and

$$K' = V_\alpha g_\beta^+ V_\gamma G_\beta^+ + V_\gamma G_0^+ V_\beta G_\beta^+, \quad (39)$$

where $G_0^+ = [E - H_0 + i\epsilon]^{-1}$ is the free-particle Green's function. Now in the Neumann series solution of the integral equation (34), the series in K

$$1 + K + K^2 \cdots \quad (40)$$

contains a subseries in K_0

$$1 + K_0 + K_0^2 \cdots, \quad (41)$$

where K and K_0 are defined by Eqs. (36) and (38), respectively. The operator K_0, which permits particle γ to propagate freely, is not completely continuous[6] and the subseries (41) diverges for the same reasons as the Born series considered by Aaron, Amado, and Lee.[5] If the Born series expansion of the two-body t matrix for the interaction V_γ diverges for any energy, then the series (41) diverges for *all* positive energies of the three-body system. As was the case in Ref. 5, the strengths of the potentials V_γ of K_0, and V_β and V_α of K' are independent, and it is reasonable to expect that the divergence of this subseries cannot be exactly compensated by any similar divergence in the remainder of the series (40) which involves the kernel K'. Moreover, there is good reason to expect K' itself to be completely continuous since it contains no disconnected diagrams. Hence the divergence of the series (40) will in general occur, and furthermore the divergence is independent of the specific choice of the distorting potential w_β made in connection with Eq. (36).

In summary, the desired amplitude $T_{\beta\alpha}^-$ cannot be obtained from the integral equation (34) since there are no techniques available for solving equations with pathological kernels which give rise to a divergent subseries like (41). As a result, the DWB inhomogeneous term of Eq. (29) cannot be regarded as the first- or lowest order approximation to $T_{\beta\alpha}^-$. In other words, the DWB model does not constitute a mathematical approximation to the transition amplitude.

The problem of finding a solvable integral equation for $U_{\beta\alpha}^-$ depends on whether the pathological part K_0 can be removed from the kernel K. This in turn depends on whether the series (41) can be explicitly summed and added to the inhomogeneous part of the original equation (34). This procedure should indeed be possible since K_0 in general contains only a single two-body potential, i.e., V_γ in Eq. (38), and the series (41) can be rewritten in terms of the t matrix for that potential.[6] With the new connected kernel, one may be able to solve for the exact transition operator $U_{\beta\alpha}^-$, or at least to find a mathematically sound first-order approximation to $U_{\beta\alpha}^-$ in the new inhomogeneous term. Explicit calculation of this term would yield an improved distorted-wave theory, and comparison of this term with the standard DWB model should provide criteria for the success or failure of the latter model. A subsequent paper[13] will attempt to deal with these problems.

[13] L. R. Dodd and K. R. Greider, Phys. Rev. 146, 675 (1966).

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Rigorous Solution of Three-Body Scattering Processes in the Distorted-Wave Formalism*

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General methods are developed for obtaining convergent solutions to the exact nonrelativistic three-body scattering amplitudes within the framework of the distorted-wave representation. A simplification of the Faddeev-Lovelace coupled integral equations is obtained when the mass of one particle is either much larger or much smaller than the other two. When this mass restriction applies, it is shown that each amplitude is determined by a single, tractable integral equation. The kernel of a typical equation is well-behaved and depends only on calculable, two-body operators. The inhomogeneous term consists of two parts, one of which is the usual distorted-wave Born term and the other, a term involving excitation of a set of two-body intermediate states. Applications are made to a variety of nuclear scattering processes, and the implications for the distorted-wave Born model discussed.

I. INTRODUCTION

IT has recently been shown that the nonrelativistic solution of three-body scattering problems in terms of the Born series[1] or the distorted-wave Born series[2] is in general divergent. The reasons for the divergence and methods for curing it have been explained in a variety of ways[1–7] and we briefly review the problem below.

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The divergence difficulties can be understood fairly easily if the exact three-body amplitude T is written in an integral equation of the Lippmann-Schwinger type, having the general form

$$T=I+KT. \tag{1}$$

In Eq. (1), the inhomogeneous term I is just the Born term of the undistorted-Born formalism of Ref. 1, while in Ref. 2, I is given by the distorted-wave Born (DWB) term. The essential difficulty is that the singular nature of the three-body kernel K of this equation invalidates the standard methods of obtaining a solution. In the language of graphs, the kernel has pathological properties because it contains the so-called dangerous diagrams or disconnected diagrams. The dangerous diagrams are those in which one of the three particles propagates freely while the other two particles interact via a two-body potential. Such diagrams will always occur in the kernel K of Eq. (1) since it can be shown[1,2] that K can be separated into two parts

$$K=K_0+K_1, \tag{2}$$

where

$$K_0=VG_0^+ \tag{3}$$

contains all the disconnected diagrams. In Eq. (3), V is one of the two-body pair interactions and G_0^+ is the Green's function for all three particles propagating freely. The divergence problem becomes apparent if we attempt to find the Neumann solution

$$T=(1+K+K^2\cdots)I \tag{4}$$

of Eq. (1). The series (4) obtained by iteration contains a subseries in K_0,

$$1+K_0+K_0^2+K_0^3\cdots \tag{5}$$

which diverges. The divergence arises since the summed series (5) is related to the t matrix for the two-body interaction V and must be singular at the energies corresponding to the bound states of the two-body system. It is evident that the divergence will not, in general, be cancelled by other terms in the series (4).[1] Furthermore, other standard methods of solving integral equations, such as the Fredholm technique, are equally futile in providing a solution of Eq. (1).

The problem of obtaining a convergent solution for three-body scattering processes has been investigated extensively by Faddeev,[3] Lovelace,[4] Weinberg,[5] Rosenberg,[6] and Amado.[7] The conclusions reached by these authors are essentially the same: In order to obtain a nondivergent solution for the three-body amplitude, it is necessary to replace the Lippmann-Schwinger equation by a set of coupled integral equations. The kernel in the coupled equations is a 3×3 matrix which, when squared, contains no dangerous diagrams. These equations, originally proposed by Faddeev, were the first that gave a mathematically sound formulation of the three-body scattering problem.

The success of these methods is due to the fact that each of the two-body pair amplitudes has been already solved. It is just this point—that the two-body amplitudes must be known—that allows meaningful approximation methods. The dangerous diagrams in K_0 of Eq. (3) can be *explicitly* removed and summed as in the subseries (5), which has physical meaning in terms of a wave operator and is calculable if K_0 contains a single two-body interaction as in Eq. (3).[8] Then the series (5) can be expressed in terms of the two-body t matrix for the interaction V. In other words, it is just the disconnected part or the troublesome part of the kernel that we know how to sum and calculate in closed form.

The application of these methods by Lovelace and others has been confined to elementary-particle processes like the three-nucleon system at low energies, where it is sufficient to include a small number of bound states and resonances in each two-body amplitude, leading to a simplification of the three basic equations. This approximation, equivalent to the introduction of separable potentials, appears to be less useful in nuclear and atomic problems which are usually formulated with local central potentials.

We present in this paper an approximation which, unlike the Faddeev-Lovelace solution, obtains just two coupled integral equations (which can be trivially solved for a single equation). However, it requires the restriction either that the mass of one particle be much larger than the other two, or that it be much smaller than the other two. Our method obtains two coupled integral equations instead of the usual three equations of Faddeev and Lovelace, because we do not require that only a single two-body potential appear in each of the Green's functions G_1 in the kernel matrix.

The Green's function used by Faddeev and Lovelace is

$$G_1^+=\frac{1}{E-H_0-V_i+i\epsilon}, \tag{6}$$

where H_0 is the three-body kinetic-energy operator and V_i is one of the two-body pair potentials ($i=\alpha,\beta,\gamma$). The subscript 1 in G_1^+ indicates the presence of only one two-body interaction (V_i in the example above). Instead, we use a Green's function which we designate G_2^+, that contains two of the pair potentials, V_i and V_j,

$$G_2^+=\frac{1}{E-H_0-V_i-V_j+i\epsilon}. \tag{7}$$

A Green's function of this type is usable or solvable in terms of known two-body states only if the kinematics of the problem are such that the kinetic-energy operator H_0 separates *exactly* into one part h_i depending only on $\mathbf{r}_i$ (the argument of V_i) plus a second part h_j depending only on $\mathbf{r}_j$ (the argument of V_j). This separation is possible, for instance, if $\mathbf{r}_i$ and $\mathbf{r}_j$ are the coordinate

[8] F. Coester, Phys. Rev. 133, B1516 (1964).

vectors of two particles relative to a heavy third particle (e.g., two electrons plus a proton.) It is also possible if one of the coordinates $\mathbf{r}_i$ gives the relative separation of two heavy particles while the other ($\mathbf{r}_j$) is the separation of the third light particle from either of the massive ones (e.g., two protons plus an electron). We will call these types of Green's functions factorable three-body Green's functions.

