^μ, then $\Omega(x,z)$ is also two-point scalar function which reduces to $\Omega_0(x,z)$ when Cartesian coordinates are chosen. Therefore, in flat space with general coordinates, the solution of the covariant wave equation (4.8) can be given covariantly as

$$\psi(x)=\frac{1}{4\pi}\int_{-\infty}^{s_0}\delta(\Omega[x,z(s)])f(s)\,ds. \tag{4.9}$$

If we now generalize our analysis to be valid also in curved space, one possible scalar wave equation[16] is given by Eq. (4.8), where the metric is now determined by the field equations (2.1) with nonvanishing $T_{\mu\nu}$. However, another equally valid generalization of the wave equation to curved space is

$$\psi_{;\alpha}{}^{;\alpha}+aR\psi=\int_{-\infty}^{\infty} f(s)\delta^4(x,z(s))\,ds, \tag{4.10}$$

where R is the curvature scalar and a is an arbitrary constant.[17] We shall choose to work with the more general wave equation (4.10). The covariant flat-space solution (4.7) does not satisfy (4.10) for any value of a. In fact, the definition of $\Omega(x,z(s))$ becomes ambiguous in a general curved space since there may be more than one geodesic path joining the points x^μ and z^μ. Therefore, as a covariant trial solution to (4.10) we take

$$\psi^{(0)}(x)=\frac{1}{4\pi}\sum_i\int_{-\infty}^{s_0^i}\delta(\Omega_i[x,z(s)])f(s)\,ds, \tag{4.11}$$

where the sum over i indicates a sum over all geodesic paths connecting the points x^μ and z^μ. In order to find to what extent this is a good approximation to the solution of (4.10), we next find $\psi^{(0)}{}_{;\alpha}{}^{;\alpha}$.

The properties of $\Omega(x,z)$ have been studied extensively by Synge.[12] If $\xi^\alpha(u)$ is the parametric equation of a geodesic path, then u is a special parameter along the geodesic path if

$$\frac{dU^\alpha}{du}+\left\{\begin{matrix}\alpha\\ \beta\ \gamma\end{matrix}\right\}U^\beta U^\gamma=0, \tag{4.12}$$

where $U^\alpha=d\xi^\alpha(u)/du$. In terms of the parameter u,

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[14] A counter-example is provided by J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945).

[15] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).

[16] One approach for finding a Green's function for this equation as well as for a vector wave equation has been given by B. S. deWitt and R. W. Brehme, Ann. Phys. (N. Y.) **9**, 220 (1960). An extension of their method to the case of the tensor potential was given by D. Robaschik, Acta. Phys. Polon. **24**, 299 (1963).

[17] If one requires that the wave equation be conformally invariant in analogy with electromagnetism, then a is determined to be $\frac{1}{6}$. See R. Penrose, in *Relativity, Groups, and Topology*, edited by B. deWitt (Gordon and Breach Science Publishers, Inc., New York, 1961).

$\Omega(x,z)$ is given by

$$\Omega(x,z)=\tfrac{1}{2}(u_1-u_0)\int_{u_0}^{u_1} g_{\alpha\beta}U^\alpha U^\beta\, du\,, \tag{4.13}$$

where $u=u_1$ at $\xi^\alpha=x^\alpha$ and $u=u_0$ at $\xi^\alpha=z^\alpha$. The derivative of $\Omega(x,z)$ with respect to x^α, keeping z^α fixed, is given by

$$\Omega_{;\alpha}=(u_1-u_0)U_\alpha\,, \tag{4.14}$$

where the $\Omega_{;\alpha}$ obey

$$\Omega_{;\alpha}\Omega^{;\alpha}=2\Omega\,. \tag{4.15}$$

From the expression for the second derivatives of $\Omega(x,z)$ given by Synge, one obtains

$$\Omega_{;\alpha}{}^{;\alpha}=4+\frac{1}{(u_1-u_0)}\int_{u_0}^{u_1}(u-u_0)^2R_{\alpha\beta}U^\alpha U^\beta\, du +O(R^2)\equiv 4+F(x,z)+O(R^2)\,, \tag{4.16}$$

where $O(R^2)$ means that squares of the Riemann tensor have been ignored in the first two terms. From (4.16) we see that $\Omega_{;\alpha}{}^{;\alpha}=4$ in flat space.

