O^{2-} ions, the Tm^{2+}-Tm^{3+} pairs then being formed by γ irradiation. This model provided the required distortion along the $\langle 110 \rangle$ axes and accounted for the need for oxygen during the growth process. However, the absence of spectra corresponding to Yb^{3+}-Yb^{3+} pairs seems to argue against this model. It is still possible that the spectra could arise from Yb^{3+}-Yb^{2+} pairs, and that the rotation of the x and y axes in the (110) plane could be produced by a migration of one of the O^{2-} ions to a more remote F^- site, but this seems to us to be rather unlikely. The construction of a detailed model for the compensation must therefore await further experiments.

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Spherical Model with Long-Range Ferromagnetic Interactions*

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The behavior of the spherical model of a ferromagnet with an interaction energy between the magnetic spins which varies with distance as $1/r^{d+\sigma}$ (where d is the dimensionality of the lattice and $\sigma>0$) is analyzed. It is shown that the model exhibits a ferromagnetic transition in one and two dimensions, providing $0<\sigma<d$. (The usual spherical model with nearest-neighbor interactions does not have a transition in one and two dimensions.) The critical-point behavior is investigated. It is found that the singularities in the specific heat and susceptibility are dependent on σ and d, but the behavior of the magnetization is independent of σ and d. In three dimensions the susceptibility diverges as $(T-T_c)^{-\gamma}$, where $\gamma=1$ for $0<\sigma<\frac{3}{2}$, $\gamma=\sigma/(3-\sigma)$ for $\frac{3}{2}<\sigma<2$ and $\gamma=2$ for $\sigma>2$. The asymptotic form of the spin-spin correlation function $\Gamma(\mathbf{r})$ is studied in the neighborhood of the critical temperature T_c. At $T=T_c$, $\Gamma(\mathbf{r})$ decays for large r as $1/r^{d-\sigma}$. Several two-dimensional models with long-range interactions falling off as $1/r^2$ in certain directions only are also investigated.

1. INTRODUCTION

In this paper, the properties of the spherical model with long-range ferromagnetic interactions between the spins, varying as $1/r^{d+\sigma}$, are determined. The spherical model may be considered as an approximate representation of the "more realistic" Ising model. Although considerable success has been achieved in the understanding of systems with certain types of long-range interactions, little detailed information is available concerning the behavior of systems with interaction potentials decaying as $1/r^{d+\sigma}$. Some of the main results so far obtained are now briefly reviewed, in order to see what light they throw on the properties of $1/r^{d+\sigma}$ interactions.

Exact results have been obtained by Kac, Uhlenbeck, and Hemmer[1–4] for a one-dimensional model of hard rods with exponential attractive interactions, and by Baker,[5,6] and Kac and Helfand[7] for the Ising model in one and two dimensions with exponential interactions in certain directions. In the limit that the exponential interaction becomes infinitely long-range, the one-dimensional gas has a phase transition, which is described exactly by the van der Waals equation (with the Maxwell equal-area rule), while the Ising models show a Bragg-Williams-type transition in this limit.[7a] Whether the properties of these systems (especially the critical properties) are characteristic of systems with more realistic interactions of the form $1/r^{d+\sigma}$ is not at present known. There is, however, as we shall see, some evidence to suggest that this is not so.

Using diagrammatic methods, Brout[8] has developed an expansion of the thermodynamic functions of the Ising model, as a power series in the reciprocal range of the interaction. It is found that the expansion reduces to the Weiss-Bragg-Williams result in the limit of infinite-range interactions. However, for finite-range interactions, the expansion breaks down in the critical region. Brout[8] and, later, Horwitz and Callen[9] have partially overcome these difficulties by obtaining self-

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2 M. Kac, G. E. Uhlenbeck, and P. C. Hemmer, J. Math. Phys. 4, 216 (1963).
3 G. E. Uhlenbeck, P. C. Hemmer, and M. Kac, J. Math. Phys. 4, 229 (1963).
4 P. C. Hemmer, M. Kac, and G. E. Uhlenbeck, J. Math. Phys. 5, 60 (1964).
5 G. A. Baker, Jr., Phys. Rev. 122, 1477 (1961).
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7a *Note added in proof.* It has been shown rigorously by J. L. Lebowitz and O. Penrose [J. Math. Phys. 7, 98 (1966)] that, for a general class of interaction potentials which have a range parameter $1/\gamma$, the van der Waals–Maxwell result is obtained in the limit $\gamma\to 0$, in any dimension.
8 R. Brout, Phys. Rev. 118, 1009 (1960).
9 G. Horwitz and H. B. Callen, Phys. Rev. 124, 1757 (1961).

consistent theories, claimed to be valid for all temperatures. Siegert[10,11] has obtained similar expansions in the reciprocal range of interaction, using a different technique, for a general class of interactions. In the critical region Siegert's method leads to similar difficulties as do the "high-density" expansions. We conclude, therefore, that these methods are not likely to be very useful for deriving the critical behavior of systems with $1/r^{d+\sigma}$ interactions.