Of course, the explicit dependence on two-body amplitudes is accomplished more generally by the Faddeev equations without any restriction on the masses but with the disadvantage that not only are there three coupled equations to be solved but that also physical insight into a meaningful first-order approximation is lost. Our method obviates the need for approximations like the separable potential to uncouple the Faddeev equations, since only one integral equation must be solved, and an evaluation of the transition amplitude by the usual, more realistic two-body amplitudes is possible. It is apparent that the approximation may work well for many problems in nuclear and atomic physics for which the mass conditions are satisfied rather well. Our method, like the Faddeev method, requires the knowledge of the two-body pair amplitudes off the energy shell, which for the remainder of this paper we assume are known and present no difficulties.

In Sec. II, the formal solution to the three-body scattering problem is given under the mass restrictions mentioned above. The resultant single integral equation for the transition amplitude is obtained in a general distorted-wave formalism. It is shown that the kernel of the equation contains no dangerous diagrams, and assuming well-behaved two-body interactions, the kernel is completely continuous.

The formulation of the three-body scattering problem of Sec. II in terms of a single, tractable, integral equation allows us to consider the very important question of what is a meaningful first-order approximation to the scattering amplitude. The fact that the usual three-body Born series or distorted-wave Born series of (4) diverges makes the validity of many previous calculations of rearrangement scattering suspect. Many of these calculations, like the distorted-wave Born approximation, are intuitive models based primarily on semiclassical concepts of direct reactions. However, it is becoming increasingly clear[4] that intuitive direct-reaction models may not afford an adequate approximation to the solution of quantum-mechanical three-body problems. We take the point of view here that a meaningful model must provide a starting point for a series of successively more accurate calculations of the scattering amplitude.

By a comparison of the inhomogeneous terms of the equations obtained in Sec. II, specialized to specific processes, and the DWB inhomogeneous term of Eq. (1), we are able to see how the usual DWB calculations must be modified to provide *true* first order solutions. In this paper, we do not attempt to show the convergence rate of any practical, iterative solutions but confine our attention to obtaining equations which are soluble by available methods. However, it seems plausible that one of the advantages of the distorted-wave representation is the rapid convergence of an iterative solution. As we shall see in Sec. III, the lowest order term in this representation often has each of the two-body subsystems interacting to *all* orders of the two-body potential.

A certain freedom is available in the choice of the inhomogeneous term by the methods of Sec. II, and Sec. III contains a description of the class of solutions which contain the DWB model as a part of the lowest order term. Section IV describes another class of solutions in which the usual DWB model does not appear at all in the lowest order term. Finally, in Sec. V, the implications of the results, and further applications of the methods are discussed.

II. FORMAL SOLUTION OF THE THREE-BODY AMPLITUDE

We consider the amplitude for a nonrelativistic three-body rearrangement process for spinless particles and use the notation of Ref. 2. [This formulation may be easily extended for a wider class of three-body reactions other than that given by Eq. (8), below.] We review briefly the results of Ref. 2, and refer the reader to that paper for the derivations. Consider three particles α, β, and γ interacting through two-body potentials V_α, V_β, and V_γ. In this notation the potential $V_i(i=\alpha, \beta, \gamma)$ is the pair interaction between those two particles *not* labeled with the index i. We assume an initial channel α in which the particle designated α is free while β and γ are bound by the interaction V_α, and a final channel β in which β is free while α and γ are bound via V_β. The rearrangement process is written schematically,

$$\alpha+(\beta+\gamma)\rightarrow\beta+(\alpha+\gamma), \tag{8}$$

where a parenthesis indicates a bound state. The complete Hamiltonian for all particles is

$$H=H_0+V_\alpha+V_\beta+V_\gamma, \tag{9}$$

where H_0 is the kinetic-energy operator for the relative motion of the three particles. The Hamiltonian for the entrance channel α is

$$H_\alpha=H_0+V_\alpha\equiv H-v_\alpha, \tag{10}$$

and for the exit channel β,

$$H_\beta=H_0+V_\beta\equiv H-v_\beta. \tag{11}$$

Equations (10) and (11) define the channel interactions v_α and v_β, as well as the channel Hamiltonians H_α and H_β. These Hamiltonians have energy eigenfunctions ϕ_α and ϕ_β, respectively, with eigenvalue E.

The exact transition amplitude for the reaction (8)

is given by Eq. (6) of Ref. 2,

$$T_{\beta\alpha}^{-}=\langle\phi_\beta|\,U_{\beta\alpha}^{-}|\,\phi_\alpha\rangle. \tag{12}$$

It was found in Ref. 2 that the transition operator $U_{\beta\alpha}^{-}$ in the distorted-wave formulation of the Lippmann-Schwinger equation is

$$U_{\beta\alpha}^{-}=\omega_\beta^{-\dagger}[(v_\alpha-w_\alpha)+(v_\beta-w_\beta^\dagger)G^{+}(v_\alpha-w_\alpha)]\omega_\alpha^{+}, \tag{13}$$

[cf. Eqs. (24) and (34) of Ref. 2].

The corresponding integral equation for $U_{\beta\alpha}^{-}$ is

$$U_{\beta\alpha}^{-}=\omega_\beta^{-\dagger}(v_\alpha-w_\alpha)\omega_\alpha^{+}+\omega_\beta^{-\dagger}(v_\beta-w_\beta^\dagger)G_\beta^{+}U_{\beta\alpha}^{-}. \tag{14}$$

Equations similar to (12), (13), and (14) hold for $T_{\beta\alpha}^{+}$ and $U_{\beta\alpha}^{+}$, and can be found in Ref. 2. For the discussion that follows, we restrict ourselves to $T_{\beta\alpha}^{-}$ and $U_{\beta\alpha}^{-}$. In Eq. (14), w_α is a model interaction which must be chosen so that it does not connect ϕ_α with ϕ_β while w_β is an interaction whose choice is completely free. The wave operator ω_α^{+} operating on ϕ_α gives scattering states due to the potential w_α. Likewise the wave operator ω_β^{-} gives the scattering of the final state ϕ_β due to w_β. The Green's function G^{+} in Eq. (13) is the complete Green's function, $G^{+}=(E-H+i\epsilon)^{-1}$, whereas G_β^{+} in Eq. (14) is defined by

$$G_\beta^{+}=[E-H_\beta+i\epsilon]^{-1}. \tag{15}$$

We note that Eq. (14) has the general form of the integral equation given in Eq. (1). The inhomogeneous term is

$$I=\omega_\beta^{-\dagger}(v_\alpha-w_\alpha)\omega_\alpha^{+}\equiv U_{\beta\alpha}^{-\mathrm{DWB}}, \tag{16}$$

and gives the distorted-wave Born amplitude for rearrangement scattering,

$$T_{\beta\alpha}^{-\mathrm{DWB}}=\langle\phi_\beta|\,U_{\beta\alpha}^{-\mathrm{DWB}}|\,\phi_\alpha\rangle$$

[see Eq. (29) of Ref. 2]. The kernel in Eq. (14),

$$K=\omega_\beta^{-\dagger}(v_\beta-w_\beta^\dagger)G_\beta^{+}, \tag{17}$$

contains the dangerous diagrams and upon iteration produces the divergence discussed in the Introduction. With this brief resumé of Ref. 2, we are ready to investigate nondivergent solutions for $U_{\beta\alpha}^{-}$, by two different, though equivalent methods, which lead to Eqs. (20) and (34) below.