$\psi^{(0)}{}_{;\alpha}{}^{;\alpha}$ can now be found to be

$$\psi^{(0)}{}_{;\alpha}{}^{;\alpha}=\frac{1}{4\pi}\sum_i\int_{-\infty}^{s_0{}^i}[2\delta''(\Omega_i)\Omega_i+4\delta'(\Omega_i)+\delta'(\Omega_i)F_i(x,z)] \times f(s)\, ds+O(R^2)\,. \tag{4.17}$$

Letting

$$\delta'(\Omega[x,z(s)])=\frac{\frac{d}{ds}\{\delta(\Omega[x,z(s)])\}}{(d/ds)[\Omega(x,z(s))]}$$

and integrating by parts with respect to s, (4.17) becomes

$$\psi^{(0)}{}_{;\alpha}{}^{;\alpha}=\int_{-\infty}^{\infty} f(s)\delta^4(x,z(s))\, ds +\frac{1}{4\pi}\sum_i\int_{-\infty}^{s_0{}^i}\delta'(\Omega_i)F_i f(s)\, ds+O(R^2)\,. \tag{4.18}$$

Comparing (4.18) with (4.10), we see that $\psi^{(0)}$ and ψ differ only in terms proportional to the Riemann tensor. Therefore, to lowest order in $R_{\alpha\beta\gamma\delta}$, we find that $\psi^{(0)}$, defined by (4.11), satisfies (4.10).

We next obtain an explicit solution of Eq. (4.10) which is valid to first order in $R_{\alpha\beta\gamma\delta}$. If we define

$$\psi^{(1)}(x)=\psi^{(0)}(x)-\frac{1}{4\pi}\sum_i\int[\surd(-g(y))]d^4y\,\delta(\Omega_i(x,y)) \times\left\{aR(y)\psi^{(0)}(y)+\frac{1}{4\pi}\sum_j\int_{-\infty}^{s_0{}^j}\delta'(\Omega_j[y,z(s)]) \times F(y,z(s))f(s)\, ds\right\}, \tag{4.19}$$

then $\psi^{(1)}$ satisfies the equation

$$\psi^{(1)}{}_{;\alpha}{}^{;\alpha}+aR\psi^{(1)}=\int_{-\infty}^{\infty} f(s)\delta^4(x,z(s))\, ds+O(R^2)\,, \tag{4.20}$$

so that $\psi^{(1)}$ is the solution of Eq. (4.8), neglecting terms involving squares or products of the Riemann tensor.

Although the terms on the right side of (4.19) appear to be complicated, in certain cases they simplify considerably. The second term arises through the inclusion of the term $aR\psi$ in the wave equation, and one may wish to set $a=0$. However, even with a nonzero a, in the case where $T_{\mu\nu}$ is localized at a point in space, the integral is trivially performed. The third term arises from the second term on the right of (4.18). An interpretation which can be given to this term is that the approximation $\psi=\psi^{(0)}$ gives rise to a spurious source at any point x^u which is connected to z^u by a null geodesic which passes through a region of nonzero $T_{\mu\nu}$. In the case of the localized $T_{\mu\nu}$, the third term of (4.19) reduces to a line integral along that particular null geodesic.

These results are best seen if we consider the case of the scalar equation in a curved space with a metric given by (3.8). The wave equation (4.8) written out to first order in ϕ becomes

$$(1-2\phi)\psi_{,00}-(1+2\phi)\psi_{,kk}=\int_{-\infty}^{\infty} f(s)\delta^4(x,z(s))\, ds\,. \tag{4.21}$$

Since the trial solution is (4.11), we next need to find explicitly $\delta(\Omega(x,z))$ in this particular metric. $\Omega(x,z)$ is given by $\frac{1}{2}(\Delta S)^2$, where

$$\Delta S=\int_{\text{geodesic path}} ds=\int_{z^0}^{t}\frac{d\xi^0}{(d\xi^0/ds)}\,, \tag{4.22}$$

where $\xi^\mu(s)$ is the equation of the geodesic path connecting z^μ with x^μ. From the geodesic equation (3.5), with $\mu=0$, we find that

$$d^2\xi^0/ds^2+2(d\phi/ds)(d\xi^0/ds)=0\,,$$

so that to order ϕ, $d\xi^0/ds=A[1-2\phi]$, where A is a constant, and thus ΔS becomes

$$\Delta S=\frac{1}{A}[t-z^0]+\frac{2}{A}\int_{z^0}^{t}\phi(\xi^0)\, d\xi^0\,. \tag{4.23}$$