The method of Yvon[12] based on a cluster-integral technique, similar to that of Mayer, has been used by Domb and Hiley[13] to derive a series of closed-form approximations to the partition function of the Ising model with nearest-neighbor interactions. They found that the extrapolated series of closed forms converged towards the correct behavior in the critical region, although the rate of convergence was rather slow. This method has been generalized by Hiley and Joyce[14] for arbitrary interactions, and a preliminary study of $1/r^{d+\sigma}$ type interactions has been made, using the first-order closed forms corresponding to a generalized Bethe approximation. Although considerable numerical computation would be required to analyze the higher order approximations for long-range interactions, the method does provide, in principle, a series of estimates for the critical behavior.

Exact series expansions at high and low temperatures have yielded (in the absence of exact solutions) the most reliable information concerning the critical behavior of lattice systems with nearest-neighbor interactions.[15] This method has been generalized for long-range interactions, and used to investigate the properties of the Ising model with $1/r^{d+\sigma}$ interactions.[16] For an interaction energy varying as $1/r^3$ in two dimensions, it is found that the high-temperature susceptibility in zero field, for T near T_c, behaves as

$$\chi_0(T)\approx B/[1-(T_c/T)]^{\gamma}, \qquad (1.1)$$

where $\gamma=1.13\pm0.01$. This result[17] differs substantially from the usual nearest-neighbor value of 1.75, and the Weiss mean-field value of 1. It would be interesting if this nonclassical behavior with long-range interactions of the form $1/r^{d+\sigma}$ could be supported qualitatively by an analytic theory.

Most approximate theories of the nearest-neighbor Ising model, such as the Weiss-Bragg-Williams and Bethe theories, effectively introduce infinite-range interactions, by the use of mean fields, thus giving the value of $\gamma=1$. However, the spherical model, first discussed by Berlin and Kac,[18] contrasts strongly with these theories, and predicts, for finite-range ferromagnetic interactions in three dimensions, $\gamma=2$. We shall find in Sec. 4, that the three-dimensional spherical model with an infinite-range interaction of the form $1/r^{3+\sigma}$ has

$$\begin{aligned}\gamma&=1, & 0<\sigma<\tfrac{3}{2}\\ &=\sigma/(3-\sigma), & \tfrac{3}{2}<\sigma<2\\ &=2, & \sigma>2.\end{aligned} \qquad (1.2)$$

In the range $\frac{3}{2}<\sigma<2$ we see that γ lies between the nearest-neighbor and mean-field values, in qualitative agreement with the results of the exact series extrapolations mentioned above. Thus the spherical model with long-range interactions is of interest. Brout[19] and, later, Baker[20] have shown that the spherical model has the same high-density limit as that of the Ising model for $T>T_c$. It is plausible, therefore, that the spherical model is a reasonable approximation to the Ising model with $1/r^{d+\sigma}$ interactions. We shall now discuss the detailed properties of the spherical model with these interactions.

In Sec. 2, the generalization of Berlin and Kac's analysis is briefly indicated and the final result for the partition function presented. We consider, in Sec. 3, the existence of a phase transition in one and two dimensions, while in Sec. 4 the thermodynamic properties are derived. The behavior of the correlation function in the critical region is obtained in Sec. 5, and finally some special two-dimensional models are defined and analyzed in Sec. 6.

2. THE PARTITION FUNCTION

We consider a lattice assembly of N sites in which the ith and jth spins, ϵ_i and ϵ_j, respectively, have an Ising interaction energy $-J_{ij}\epsilon_i\epsilon_j$, where J_{ij} is assumed to be of finite-range (the number of spins interacting with any chosen spin is finite) and positive within the range of interaction, and $\epsilon_i\epsilon_j=\pm1$. The range of the interaction will be extended to infinity, only after the limit $N\to\infty$ has been taken. The Hamiltonian for this assembly, in the presence of an external magnetic field H, is

$$\mathcal{H}=-\tfrac{1}{2}\sum_{i,j}J_{ij}\epsilon_i\epsilon_j-\mu H\sum_i\epsilon_i, \qquad (2.1)$$

where the first summation is taken over all i and j with $J_{ii}=0$ and μ is the magnetic moment per spin. Defining $\beta=1/k_BT$ and $W=\beta\mu H$, we can write the partition function as

$$Q_N(I)=2^{-N}\sum_{\epsilon_1=\pm1}\cdots\sum_{\epsilon_N=\pm1}\exp\{(\beta/2)\sum_{i,j}J_{ij}\epsilon_i\epsilon_j+W\sum_i\epsilon_i\}. \qquad (2.2)$$

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[20] G. A. Baker, Jr., Phys. Rev. **126**, 2071 (1962).