An obvious method of overcoming the difficulties that arise from the disconnected diagrams in the kernel of Eq. (17) is to explicitly subtract all the dangerous diagrams from K. This procedure is not as systematic as the solution in terms of two coupled integral equations introduced later, but it exhibits the main principle of our method. (We use the distorted-wave formalism of Ref. 2, although the methods and results are equally applicable to the ordinary nondistorted Born formulation for which $w_\alpha=w_\beta=0$.) We rewrite Eq. (14) as

$$U_{\beta\alpha}^{-}=U_{\beta\alpha}^{-\mathrm{DWB}}+(K-K_0)U_{\beta\alpha}^{-}+K_0U_{\beta\alpha}^{-},$$

or

$$(1-K_0)U_{\beta\alpha}^{-}=U_{\beta\alpha}^{-\mathrm{DWB}}+(K-K_0)U_{\beta\alpha}^{-}. \tag{18}$$

The choice of the kernel K_0 is free, except that it must *at least* contain the pathological part of K. For instance, in connection with Eq. (17), or Eq. (38) of Ref. 2, K_0 could be taken as

$$K_0=(v_\beta-w_\beta^\dagger)G_0^{+}, \tag{19}$$

where G_0^{+} is the free three-particle Green's function. By eliminating all the dangerous diagrams in this way, we assume that the new operator $(K-K_0)$ is completely continuous.[4,5] If we multiply Eq. (18) on the left by $(1-K_0)^{-1}$, we find a new integral equation and our first solution for $U_{\beta\alpha}^{-}$,

$$U_{\beta\alpha}^{-}=(1-K_0)^{-1}U_{\beta\alpha}^{-\mathrm{DWB}}+(1-K_0)^{-1}(K-K_0)U_{\beta\alpha}^{-}. \tag{20}$$

The operator $(1-K_0)^{-1}$ is rather simple if K_0 has the general form of Eq. (19), since

$$(1-K_0)^{-1}=(1-(v_\beta-w_\beta^\dagger)G_0)^{-1}$$

$$=\left[1+(v_\beta-w_\beta^\dagger)\frac{1}{E-H_0-(v_\beta-w_\beta^\dagger)+i\epsilon}\right] \tag{21}$$

is just the wave operator for the interaction $(v_\beta-w_\beta^\dagger)$ and, with a suitable choice for w_β, should be calculable without encountering any pathologies.[8] If the distorting potential w_β is chosen such that either (a) only a single two-body potential appears in $(1-K_0)^{-1}$, or (b) the Green's function in $(1-K_0)^{-1}$ is factorable, then the operator $(1-K_0)^{-1}$ is a bounded operator,[4,5] and its product with the completely continuous (or Schmidt) operator $(K-K_0)$ yields another Schmidt operator K', where

$$K'=(1-K_0)^{-1}(K-K_0). \tag{22}$$

A specific example of this procedure will be considered in Sec. IV.

A meaningful solution to the scattering problem is now possible either by iteration of the new inhomogeneous term $(1-K_0)^{-1}U_{\beta\alpha}^{-\mathrm{DWB}}$ or by other methods that apply to integral equations for which the kernel is completely continuous.[5] In any case, the new inhomogeneous term

$$I=(1-K_0)^{-1}U_{\beta\alpha}^{-\mathrm{DWB}} \tag{23}$$

has at least a chance of being shown to be a true first approximation to the exact amplitude, whereas the pure distorted-wave Born term $U_{\beta\alpha}^{-\mathrm{DWB}}$ does not.[9]

There may be applications for which $(1-K_0)^{-1}$ has little effect and the inhomogeneous term of Eq. (23) is well represented by $U_{\beta\alpha}^{-\mathrm{DWB}}$. However, Eq. (23) or its more general form Eq. (34), permits an explicit calculation of a genuine first approximation to $U_{\beta\alpha}^{-}$, and by obtaining a quantitative evaluation of this term, the accuracy and applicability of the distorted-wave Born model for the scattering process is determied.

Our second solution for $U_{\beta\alpha}^{-}$, equivalent to Eq. (20),

[9] B. Buck and J. R. Rook, Nucl. Phys. **67**, 504 (1965).

is obtained in terms of coupled integral equations. The general form of the Faddeev-Lovelace integral equations for (channel α) $\rightarrow$ (channel β) is obtained by expressing the operator $U_{\beta\alpha}^{-}$ in terms of the operators $U_{\alpha\alpha}^{-}$ for (channel α) $\rightarrow$ (channel α), and $U_{\gamma\alpha}^{-}$ for (channel α) $\rightarrow$ (channel γ), and if necessary, $U_{0\alpha}^{-}$ for (channel α) $\rightarrow$ (channel 0), channel 0 being the channel in which all particles propagate freely. This procedure leads to the coupled equations of Refs. 3, 4, and 5. However, under the mass conditions set down above, it is only necessary to write $U_{\beta\alpha}^{-}$ in terms of one other operator, which we call $U_{x\alpha}^{-}$. The channel "x" is as yet unspecified, and, as we shall see, its choice leads to a certain freedom for the kernel of the final single integral equation for $U_{\beta\alpha}^{-}$. This freedom in the kernel often allows one to choose between several sets of the two-body intermediate states that will eventually appear in the inhomogeneous term.

We return to Eq. (13) for the operator $U_{\beta\alpha}^{-}$,

$$U_{\beta\alpha}^{-}=\omega_\beta^{-\dagger}[(v_\alpha-w_\alpha)+(v_\beta-w_\beta^\dagger)G^+(v_\alpha-w_\alpha)]\omega_\alpha^+, \quad (24)$$

and write the general operator identity

$$G^+=g_x^+[1+v_xG^+], \quad (25)$$

where v_x is a potential or sum of potentials as yet unspecified. From Eq. (25) it is evident that

$$g_x^+=[E-H+v_x+i\epsilon]^{-1}$$
$$=[E-H_0-V_\alpha-V_\beta-V_\gamma+v_x+i\epsilon]^{-1}. \quad (26)$$

Substitution of Eq. (25) into Eq. (24) obtains the first of the two integral equations,

$$U_{\beta\alpha}^{-}=\omega_\beta^{-\dagger}(v_\alpha-w_\alpha)\omega_\alpha^+ +\omega_\beta^{-\dagger}(v_\beta-w_\beta^\dagger)g_x^+U_{x\alpha}^{-}, \quad (27)$$

where

$$U_{x\alpha}^{-}=(v_\alpha-w_\alpha)\omega_\alpha^+ +v_xG^+(v_\alpha-w_\alpha)\omega_\alpha^+. \quad (28)$$

By repeating this same procedure, we can express $U_{x\alpha}^{-}$ in terms of $U_{\beta\alpha}^{-}$:

$$U_{x\alpha}^{-}=(v_\alpha-w_\alpha)\omega_\alpha^+ +v_xg_\beta^+[1+(v_\beta-w_\beta^\dagger)G^+](v_\alpha-w_\alpha)\omega_\alpha^+$$
$$=(v_\alpha-w_\alpha)\omega_\alpha^+ +v_xG_\beta^+\omega_\beta^{-\dagger}$$
$$\times[1+(v_\beta-w_\beta^\dagger)G^+](w_\alpha-w_\alpha)\omega_\alpha^+, \quad (29)$$

where

$$G_\beta^+\omega_\beta^{-\dagger}=g_\beta^+. \quad (30)$$

Equation (30) follows from Eq. (15) and from the definitions of g_β^+,

$$g_\beta^+=\frac{1}{E-H_0-V_\beta-w_\beta^\dagger+i\epsilon}, \quad (31)$$

and the distorted-wave operator ω_β^-,

$$\omega_\beta^-=1+\frac{1}{E-H_0-V_\beta-w_\beta-i\epsilon}w_\beta. \quad (32)$$

Comparison of Eq. (29) with Eq. (24) yields the second of the two integral equations,

$$U_{x\alpha}^{-}=(v_\alpha-w_\alpha)\omega_\alpha^+ +v_xG_\beta^+U_{\beta\alpha}^{-}. \quad (33)$$

Equations (27) and (33) are the two coupled integral equations for $U_{\beta\alpha}^{-}$ which can be solved if the kernel is completely continuous and if the Green's functions G_β^+ and g_x^+ are manageable. The first of these, G_β^+, given in Eq. (15), is just the three-body Green's function with the single potential V_β, and requires only the knowledge of the $(\alpha+\gamma)$ system, which we have assumed already solved. The other Green's function g_x^+ will in general contain two potentials, and can be solved for those cases that satisfy the mass condition discussed earlier.