We are interested in the expression for ΔS when ΔS is small, or equivalently, when the geodesic path is nearly null. In this case we can determine A by computing its value from an extension of the geodesic path to large distances, where $\phi\approx 0$. In the second term of (4.23) we can approximate the path by a straight line, since deviations from this will be of order ϕ^2 and thus negligible in (4.23).

Since

$$g_{\mu\nu}(d\xi^\mu/ds)(d\xi^\nu/ds)=1\,,$$

we have that

$$(ds/d\xi^0)^2=(1/A^2)(1+4\phi)=(1+2\phi)-v^2(1-2\phi)\,,\quad(4.24)$$

where $\mathbf{v}=d\boldsymbol{\xi}/d\xi^0$ and $v^2=\mathbf{v}\cdot\mathbf{v}$. If we extend the geodesic path passing through $(\mathbf{x},t)$ and $(\mathbf{z},z^0)$ to large $|\boldsymbol{\xi}|$, we find that $A^{-2}=1-v_A{}^2$, where v_A is the asymptotic value of v. To zeroth order in ϕ, $v_A=R/(t-z^0)$, where $R=|\mathbf{x}-\mathbf{z}|$. This gives $\Delta S=[(t-z^0)^2-R^2]^{1/2}$, in agreement with the special relativistic result. To first order in ϕ, the path may still be considered straight, but with v changing with position. From (4.24) we find that, to order ϕ, $v^2=v_A{}^2+4\phi$ or $v=v_A+2\phi$, where we replace v by c in the terms of order ϕ since deviations from this will be of higher order in ϕ. The condition that the path go through $(\mathbf{x},t)$ and $(\mathbf{z},z^0)$ is

$$R=\int_{z^0}^{t}v(\xi^0)\,d\xi^0=v_A(t-z^0)+2\int_{z^0}^{t}\phi(\xi^0)\,d\xi^0.$$

Solving for v_A gives

$$v_A=\frac{R}{t-z^0}-\frac{2}{t-z^0}\int_{z^0}^{t}\phi(\xi^0)\,d\xi^0.$$

We then substitute $1/A=(1-v_A{}^2)^{1/2}$ into Eq. (4.23) and carry out the integrations over the ϕ terms, assuming again that the path is straight and that $v\cong c$. This yields the expression for $(\Delta S)^2$:

$$(\Delta S)^2\approx[(t-z^0)^2-R^2-4GMR\,\Gamma][1-(4GM/R)\Gamma]\,,$$

where $\mathbf{R}=\mathbf{r}-\mathbf{z}$, $R=|\mathbf{R}|$, and $\Gamma(\mathbf{r},\mathbf{z})$ is

$$\Gamma(\mathbf{r},\mathbf{z})=\ln\left(\frac{rR+\mathbf{r}\cdot\mathbf{R}}{|\mathbf{z}|R+\mathbf{z}\cdot\mathbf{R}}\right).\quad(4.25)$$

$(\Delta S)^2$ is factorable into retarded and advanced parts, so that $\delta(\Omega)$ becomes a sum of retarded and advanced parts. By our choice of s_0 in (4.11), the only contribution comes from the retarded part, and we can write[18]

$$\psi^{(0)}(x)\equiv\int_{-\infty}^{s_0}\delta(\Omega[x,z(s)])f(s)\,ds$$

$$=\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\delta(z^0-t+R+2GM\Gamma)}{R}f(s)\,ds.\quad(4.26)$$