The two integral equations (27) and (33) provide our second solution for $U_{\beta\alpha}^{-}$. However, it is more instructive to combine them and obtain a single integral equation of the form of Eq. (1) for the transition matrix. Substituting Eq. (33) into Eq. (27), we obtain finally,

$$U_{\beta\alpha}^{-}=\omega_\beta^{-\dagger}(v_\alpha-w_\alpha)\omega_\alpha^+ +\omega_\beta^{-\dagger}(v_\beta-w_\beta^\dagger)g_x^+(v_\alpha-w_\alpha)\omega_\alpha^+$$
$$+\omega_\beta^{-\dagger}(v_\beta-w_\beta^\dagger)g_x^+v_xG_\beta^+U_{\beta\alpha}^{-}. \quad (34)$$

This integral equation is the desired end result of our manipulations, and should be compared with our first solution given by Eq. (20).

We conclude our derivation with a brief discussion of the terms in Eq. (34). The inhomogeneous term in this equation is

$$I=\omega_\beta^{-\dagger}[(v_\alpha-w_\alpha)+(v_\beta-w_\beta^\dagger)g_x^+(v_\alpha-w_\alpha)]\omega_\alpha^+, \quad (35)$$

and is similar to the inhomogeneous term of Eq. (23), except that the intermediate states represented by g_x^+, as yet unspecified, are explicitly displayed in Eq. (35). The important thing to note is that the direct reaction term $\omega_\beta^{-\dagger}(v_\alpha-w_\alpha)\omega_\alpha^+$ always appears with the term $\omega_\beta^{-\dagger}[(v_\beta-w_\beta^\dagger)g_x^+(v_\alpha-w_\alpha)]\omega_\alpha^+$, representing excitations to the two-body intermediate states in the spectrum of g_x^+. Furthermore, it is this combination of terms which is the mathematically meaningful first approximation to the exact amplitude, rather than the Born term alone.[1,2]

The kernel of the integral equation (34) is

$$K=\omega_\beta^{-\dagger}(v_\beta-w_\beta^\dagger)g_x^+v_xG_\beta^+, \quad (36)$$

and by suitable choice of the potentials w_β and v_x, each of which is completely arbitrary,[2] K can be made to be connected. To examine this point further, we look at that part of K that could be most troublesome, by taking only the free Green's function G_0^+ in the expansion of both g_x^+ and G_β^+, and by taking only the unit operator in the wave operator $\omega_\beta^{-\dagger}$. Then the pertinent part of K is

$$\bar{K}_0=(v_\beta-w_\beta^\dagger)G_0^+v_xG_0^+. \quad (37)$$

Certainly if $\bar{K}_0$ is well-behaved, the remainder, $(K-\bar{K}_0)$, will also in general be well-behaved. The divergence difficulties of Refs. 1 and 2 can now be systematically eliminated by requiring that $\bar{K}_0$ contain no disconnected diagrams. This condition is met if no two-body potential in $(v_\beta-w_\beta^\dagger)$ is repeated in v_x. It is generally possible, under the mass approximation already made, to find

at least one set of potentials w_β and v_x such that: (1) $\bar{K}_0$ is connected and (2) all Green's functions are calculable. That is, the Green's functions must obey the conditions set down following Eq. (21). They must either be of the two-body form of Eq. (6) in which case they present no problem, or they must be factorable three-body Green's functions, of the form of Eq. (7), and obeying the mass restrictions made following Eq. (7).

If it turns out that for a particular reaction, the conditions (1) and (2) above are met with a variety of potentials w_β and v_x, then the choice between the several sets of potentials can be determined by the rapidity of convergence of the iteration procedure for a particular set, or can be made on physical grounds based on the importance of particular two-body intermediate states in the spectrum of g_x^+, appearing in the inhomogeneous term, Eq. (35).

To complete this section, we write without derivation the relevant integral equations for the opeartor $U_{\beta\alpha}^+$ that obtains the transition amplitude $T_{\beta\alpha}^+$ [See Eq. (27) of Ref. 2],

$$U_{\beta\alpha}^+=\omega_\beta^{-\dagger}(v_\beta-w_\beta^\dagger)\omega_\alpha^+ + U_{\beta x}^+ g_x^+(v_\alpha-w_\alpha)\omega_\alpha^+, \quad (38)$$

and

$$U_{\beta x}^+=\omega_\beta^{-\dagger}(v_\beta-w_\beta^\dagger)+U_{\beta\alpha}^+ G_\alpha^+ v_x. \quad (39)$$

These two coupled equations combine to yield the single integral equation for $U_{\beta\alpha}^+$,

$$U_{\beta\alpha}^+=\omega_\beta^{-\dagger}(v_\beta-w_\beta^\dagger)\omega_\alpha^+ + \omega_\beta^{-\dagger}(v_\beta-w_\beta^\dagger)g_x^+(v_\alpha-w_\alpha)\omega_\alpha^+ + U_{\beta\alpha}^+ G_\alpha^+ v_x g_x^+(v_\alpha-w_\alpha)\omega_\alpha^+. \quad (40)$$

The inhomogeneous part

$$I'=\omega_\beta^{-\dagger}[(v_\beta-w_\beta^\dagger)+(v_\beta-w_\beta^\dagger)g_x^+(v_\alpha-w_\alpha)]\omega_\alpha^+ \quad (41)$$

and the kernel

$$K'=G_\alpha^+ v_x g_x^+(v_\alpha-w_\alpha)\omega_\alpha^+ \quad (42)$$

can be compared with I and K of Eqs. (35) and (36), respectively. Note that the second term (added to the DWB term) in Eq. (41) has apparently the same form as the corresponding second term in Eq. (35). The difference between these two forms arises from the fact that in Eq. (35), w_α must be chosen so as not to lead to the rearranged state ϕ_β while w_β is completely free. The opposite is true of Eq. (41): w_β must not lead to the rearranged state ϕ_α while w_α is free. Whether one chooses the post or prior forms of the integral equations (34) or (40) will depend on convenience of calculation for the specific physical process considered.

However, in either case, one must choose v_x and w_β in Eq. (34), or v_x and w_α in Eq. (40) so that the kernel of the integral equation contains no disconnected diagrams. Then one must either try to solve for the exact amplitude directly,[4,5] or use the approximation in which the inhomogeneous term is the lowest order approximation to the total amplitude. If the latter view is taken, the particular choice of v_x and w_β that gave the connected kernel will also determine the mathematically meaningful distorted-wave approximation. The following two sections give specific examples of this latter procedure.

III. REARRANGEMENT SCATTERING EXPRESSED IN TERMS OF THE DWB AMPLITUDE

The integral equation (34) for $U_{\beta\alpha}^-$ can be formulated in terms of the usual distorted-wave methods[10] by choosing ω_β^- as the exit-channel elastic-scattering operator, and ω_α^+ as the entrance-channel elastic-scattering operator. This choice for ω_α^+ satisfies the condition (20) of Ref. 2 in that ω_α^+ (or w_α) does not lead to the rearranged state β. On the other hand, since w_β is completely arbitrary in this formulation, the usual distorted-wave method picks out a *specific model* for ω_β^- (or w_β), in requiring that it represent elastic scattering. The elastic-scattering operators ω_α^+ and ω_β^- are then obtained from optical-model potentials w_α and w_β, respectively, and these potentials are usually chosen to produce the best fit to the measured elastic scattering in the particular channel.[11]

Since a part of v_β is cancelled by w_β in the kernel Eq. (36) of the integral equation (34), one can only choose the potential v_x after knowing what part of $v_\beta=V_\gamma+V_\alpha$ remains in the expression $(v_\beta-w_\beta^\dagger)$. This is only possible if w_β is known, which means that the particular exit channel β must be specified. Therefore, it is apparent that the preceding generality must be abandoned and particular reactions considered.

A. The (p,n) Knockout Reaction

We choose as the first example the (p,n) knockout reaction on a heavy nucleus, and ignore spin and charge exchange effects. The three particles in this model are, α=proton, β=neutron, γ=inert core. The distorted-wave theory approximates the exit-channel (neutron +residual nucleus) interaction by V_α, the neutron-core interaction. The other potentials are V_β=proton-core interaction, and V_γ=neutron-proton potential. Under this approximation, $(v_\beta-w_\beta^\dagger)=V_\gamma$, which leaves three obvious choices for v_x such that K in Eq. (36) remains connected, i.e., such that V_γ is not repeated in K:

$$\begin{aligned}&\text{(a)}\ v_x=V_\alpha,\\ &\text{(b)}\ v_x=V_\beta,\\ &\text{(c)}\ v_x=V_\alpha+V_\beta.\end{aligned} \quad (43)$$

The corresponding Green's functions are from Eq. (26):

$$\begin{aligned}&\text{(a)}\ g_x^+=(E-H_0-V_\beta-V_\gamma+i\epsilon)^{-1},\\ &\text{(b)}\ g_x^+=(E-H_0-V_\alpha-V_\gamma+i\epsilon)^{-1},\\ &\text{(c)}\ g_x^+=(E-H_0-V_\gamma+i\epsilon)^{-1}.\end{aligned} \quad (44)$$

[10] W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford University Press, London, 1961).

[11] See, for example, N. Austern, *Selected Topics in Nuclear Theory* (International Atomic Energy Agency, Vienna, 1963).

It is evident that the Green's functions (a) and (b) above are nonfactorable three-body Green's functions and are not calculable even under the mass approximations made earlier. In each case, the two potentials $(V_\alpha+V_\gamma)$ or $(V_\beta+V_\gamma)$ are functions of coordinates in which the kinetic-energy operator does not separate. Thus the choice (c) gives the only calculable one for use in our integral equation.

With the choice (c) of Eq. (43), the inhomogeneous term in Eq. (35) becomes

$$I=\omega_\beta^{-\dagger}(V_\gamma+V_\gamma G_\gamma^+ V_\gamma)\omega_\alpha^+, \tag{45}$$

since in the distorted-wave model $w_\alpha \sim V_\beta$ and hence $(v_\alpha-w_\alpha)=V_\gamma$. The operator appearing in the parentheses of Eq. (45) is just t_γ, the two-body neutron-proton t matrix in which the particle γ (i.e., the core) propagates freely. As a consequence, for the knockout reaction we find the not too surprising result that it is t_γ (rather than the V_γ of the DWB) that gives the mathematically meaningful first-order term in the distorted-wave picture. It should be emphasized that t_γ is the *free* neutron-proton t matrix, for which neither the neutron nor proton interacts with the core.

Insofar as V_γ accurately represents t_γ, the DWB model will be successful in providing a first-order approximation. This is apparently the case for electron-hydrogen scattering at high energies.[12] However, for the strong interactions found in the neutron-proton system, it is unlikely that V_γ is ever a good approximation to t_γ; that is, the Born approximation is rather bad for strong interactions at nonrelativistic energies.[13] Certainly if the energy of the reaction is low enough so that the bound triplet deuteron and the singlet n-p resonance are important in intermediate states, the Born term is manifestly inadequate. Even in the higher energy regions, the normalization of the DWB amplitude depends on $V_\gamma(=V_{np})$ which, in general, is quite different from the normalization based on t_γ, and again the Born approximation may fail. In any case, it should not be difficult to calculate the new inhomogeneous term, Eq. (45), by using one of the various models for the free n-p t matrix off the energy shell proposed by several authors.[4,14]

B. Deuteron Stripping and Pickup Reactions

As our second example we consider the well-known and often-used deuteron pickup reaction. [The analogous deuteron stripping reaction can be obtained in its usual distorted-wave form from the amplitude $T_{\beta\alpha}^+$ by applying the results below to the integral equation for $U_{\beta\alpha}^+$, Eq. (40)]. As in the knockout reaction, the entrance-channel elastic interaction w_α is approximated by the proton-core potential, which for pickup is V_γ. The neutron-core potential is V_α, and the neutron-proton potential V_β. The usual distorted-wave method then chooses the exit-channel potential w_β as the elastic deuteron-core potential. With this choice, we find from Eq. (37) that the possible pathological part of the kernel is

$$\bar{K}_0=(V_\gamma+V_\alpha-w_\beta^\dagger)G_0^+ v_x G_0^+, \tag{46}$$

from which it is apparent that v_x must be set equal to V_β, the neutron-proton potential, if $\bar{K}_0$ is to be well-behaved, i.e., contain no disconnected diagrams. Then the Green's function g_x^+ is

$$g_x^+=(E-H_0-V_\alpha-V_\gamma+i\epsilon)^{-1}. \tag{47}$$

The new meaningful first approximation to the transition operator $U_{\beta\alpha}^-$ is, by Eq. (35),

$$I=\omega_\beta^{-\dagger}\left[V_\beta+(V_\gamma+V_\alpha-w_\beta^\dagger)\times\frac{1}{E-H_0-V_\alpha-V_\gamma+i\epsilon}V_\beta\right]\omega_\alpha^+. \tag{48}$$

The spectrum of intermediate states contained in the second term of Eq. (48) are those of proton-core (interacting via V_α) and neutron+core (interacting via V_γ). Although both V_α and V_γ appear in the Green's function, this operator is of the factorable three-body type under the mass condition of an infinitely heavy core made in the previous sections. Consequently, the spectrum of states is calculable in terms of the separate neutron-core and proton-core Green's functions. It is not at all evident how important these intermediate states are, compared with the pure DWB term, $\omega_\beta^{-\dagger}V_\beta\omega_\alpha^+$. However, a calculation, or at least an estimate, is now possible with Eq. (48).

A calculational simplification of the second term may be reasonable if it is assumed that the deuteron-core interaction w_β approximately cancels part of the potentials $(V_\alpha+V_\gamma)$. For example, at very low energies, the important part of w_β is the deuteron-core Coulomb potential which is approximately cancelled by the proton-core Coulomb potential in V_γ. The errors in this approximation depend on the polarizability of the deuteron, which has been shown to be small by Clement[15] and by Kerman and Gibson.[16]

At higher energies, the difficulty of incomplete cancellation of the elastic deuteron-core and nucleon-core potentials inherent in the usual distorted-wave theory is overcome in several ways. First, the deuteron-core potential w_β could be chosen so as to provide the *maximum* cancellation in $(V_\gamma+V_\alpha-w_\beta^\dagger)$ of Eq. (48), thus minimizing the contribution of the intermediate states. This choice of w_β would undoubtedly not fit the experimental elastic scattering in the β channel; however, as we have seen, w_β is completely arbitrary anyway and

[12] E. Gerjuoy, Rev. Mod. Phys. **33**, 544 (1961).
[13] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, New York, 1964), p. 307.
[14] K. L. Kowalski, Phys. Rev. Letters **15**, 798 (1965); H. P. Noyes, *ibid.* **15**, 538 (1965).
[15] C. F. Clement, Phys. Rev. **128**, 2728 (1962).
[16] A. K. Kerman and F. P. Gibson, Argonne National Laboratory Report No. ANL-6848, p. 43 (unpublished).

need not represent the β-channel optical-model potential. Calculations with this choice of w_β are in progress and the results appear quite promising. The second obvious way to avoid the incomplete cancellation of w_β is to never introduce the deuteron-core potential in the first place, i.e., let $w_\beta=0$. By avoiding such artificial potentials between two composite systems (like the deuteron and the core), we can avoid both the difficulty of the theoretical justification of the existence of equivalent local potentials of this type, as well as the problem of incomplete cancellation of potentials. This point will be discussed further in Sec. IV.

C. The Breakup Reaction

As the next example of a specific reaction described by the usual distorted-wave formalism, we consider the (p,pn) reaction on a heavy nucleus. The particles are called by the same names here as in the (p,d) reaction: α=proton, β=core, γ=neutron. The transition operator $U_{\beta\alpha}^-$ now becomes $U_{0\alpha}^-$, the final state being designated "0" when all three particles are free. The integral equation (34) for $U_{\beta\alpha}^-$ holds also for $U_{0\alpha}^-$ if "0" is substituted everywhere for "β".[2]

The distorted-wave theory again approximates the entrance-channel elastic-scattering potential w_α by the proton-core potential V_γ. The exit channel is now distorted via ω_0^- which contains both the proton-core potential V_γ as well as the neutron-core potential V_α. That is,

$$\omega_0^-=[1+(E-H_0-V_\alpha-V_\gamma-i\epsilon)^{-1}(V_\alpha+V_\gamma)], \quad (49)$$

and this operator factors into the product of the separate neutron-core scattering operator and the proton-core operator. [See Eq. (56).] Since v_0 contains all potentials and $w_0=V_\alpha+V_\gamma$, the kernel K of Eq. (36) is now

$$\begin{aligned}K&=\omega_0^{-\dagger}(v_0-w_0)g_x^+v_xG_\beta^+\\&=\omega_0^{-\dagger}V_\beta g_x^+v_xG_0^+.\end{aligned} \quad (50)$$

As with the (p,n) reaction, there are apparently three choices for v_x: V_γ, V_α, or $V_\gamma+V_\alpha$. But, as was the case in the former process, only the choice $v_x=V_\gamma+V_\alpha$ permits a factorable or calculable three-body Green's function, $g_x^+=(E-H_0-V_\beta+i\epsilon)^{-1}=G_\beta^+$.