By explicit calculation, one finds that $\Box\psi^{(0)}$ is

$$\Box\psi^{(0)}=\int\delta(t-z_0(s))\delta^3(\mathbf{r}-\mathbf{z}(s))f(s)\,ds+4\phi\frac{\partial^2\psi^{(0)}}{\partial t^2}+I(\mathbf{r},\mathbf{z})\,,\quad(4.27)$$

where $I(\mathbf{r},\mathbf{z})$ is zero except along the line $\mathbf{r}\cdot\mathbf{z}=-r|\mathbf{z}|$. Aside from the term $I(\mathbf{r},\mathbf{z})$ which arises from the second term on the right of (4.18), this is identical to Eq. (4.21) when one replaces

$$\int\delta(t-z^0(s))\delta^3(\mathbf{r}-\mathbf{z}(s))f(s)\,ds$$

$$=\int\delta^4(x,z)[\sqrt{(-g)}]f(s)\,ds\approx\int\delta^4(x,z)f(s)(1-2\phi)\,ds$$

$$\approx\int\delta^4(x,z)f(s)\,ds-2\phi\psi^{(0)}{}_{,00}+2\phi\psi^{(0)}{}_{,kk}\quad(4.28)$$

using the definition of $\delta^4(x,z)$.

In order to find the next order solution $\psi^{(1)}$, we let $\delta\psi$ be defined so that

$$\Box\delta\psi=-I(\mathbf{r},\mathbf{z})\,.$$

Then $\delta\psi$ has the explicit solution

$$\delta\psi(\mathbf{r},t)=\frac{GM}{2\pi}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}ds\int_0^{\infty}du\times\frac{\delta(z^0(s)-t+u+|\mathbf{z}(s)|+\rho(u))}{[u+|\mathbf{z}(s)|]\rho(u)}\,,\quad(4.29)$$

where

$$\rho(u)=\left[r^2-\left(\frac{\mathbf{r}\cdot\mathbf{z}}{|\mathbf{z}|}\right)^2+\left(\frac{\mathbf{r}\cdot\mathbf{z}}{|\mathbf{z}|}+u\right)^2\right]^{1/2}.\quad(4.30)$$

Therefore $\psi^{(1)}\equiv\psi^{(0)}+\delta\psi$ is the solution to order ϕ of the scalar wave equation (4.21).

V. SOLUTION OF THE PERTURBED FIELD EQUATIONS

Having found the solution of the scalar wave equation, we now proceed to solve Eq. (3.13) for the $\bar{h}_{ij}$. In analogy with the scalar equation, we choose our trial solution for $\bar{h}_{ij}$ to be the same as (4.26) and (4.29) with $f(s)$ replaced by $-16\pi Gm(1-2\phi)(dz^i/ds)(dz^j/ds)$. The resulting function, which we call $\bar{h}^{(0)}{}_{ij}(1+2\phi)$, satisfies the equation

$$\Box[\bar{h}^{(0)}{}_{ij}(1+2\phi)]-4\phi\bar{h}^{(0)}{}_{ij,00}=-16\pi G\delta T_{ij}.\quad(5.1)$$

This only gives some of the terms of Eq. (3.13) if $\bar{h}^{(0)}{}_{ij}$ is taken to be equal to $\bar{h}_{ij}$. We note that the terms which are missing in (5.1) involve components of the perturbed metric other than $\bar{h}_{ij}$. This problem, of course, did not arise in the discussion of the scalar equation since in that case there was only one component.

We can see, at least qualitatively, the reason why components of $h_{\mu\nu}$ are present in Eq. (3.13) which are not present in (5.1). The solution to the scalar equation, upon which (5.1) is based, took into account only the

[18] The argument of the δ function differs from that of flat space by the term $2GM\Gamma$. This gives rise to the time delay in the radar reflection experiment proposed by I. I. Shapiro, Phys. Rev. Letters 13, 789 (1964). Also see the analysis of this experiment by I. I. Shapiro, Phys. Rev. 141, 1219 (1966), and by D. K. Ross and L. I. Schiff, *ibid.* 141, 1215 (1966).

fact that the signal from the source was received at a time determined by the zero of the δ function of Eq. (4.26) rather than the zero of the δ function of Eq. (4.2). If the source has a direction associated with it, e.g., a velocity component, the components of the potential generated by this source will not be simply related to the components of the source because the signal will have been deflected in its propagation from the source to receiver. Thus we may expect terms in the equation for the perturbed metric which mix different velocity components. We will not, however, analyze this bending phenomenon quantitatively in order to obtain a better approximation to $\bar{h}_{ij}$. Rather, the solution of the scalar wave equation will be used to generate an expression for $\bar{h}_{ij}$ which satisfies Eq. (3.13).