Thus the inhomogeneous term, the first-order approximation to $U_{0\alpha}^-$, is [by Eq. (35)]

$$\begin{aligned}I&=\omega_0^{-\dagger}(V_\beta+V_\beta G_\beta^+V_\beta)\omega_\alpha^+\\&=\omega_0^{-\dagger}t_\beta\omega_\alpha^+.\end{aligned} \quad (51)$$

Thus the lowest order amplitude is expressed in terms of t_β, the neutron-proton t matrix which appeared before in Eq. (45) for the (p,pn) knockout process. As was the case for the (p,pn) reaction, this is also not a surprising result for the breakup reaction. The probable importance of using t_β, rather than the more usual Born term V_β, has already been conjectured and discussed by McCarthy for $(p,2p)$ reactions[17]; what we have shown here is that t_β *automatically appears* in the lowest order term if one is interested in a rigorous approximation scheme. [However, see Eq. (64) for a somewhat different result for the lowest order term.]

D. Inelastic Excitation

As the last example of these methods, we consider the three-body description of inelastic scattering for which both the entrance- and exit-channel states are eigenfunctions of the same channel Hamiltonian H_α. Particle α is incident on the bound $(\beta+\gamma)$ system (in its ground state) in the entrance channel, and the same particle α leaves the $(\beta+\gamma)$ system in a specific excited state called α' in the exit channel. This three-body model would apply in the "microscopic" description of (p,p') reactions, for instance, for which β is a nucleon bound to the massive core γ.

The exact matrix element $T_{\alpha'\alpha}^-$ is obtained from the exact three-body transition operator $U_{\alpha'\alpha}^-$, in a manner similar to Eq. (12),

$$T_{\alpha'\alpha}^-=\langle\phi_{\alpha'}|U_{\alpha'\alpha}^-|\phi_\alpha\rangle. \quad (52)$$

The prime serves to indicate that the final state $\phi_{\alpha'}$ and the initial state ϕ_α are different eigenfunctions of the channel Hamiltonian H_α. The transition operator is given by Eq. (13), if β is replaced by α':

$$\begin{aligned}U_{\alpha'\alpha}^-=&\omega_{\alpha'}^{-\dagger}(v_\alpha-w_\alpha)\omega_\alpha^+\\&+\omega_{\alpha'}^{-\dagger}(v_\alpha-w_{\alpha'}{}^\dagger)G^+(v_\alpha-w_\alpha)\omega_\alpha^+.\end{aligned} \quad (53)$$

As was the case for rearrangement scattering, the model potentials w_α and $w_{\alpha'}$ differ in that w_α (or the wave operator ω_α^+) must not connect ϕ_α with $\phi_{\alpha'}$, while $w_{\alpha'}$ is completely free. It is now possible to write two coupled integral equations for $U_{\alpha'\alpha}^-$ and $U_{x\alpha}$, where the channel x is as yet unspecified, similar to Eqs. (27) and (33) for the rearrangement case. These two integral equations then lead to the single integral equation

$$\begin{aligned}U_{\alpha'\alpha}^-=&\omega_{\alpha'}^{-\dagger}(v_\alpha-w_\alpha)\omega_\alpha^+\\&+\omega_{\alpha'}^{-\dagger}(v_\alpha-w_{\alpha'}{}^\dagger)g_x^+(v_\alpha-w_\alpha)\omega_\alpha^+\\&+\omega_{\alpha'}^{-\dagger}(v_\alpha-w_{\alpha'}{}^\dagger)g_x^+v_xG_\alpha^+U_{\alpha'\alpha}^-.\end{aligned} \quad (54)$$

Again, in analogy with the rearrangement case, the spectrum of states g_x^+ in the inhomogeneous term depends on the choice of v_x in the kernel of Eq. (54). For the (p,p') case considered here as an example, we take $w_{\alpha'}$ as the proton-core potential V_β, so that $(v_\alpha-w_{\alpha'})=V_\gamma$, the proton-nucleon potential of the usual DWB. Then to produce a connected kernel, the choices for v_x and the corresponding Green's function g_x^+ are the same as in Eqs. (43) and (44) for the knockout reaction. Again, only the choice $v_x=V_\beta+V_\alpha$ yields a calculable Green's function, $g_x^+=(E-H_0-V_\gamma+i\epsilon)^{-1}$

[17] K. L. Lim and I. E. McCarthy, Phys. Rev. Letters **13**, 446 (1964).

$=G_\gamma^+$. Hence, the inhomogeneous term

$$I=\omega_{\alpha'}^{-\dagger}(V_\gamma+V_\gamma G_\gamma^+ V_\gamma)\omega_\alpha^+=\omega_{\alpha'}^{-\dagger}t_\gamma\omega_\alpha^+ \quad (55)$$

gives the mathematically meaningful first-order term for inelastic scattering. Here again, the mathematically correct lowest order interaction operator is t_γ, the free proton-nucleon t matrix, rather than the potential V_γ of the DWB model.

It should be noted finally that the form of the first-order term of Eq. (51) for the (p,pn) reaction as well as Eq. (48) for the (p,d) process, Eq. (45) for the (p,n) reaction, and Eq. (55) for (p,p') inelastic scattering is not unique. The next section will obtain forms for the inhomogeneous term in the (connected) integral equation different from those above, by relaxing the condition that demands the usual elastic exit-channel distorted-wave choice for w_β (or w_0).

IV. REARRANGEMENT SCATTERING EXPRESSED IN TERMS OF THE BREAKUP AMPLITUDE

In the preceding section the final-state wave operators ω_β^- and ω_0^- were chosen to conform with the usual DWB picture of rearrangement and breakup reactions, respectively. As has been indicated,[2] these wave operators and the corresponding potentials w_β and w_0 are, however, quite arbitrary in the post forms of the equations, and in this section, we show that other choices for ω_β^- and ω_0^- lead to integral equations for $U_{\beta\alpha}^-$ and $U_{0\alpha}^-$ with well-behaved kernels. In general, there are many different ways of constructing an acceptable kernel. In the specific case of pickup (or stripping, if $U_{\beta\alpha}^+$ is used) the equations derived below appear to have valuable advantages over the usual DWB treatment. The method described in this section is characterized by setting $w_\beta=0$, thus avoiding ambiguous potentials such as the deuteron-nucleus potential. As a result, we find the DWB amplitude for a *breakup process* appearing as the first term in the iterative expansion for *the rearrangement amplitude*, and find a general relation between the transition operators for rearrangement and breakup.

A. Rearrangement Scattering (Deuteron Pickup)

We recall from Sec. II that the essential point in dealing with the divergence problem was to sum those disconnected diagrams, for which one particle does not interact with the other two particles, into a wave operator (21). This operator was required to be calculable in terms of known two-body operators. Now if particle β is very massive compared with particles α and γ (and this is the essential restriction of this section), it is not necessary to introduce an auxiliary potential w_β to cancel a part of the channel interaction v_β, since the wave operator

$$\left[1+\frac{1}{E-H_0-v_\beta+i\epsilon}v_\beta\right]=\left[1+\frac{1}{E-H_0-V_\alpha-V_\gamma+i\epsilon}(V_\alpha+V_\gamma)\right]$$

is already factorable. This follows from the exact separation of the kinetic-energy operator H_0 into two parts h_α and h_γ for the kinetic energy of particles α and γ relative to the center of mass of particle β. Thus

$$\left[1+\frac{1}{E-H_0-v_\beta+ie}v_\beta\right]=\left[1+\frac{1}{E-h_\alpha-h_\gamma-V_\alpha-V_\gamma+i\epsilon}V_\alpha\right]\times\left[1+\frac{1}{E-h_\alpha-h_\gamma-V_\gamma+i\epsilon}V_\gamma\right]. \quad (56)$$

When this operator acts on a state $|\phi_0\rangle=|\alpha,\gamma\rangle$, where all three particles are free, the resulting state is the product of two distorted waves describing particles α and γ scattering independently on β via the potentials V_γ and V_α, respectively. Therefore, the three-body operator (56) may be replaced by the product of two-body wave operators, evaluated off the energy shell:

$$\left[1+\frac{1}{E-H_0-v_\beta+i\epsilon}v_\beta\right]|\alpha,\gamma\rangle=\lambda_\alpha^+(E-E_\gamma)|\alpha\rangle\,\lambda_\gamma^+(E-E_\alpha)|\gamma\rangle, \quad (57)$$

where

$$\lambda_\alpha^+(E)=\left[1+\frac{1}{E-h_\alpha-V_\alpha+i\epsilon}V_\alpha\right], \quad (58)$$

and E_γ and E_α are the kinetic energies of γ and α relative to the center of mass. Thus if the two-body problems have been solved completely, and in the limit that the mass of particle β becomes infinite, the operator (56) can be calculated and there is no need to introduce a distorting potential w_β.

This choice, $w_\beta=0$, gives $\omega_\beta^{-\dagger}=1$, and the inhomogeneous term of the integral equation (20) for $U_{\beta\alpha}^-$ becomes

$$I=(1-K_0)^{-1}U_{\beta\alpha}^{-\mathrm{DWB}}=\lambda_\alpha^{-\dagger}\lambda_\gamma^{-\dagger}(v_\alpha-w_\alpha)\omega_\alpha^+. \quad (59)$$

The new kernel is

$$K=(1-K_0)^{-1}(K-K_0)=\lambda_\alpha^{-\dagger}\lambda_\gamma^{-\dagger}v_\beta G_0^+ V_\beta G_\beta^+=[\lambda_\alpha^{-\dagger}\lambda_\gamma^{-\dagger}-1]t_\beta G_0^+, \quad (60)$$

where the two-body wave operators $\lambda_\alpha^{-\dagger}$, $\lambda_\gamma^{-\dagger}$ of Eq. (59) and the two-body transition operator t_β for the scattering of particles α and γ are to be evaluated off the energy shell.

The kernel (60) obviously satisfies the requirement that it contain no disconnected diagrams since only the scattered parts of the distorted waves (and interference terms) are included. It was just the unit operator in the wave operator $\omega_\beta^{-\dagger}$ that was troublesome in the kernel of our original equation (14). This result may also be obtained directly from the general equation for the kernel, Eq. (36), by noticing that if $w_\beta=0$ and $\omega_\beta^{-\dagger}=1$, then the choice $v_x=V_\beta$ will not repeat a potential.

The physical interpretation of the inhomogeneous term and the kernel may be readily seen by introducing a complete set of free α and γ states in the expression (12) for the transition amplitude. For convenience let us consider a deuteron pickup, (p,d) reaction, for which particle α is a proton, γ a neutron, and β the massive core. Introducing a complete set of plane-wave proton and neutron states relative to the core, and specifying the distorting potential w_α in the initial channel as the proton-core interaction, we have

$$T_{\beta\alpha}^{-}=\sum_{\alpha,\gamma}\langle\phi_\beta|\alpha,\gamma\rangle\langle\alpha,\gamma|U_{\beta\alpha}^{-}|\phi_\alpha\rangle. \tag{61}$$

The amplitude $\langle\alpha,\gamma|U_{\beta\alpha}^{-}|\phi_\alpha\rangle$ satisfies the equation

$$\langle\alpha,\gamma|U_{\beta\alpha}^{-}|\phi_\alpha\rangle=\langle\alpha,\gamma|\lambda_\alpha^{-\dagger}\lambda_\gamma^{-\dagger}V_\beta\omega_\alpha^{+}|\phi_\alpha\rangle + \sum_{\alpha',\gamma'}\langle\alpha,\gamma|[\lambda_\alpha^{-\dagger}\lambda_\gamma^{-\dagger}-1]t_\beta G_0^{+}|\alpha',\gamma'\rangle \times\langle\alpha',\gamma'|U_{\beta\alpha}^{-}|\phi_\alpha\rangle. \tag{62}$$

The lowest order term is the first term on the right-hand side of Eq. (62) and has the following meaning: The incident proton scatters elastically from the core through the wave operator ω_α^{+}, then interacts with the bound neutron via the potential V_β; the outgoing proton and neutron scatter independently on the core via the wave operators $\lambda_\alpha^{-\dagger}$, $\lambda_\gamma^{-\dagger}$. The probability amplitude that they will then have the correct momenta, etc., to form a deuteron in the final state is taken into account in Eq. (61) by appropriately weighting the breakup amplitude with the projection of the free proton and neutron states on the wave function of the deuteron. The wave operators are evaluated off the energy shell since the breakup is unobserved. Successive iterations of the kernel then allow the proton and neutron to interact between scatterings from the core to all orders through the two-body t matrix t_β taken off the energy shell.

One important advantage of the present formulation is that in the lowest order approximation, Eq. (59), the deuteron is not scattered as a point particle as it is in the DWB approximation, which introduces the deuteron-core distorting potential. Such an optical-model potential is not only ambiguous, but more important, it does not reproduce the *separate* neutron-core and proton-core scattering which, owing to the loose structure of the deuteron, are undoubtedly important in physical pickup or stripping reactions. On the other hand, the picture given here is that, in lowest order, the neutron and proton scatter individually on the core, without rescattering on each other via V_β. Thus the DWB model [the first term in Eq. (48)] and the inhomogeneous term of Eq. (59) represent two extreme viewpoints of deuteron scattering. In the former, the deuteron never breaks up while scattering on the core (although breakup is crudely described to some extent by the imaginary part of the optical potential), whereas, in the latter, the deuteron always dissociates. The actual deuteron scattering lies between these two extremes and is obtained approximately from calculation of the term involving neutron and proton intermediate states in Eq. (48). It is obtained exactly by solving either form of the integral equation for $U_{\beta\alpha}^{-}$.

Recently, Butler,[18] Tanifuji,[19] and Austern[20] have independently derived deuteron stripping amplitudes based on a physical picture involving the free neutron and proton scattering, similar to the one in this section. However, they do not use the integral equation techniques given here, and consequently they obtain calculable matrix elements only by direct simplification of the three-body amplitudes. Their arguments, like those used to give the DWB model, are difficult to justify since estimates of the errors incurred are not possible. However, Butler's fit to experiment[21] for $Ca^{40}(d,p)Ca^{41}$ is in good agreement with the usual DWB analysis.[22]

B. The Breakup Reaction

The integral equation for the amplitude $U_{0\alpha}^{-}$ for a (p,pn) reaction was given in Sec. III.C in terms of the usual distorted-wave formalism. One can alternatively obtain a different expression for the breakup amplitude which is closely related to the amplitude for pickup given by Eq. (61). We take the distorting potential w_0 in the final state as the interaction between the two light particles (proton and neutron), $w_0=V_\beta$, instead of the customary interaction of each light particle with the core [$w_0=v_\beta$, cf. Eq. (49)]. Then a well-behaved kernel is obtained from Eq. (36) by taking $v_x=V_\beta$, and the corresponding Green's function is

$$g_x^{+}=\frac{1}{E-H_0-V_\gamma-V_\alpha+i\epsilon}, \tag{63}$$

which is a factorable three-body Green's function. The resulting integral equation for $U_{0\alpha}^{-}$ has the inhomogeneous term

$$I=\lambda_\beta^{-\dagger}\lambda_\alpha^{-\dagger}\lambda_\gamma^{-\dagger}V_\beta\omega_\alpha^{+}, \tag{64}$$

[18] S. T. Butler, Nature **207**, 1346 (1965).
[19] M. Tanifuji, Nucl. Phys. **58**, 81 (1964).
[20] N. Austern, Brookhaven National Laboratory Report No. BNL 948, 1965, p. 539 (unpublished).
[21] S. T. Butler, R. G. Hewitt, and R. M. May, Phys. Rev. Letters **15**, 1033 (1965).
[22] L. L. Lee, Jr., J. P. Schiffer, B. Zeidman, G. R. Satchler, R. M. Drisco, and R. H. Bassel, Phys. Rev. **136**, B971 (1964).

and the kernel is

$$K=\lambda_\beta^{-\dagger}[\lambda_\alpha^{-\dagger}\lambda_\gamma^{-\dagger}-1]V_\beta G_0^{+}. \tag{65}$$

It is interesting to note that the lowest order term, Eq. (64), is obtained from that of the usual distorted-wave method, Eq. (51), by commutation of the operators $\lambda_\beta^{-\dagger}$ and $\lambda_\alpha^{-\dagger}\lambda_\gamma^{-\dagger}(=\omega_0^{-\dagger})$. Note that $\lambda_\beta^{-\dagger}$ in Eq. (51) acts on V_β to yield the neutron-proton t matrix t_β *before* the proton-core and neutron-core final-state scattering occurs, whereas in Eq. (64) $\lambda_\beta^{-\dagger}$ serves as a final-state interaction *after* the neutron and proton have scattered separately on the core via $\lambda_\alpha^{-\dagger}$ and $\lambda_\gamma^{-\dagger}$.