Since Eq. (4.21) is linear in ψ with coefficients independent of time, we may Fourier transform (4.21) with respect to time and consider $\psi(\mathbf{r},\omega)$ rather than $\psi(\mathbf{r},t)$. Equation (4.26) then becomes

$$\psi^{(0)}(\mathbf{r},\omega)=\frac{1}{4\pi}\int\frac{e^{i\omega[z^0+R+2GM\Gamma]}}{R}f(s)\,ds$$
$$\approx\frac{1}{4\pi}\int\frac{e^{i\omega(z^0+R)}}{R}[1+2i\omega GM\Gamma]f(s)\,ds. \quad (5.2)$$

Comparison of (5.2) with the Fourier transforms of (4.27) and (4.29) shows that the quantity

$$G(\mathbf{r},\mathbf{z},\omega)=\frac{GMi}{2\omega}\left[\frac{e^{i\omega R}}{R}\ln\frac{rR+\mathbf{r}\cdot\mathbf{R}}{|\mathbf{z}|R+\mathbf{z}\cdot\mathbf{R}}-\int_0^\infty\frac{du\;e^{i\omega(u+|\mathbf{z}|+\rho)}}{(u+|\mathbf{z}|)\rho}\right], \quad (5.3)$$

with $\rho(\mathbf{r},\mathbf{z},u)$ given by Eq. (4.30), satisfies the equation

$$\Box G(\mathbf{r},\mathbf{z},\omega)=-(\omega^2+\nabla^2)G(\mathbf{r},\mathbf{z},\omega)=\phi(r)e^{i\omega R}/R. \quad (5.4)$$

From Eq. (5.3) we may generate other useful relations. If we operate on Eq. (5.4) with the operator

$$\nabla_k\equiv\partial/\partial x^k+\partial/\partial z^k, \quad (5.5)$$

then, since

$$(\partial/\partial x^k+\partial/\partial z^k)f(R)=0,$$

we find that

$$\Box\nabla_kG(\mathbf{r},\omega)\equiv\Box(\partial/\partial x^k+\partial/\partial z^k)G(\mathbf{r},\mathbf{z},\omega)=\phi_{,k}(r)e^{i\omega R}/R. \quad (5.6)$$

Operating on (5.6) with ∇_l gives

$$\Box(\nabla_k\nabla_lG(\mathbf{r},\mathbf{z},\omega))=\phi_{,kl}e^{i\omega R}/R. \quad (5.7)$$

Therefore, by operating on the function $G(\mathbf{r},\mathbf{z},\omega)$ with various combinations of ∇_i, we generate solutions to different wave equations. We shall find the expressions (5.6) and (5.7) together with (5.3) to be sufficient to solve the problem of the extra terms in (3.13).

To lowest order in ϕ, $\bar{h}_{ij}$ is given by the trial solution $\bar{h}^{(0)}{}_{ij}$, which is approximately

$$\bar{h}_{ij}(\mathbf{r},t)\approx-4Gm\int ds\frac{\delta(z^0-t+R)}{R}\frac{dz^i}{ds}\frac{dz^j}{ds}. \quad (5.8)$$

Taking the Fourier transform of (5.8) and changing the variable of integration to z^0, we obtain

$$\bar{h}_{ij}(\mathbf{r},\omega)\approx-4Gm\int dz^0\frac{e^{i\omega[z^0+R]}}{R}\frac{dz^i}{dz^0}\frac{dz^j}{dz^0}\left(\frac{dz^0}{ds}\right). \quad (5.9)$$

From (3.14) $\bar{h}_{ij,j}\approx-i\omega\bar{h}_{i0}$, and (5.9) then implies that

$$\bar{h}_{ij,j}\approx-i\omega\bar{h}_{i0}\approx+4Gi\omega m\int dz^0\frac{e^{i\omega(z^0+R)}}{R}\frac{dz^i}{dz^0}\left(\frac{dz^0}{ds}\right), \quad (5.10)$$

where terms of order ϕ arising from d^2z^i/dz^{02} and dz^0/ds have been ignored. One can similarly express $\bar{h}_{0i,i}$ in terms of ω and dz^0/ds. Since (3.13) differs from (5.1) only in terms proportional to ϕh, we may use the relations (3.14) and (5.10) to give the $h_{\mu\nu}$ in the ϕh terms explicitly in terms of the velocity components $v^i\equiv dz^i/dz^0$.

If we now take the Fourier transform of Eq. (3.13), we obtain

$$\Box[(1+2\phi)\bar{h}_{ij}]+4\omega^2\phi\bar{h}_{ij}+4i\omega[\phi_{,i}\bar{h}_{0j}+\phi_{,j}\bar{h}_{0i}]-4(\phi_{,ij}-\tfrac{1}{2}\delta_{ij}\phi_{,kk})h_{00}$$
$$=-16\pi Gm\int\frac{dz^0}{1+2\phi}\delta(t-z^0)\delta^3(\mathbf{r}-\mathbf{z})v^iv^j\frac{dz^0}{ds}, \quad (5.11)$$

where we have used Eq. (4.28) for $\delta^4(x,z)$. From Eqs. (5.4)–(5.10) we then have the solution for $\bar{h}_{ij}(\mathbf{r},\omega)$, valid to order ϕ,

$$\bar{h}_{ij}(\mathbf{r},\omega)=-\frac{4Gm}{(1+2\phi)}\int\frac{dz^0}{1+2\phi}\left(\frac{dz^0}{ds}\right)v^iv^j\frac{e^{i\omega(z^0+R)}}{R}$$
$$+16Gm\int dz^0\left(\frac{dz^0}{ds}\right)e^{i\omega z^0}\int d^3r'\,\delta^3(\mathbf{r}'-\mathbf{z}(z^0))$$
$$\times\{\omega^2v^iv^j-i\omega(v^i\nabla_j+v^j\nabla_i)-\tfrac{1}{2}(1+v^2)\times(\nabla_i\nabla_j-\tfrac{1}{2}\delta_{ij}\nabla^2)\}G(\mathbf{r},\mathbf{r}',\omega), \quad (5.12)$$

where $\nabla_i=\partial/\partial x_i+\partial/\partial x_i'$ and $G(\mathbf{r},\mathbf{r}',\omega)$ is given explicitly by Eq. (5.3). The components $\bar{h}_{0i}$ and $\bar{h}_{00}$ may be found through Eq. (3.14).

In particular cases (5.12) reduces considerably. If we consider the case where $r\gg|\mathbf{z}|$, then the only part of

$G(\mathbf{r},\mathbf{r}',\omega)$ which contributes in order $1/r$ is

$$G(\mathbf{r},\mathbf{r}',\omega) \underset{r>>r'}{\sim} -\frac{GMi}{2\omega}\left[\frac{e^{i\omega[r-\mathbf{n}\cdot\mathbf{r}']}}{r}\right.$$
$$\left.\times\left(\ln(r'+\mathbf{n}\cdot\mathbf{r}')+\int_{r'+\mathbf{n}\cdot\mathbf{r}'}^{\infty}\frac{e^{i\omega u}}{u}du\right]\right), \quad (5.13)$$

where $\mathbf{n}=\mathbf{r}/r$. Operating on (5.13) with ∇_k gives an especially simple result

$$\nabla_k[G(\mathbf{r},\mathbf{r}',\omega)] \underset{r>>r'}{\sim} -\frac{GMi}{2\omega}\frac{e^{i\omega r}}{r}$$
$$\times[e^{-i\omega\mathbf{n}\cdot\mathbf{r}'}-e^{i\omega r'}]\frac{x_k'/r'+n_k}{r'+\mathbf{n}\cdot\mathbf{r}'}. \quad (5.14)$$

If we now consider the case $\omega r'\gg 1$ (quadrupole approximation) as well as $r\gg r'$, only the first and last term on the right side of (5.12) survive and we obtain

$$\bar{h}_{ij}(\mathbf{r},\omega) \underset{r>>r',\omega r'<<1}{\sim} -\frac{4Gm}{r}e^{i\omega r}\int dz^0$$
$$\times e^{i\omega z^0}\left(v^iv^j+\frac{GMz_iz_j}{|\mathbf{z}|^3}\right). \quad (5.15)$$