The connection between the breakup amplitude with the pickup amplitude is obtained by comparing Eqs. (64) and (65) for (p,pn) with Eqs. (59) and (60) for (p,d). This comparison yields the identity

$$U_{0\alpha}^{-}=\lambda_\beta^{-\dagger}U_{\beta\alpha}^{-}, \tag{66}$$

where $\lambda_\beta^{-\dagger}$ is defined by Eq. (58). Thus if the pickup amplitude (or a lowest order approximation to it) is known, the breakup amplitude can be calculated by application of the final-state neutron-proton interaction operator $\lambda_\beta^{-\dagger}$. Conversely, a knowledge of the breakup amplitude gives, as we have already seen in Eq. (59), the amplitude for pickup.

V. DISCUSSION

The integral equations derived in the preceding sections afford a calculable method of obtaining the three-body transition amplitude for a wide variety of interesting reactions. This simplification of the usual three-body Faddeev-Lovelace equations is obtained here for two reasons: First, the mass restriction allows the use of factorable three body Green's functions which would be uncalculable without this restriction. Second, in many cases, the distorted-wave formalism permits cancellations in the potentials $(v_\beta-w_\beta^{\dagger})$ of the general kernel in Eq. (36), and as a consequence, the choice of v_x (and hence of the intermediate states of g_x^{+}) is not as limited as it would have been without the use of distorted waves. Both the mass restriction and the distorted-wave formalism should also prove useful in simplifying the connected integral equations for four-body and more complicated many-body reactions.

However, it must be emphasized that the pioneering work of Faddeev showed the way to the most general solution of the three-body problem in terms of tractable integral equations. Furthermore, Lovelace, Weinberg, Amado and collaborators, and Rosenberg developed and refined these general methods, in particular, for the scattering of three bodies of comparable masses. Unfortunately, the significance of the work of these authors has not been as widely recognized as it should, particularly in the field of nuclear physics.

One reason for this perhaps is that the divergence of the Born series or of the distorted-wave Born series is not sufficient to prove the Born approximation or the distorted-wave Born approximation invalid. It is possible that the Born or distorted-wave Born amplitude may be shown to be valid approximations by methods of solution which have yet to be formulated. For example, it has been conjectured that the Born amplitudes are perhaps first-order terms in asymptotic expansions of the exact amplitude.[12] However, in the absence of such a scheme it seems reasonable and necessary in the authors' opinion, to at least question the validity of these often-used "approximations." A general and rigorous mathematical proof that the distorted-wave Born (or the ordinary Born) model is, or is not valid, would be extremely valuable, but probably difficult to find.

A most promising approach to this problem is the direct comparison of the DWB amplitude with an *exact* calculation of the three-body amplitude for a simplified model process, which is now possible using the Faddeev equations, and extensions of these equations, such as those developed in this paper. Of course, the work involved in the exact calculation of real physical interest is considerably greater than that in the DWB model. One must know off-energy shell amplitudes for which the potentials at present are only defined phenomenologically on the energy shell (optical-model potentials). Even if we are given these as yet unknown amplitudes, the computer problem in solving for the exact three-body amplitude is enormous.

So one might reasonably ask again from a general standpoint to what extent the present DWB calculations are valid. The integral equations technique presented here attempts to answer this question by putting the DWB model into a wider and more rigorous mathematical context than is customary. Under the assumed mass conditions, we have arrived at rather simple forms for the inhomogeneous part of the distorted-wave integral equation for specific rearrangement processes. Since the kernel of the integral equation was chosen in each case to be connected, there is a strong possibility that the inhomogeneous term is a reasonable first approximation to the exact amplitude. Although the convergence of the iterative solution must be examined for each specific process, well-known methods of improving the convergence rate are available when the integral equation has a connected or completely continuous kernel.[23] As a consequence, we have at least found a firm starting point for accurate calculations.

The inhomogeneous term contains not only the DWB amplitude, but also a term describing virtual excitations in one (or at most two) of the two-body subsystems. This additional term is like a second-order Born term, and in principle is not difficult to calculate, especially if the intermediate states are those of the nucleon-nucleon system, which have received some theoretical

[23] See Ref. 8, and also S. Weinberg, Phys. Rev. **131**, 440 (1963); R. Sugar and R. Blankenbecler, *ibid.* **136**, B472 (1964).

attention lately.[4,14] In any case, the magnitude of the second-order term can now be estimated. The important points are the following: It is the sum of the first Born term and a term involving excitations of intermediate states that together afford the meaningful first approximation in the distorted-wave method, and a necessary condition for the DWB model to be a realistic approximation is that the latter term be shown to be negligible.

A result similar to this is obtained from variational methods in rearrangement processes.[24] In order that the three-body transition amplitude Eq. (13) be stationary for small variations of the wave functions, the correct lowest order variational amplitude is the sum of the first-order Born (or distorted-wave Born) amplitude and a second-order term that involves the spectrum of intermediate states of one of the two-body subsystems. This result is very similar to our own and strengthens our argument that both the direct-reaction DWB approximation and the second-order DWB approximation *must* be calculated to obtain a meaningful approximation in lowest order. An approximate calculation of the second-order term for a deuteron stripping reaction in the undistorted-wave representation has been made by Joachain, who finds its magnitude is of the same order as that of the first Born term.[25] His result also lends concrete calculational support to the argument that the pure direct-reaction term, or the Born term, is not a general, demonstrable first approximation to the three-body amplitude.

It has been recognized for several years that second-order terms of this kind were necessary to understand the inelastic excitations of collective nuclear states induced by proton and neutron scattering. It was found that the virtual excitation and de-excitation of strongly coupled low-lying rotational states must be calculated in addition to the usual direct excitation, in order to explain experimental results. Although the general spirit of these methods is similar to our own, there are several differences, however, between these strong-coupling calculations and the three-body model for inelastic scattering of Sec. III.D. First, the usual strong-coupling methods use a "macroscopic" reaction model which is, in effect, a two-body collective-model approximation to the actual many-body problem.[26–28] As a consequence, this model avoids the divergence difficulties inherent in actual three- (and more) body situations since the graphs for two-body kernels are not disconnected. Second, even in the more realistic many-body or "microscopic" models developed recently,[29] the strong-coupling approximation would explicitly put a truncated subset of target intermediate states, corresponding to a discrete set of bound states in the spectrum of G_α^+ in our Eq. (54) and transitions between these states would be caused by the potential V_γ.

However, the microscopic model, like the macroscopic model, again avoids the divergence problems simply by ignoring the disconnected diagrams which result from the unbound scattering states in the spectrum G_α^+. In other words, by truncation of the spectrum of G_α^+, all the disconnected diagrams are discarded. Our philosophy is that one should not throw away these dangerous diagrams; instead, the divergence should first be cured (by application of Faddeev-like methods), and *then* approximations, like the truncation of a set of states, can be made with a great deal more mathematical justification.

Specifically, for this case, we show in Eq. (55) the effects of keeping and curing the divergences. The mathematically meaningful calculation of the amplitude is possible only if *all* the diagrams involving the two-body interaction V_γ have been previously summed into the interaction operator t_γ, the t matrix for the γ subsystem. It is this operator rather than V_γ which appears in the kernel and causes transitions among the states in the spectrum of G_α^+ when the complete integral equation is solved.

We should like to point out that the mass restrictions of this paper may not yield a good approximation in some physical processes, particularly at high energies. It was found earlier[30] that these same mass restrictions, when used in a model for medium and high-energy nuclear-rearrangement processes, gave results in striking disagreement with experiment. When the recoil terms (terms of the order of the mass ratio of the light particle to the heavy particle) were included, it was found that they were not negligible and that they changed the entire character of both the angular distribution as well as the magnitude of the reaction cross sections. Furthermore, reasonable agreement with experiment was obtained by the inclusion of recoil effects. Although this previous work was carried out in the framework of the DWB model, we would expect the same general results to obtain in the framework of the more exact first-order calculation advocated here. Since the recoil effects become important only at high energies, our neglect of them in this paper is probably justifiable for low-energy nuclear reactions, $E<50$ MeV.

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