Since $dv^i/dz^0\cong -GMz_i/|\mathbf{z}|^3$ and $z_i\cong -z^i$, Eq. (5.15) becomes

$$\bar{h}_{ij}(\mathbf{r},\omega) \underset{r>>r',\omega r'<<1}{\sim} -\frac{2G}{r}e^{i\omega r}\int dz^0\, e^{i\omega z^0}\frac{d^2}{dz^{02}}[mz^iz^j], \quad (5.16)$$

which agrees with the Fourier transform of the usual expression for $\bar{h}_{ij}$ in the large r, small $\omega r'$ limit in terms of time derivatives of the mass quadrupole tensor of the system.[3]

In the limit $R\rightarrow 0$, keeping $|\mathbf{z}|$ finite, we obtain an expression for $\bar{h}_{ij}$ which is valid near the particle m. In this limit, $G(\mathbf{r},\mathbf{r}',\omega)$ and $\nabla_k G(\mathbf{r},\mathbf{r}',\omega)$ become

$$G(\mathbf{r},\mathbf{r}',\omega)=\frac{GMi}{2\omega}\left(\frac{1}{r'}-\int_{r'}^{\infty}\frac{du\, e^{2i\omega u}}{u^2}\right)+O(R),$$
$$\nabla_k G(\mathbf{r},\mathbf{r}',\omega)=-\frac{GMi}{2\omega}\left(\frac{x_k'}{r'^3}\right)(1-e^{2i\omega r'})+O(R). \quad (5.17)$$

If we now consider $\omega r'$ small as well as $R\ll r'$, then

$$\nabla_k G(\mathbf{r},\mathbf{r}',\omega) \underset{R<<r,\omega r'<<1}{\sim} -GMx_k'/r'^2. \quad (5.18)$$

Although the first term on the right side of (5.12) dominates the expression for small R, some effects, such as gravitational radiation reaction, depend on parts of the $h_{\mu\nu}$ which remain finite at the position of the particle, and the contributions of (5.17) and (5.18) cannot be neglected in such problems.

The last limiting case we shall consider is that near the region of the heavy mass M. The behavior of $G(\mathbf{r},\mathbf{r}',\omega)$ for $r\ll r'$ is found to be

$$G(\mathbf{r},\mathbf{r}',\omega) \underset{r<<r'}{\sim} -\frac{GMi}{2\omega}\left[\frac{e^{i\omega r'}}{r'}\ln\left(\frac{rr'+\mathbf{r}\cdot\mathbf{r}'}{2r'^2}\right)\right.$$
$$\left.+\int_0^{\infty}\frac{e^{i\omega(u+r'+\rho)}\,du}{(u+r')\rho}\right].$$

Although this looks divergent at $r=0$, if we let $dv=[1/\rho(u)]du$ in the second term and integrate by parts, we find

$$G(\mathbf{r},\mathbf{r}',\omega) \underset{r<<r'}{\sim} -\frac{GMi}{2\omega}\left[\frac{e^{i\omega r'}}{r'}\ln(2r'^2)\right.$$
$$\left.-\int_0^{\infty}du\,\ln(2u)\frac{d}{du}\left(\frac{e^{i\omega(2u+r')}}{(r'+u)}\right)\right], \quad (5.19)$$

which is finite. More generally, we find that $G(\mathbf{r},\mathbf{r}'\omega)$ is finite along the line $\mathbf{r}\cdot\mathbf{r}'=-rr'$, even though each term of (5.3) diverges for $\mathbf{r}\cdot\mathbf{r}'=-rr'$.

VI. CONCLUSION

In summary, we have found that the expanded field equations (5.11) have an explicit solution (5.12) in terms of the Green's function (5.3). This Green's function has the property that the effect at x of a source at y is not only felt when x and y are joined by a null geodesic, but also when x is joined by a null geodesic to any point z which is itself joined to y by a null geodesic which passes through a region of nonzero stress-energy. These equations serve to give the gravitational analog of the Lienard-Wiechert potentials in the presence of a massive body. Unlike the solutions of the linearized field equations, this solution takes into account the nonlinear terms ϕh, which represent the important fact that gravity is itself a source of gravity. From the approximation (5.13) one can show that these nonlinear terms yield a $1/r$ contribution at large distances which contribute to the radiation problem in the same order of magnitude as that arising directly from the perturbing matter. Also from the approximation (5.17) we see that the nonlinear terms give a finite contribution at the position of the perturbing particles, and therefore the effects of such terms should be included in any discussion of the gravitational radiation reaction problem.