This partition function is now evaluated approximately, using the spherical model. The methods developed by Berlin and Kac[18] for the nearest-neighbor spherical model can be used with minor modification to deal with further-neighbor interactions. Therefore the complete details of the derivation of the partition function are not given, but the steps in the argument are indicated.

In the spherical model the spins are allowed to have all values $-\infty<\epsilon_j<\infty$ for all ϵ_j, with the restriction

$$\sum_{j=1}^{N}\epsilon_j^2=N. \tag{2.3}$$

Introducing the δ function

$$\delta(N-\sum_{j=1}^{N}\epsilon_j^2)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\exp\{y(N-\sum_{j=1}^{N}\epsilon_j^2)\}dy, \tag{2.4}$$

we find that the spherical-model partition function can be written as

$$Q_N(S)=\frac{A_N^{-1}}{2\pi i}\int_{\alpha_0-i\infty}^{\alpha_0+i\infty}ds\,\exp(Ns)\int_{-\infty}^{+\infty}\cdots\int d\epsilon_1\cdots d\epsilon_N$$
$$\times\exp\{(\beta/2)\sum_{i,j}J_{ij}\epsilon_i\epsilon_j-s\sum_{j=1}^{N}\epsilon_j^2+W\sum_{j=1}^{N}\epsilon_j\}, \tag{2.5}$$

where

$$A_N=2\pi^{N/2}N^{\frac{1}{2}(N-1)}/\Gamma(N/2).$$

An orthogonal transformation of the variables $\{\epsilon_j\}$ reduces the multiple integral to a product of Gaussian integrals, which yields

$$Q_N(S)=\frac{A_N^{-1}\pi^{N/2}}{2\pi i}\int_{\alpha_0-i\infty}^{\alpha_0+i\infty}ds\,\exp N\left\{s-\frac{1}{2N}\sum_{j=1}^{N}\ln(s-\tfrac{1}{2}\beta\lambda_j)+W^2/4(s-\tfrac{1}{2}\beta\lambda_1)\right\}. \tag{2.6}$$

The determination of the eigenvalues λ_j of the interaction matrix J_{ij} and the derivation of the

$$\lim_{N\to\infty}N^{-1}\sum_{j=1}^{N}\ln(s-\tfrac{1}{2}\beta\lambda_j)$$

were carried out by Berlin and Kac[18] for a d-dimensional simple cubic lattice, by imposing cyclic boundary conditions. Their analysis can be readily generalized to further-neighbor interactions providing the range of the interaction is finite and not too large. (This latter restriction is necessary because of the cyclic boundary conditions.) The integral in (2.6) is then evaluated in the limit $N\to\infty$ by the method of steepest descent. Finally allowing the range of interaction to become infinite, we find, for a d-dimensional "simple cubic" lattice, that

$$\ln Q\equiv\lim_{N\to\infty}N^{-1}\ln Q_N=-\tfrac{1}{2}-\tfrac{1}{2}\ln(\beta\phi)+\beta\phi z_s/2$$
$$+W^2/2\beta\phi(z_s-1)-\frac{1}{2(2\pi)^d}\int_0^{2\pi}\cdots\int d\omega_1\cdots d\omega_d$$
$$\times\ln[z_s-\phi^{-1}\sum_{l_1}\cdots\sum_{-\infty}^{+\infty}{}_{l_d}{}'J_{l_1\cdots l_d}\cos(l_1\omega_1+\cdots+l_d\omega_d)], \tag{2.7}$$

where $J_{l_1\cdots l_d}$ is obtained from J_{ij} by taking the ith spin as an origin for the coordinates $(l_1a,\cdots,l_da)$ of the jth spin. (a is the lattice spacing.) The quantity ϕ is defined as

$$\phi=\sum_{l_1}\cdots\sum_{-\infty}^{+\infty}{}_{l_d}{}'J_{l_1\cdots l_d}<\infty. \tag{2.8}$$

The parameter z_s can be calculated as a function of temperature and external field via the saddle-point equation,

$$\beta\phi=\frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int d\omega_1\cdots d\omega_d$$
$$\times[z_s-\phi^{-1}\sum_{l_1}\cdots\sum_{-\infty}^{\infty}{}_{l_d}{}'J_{l_1\cdots l_d}\cos(l_1\omega_1+\cdots+l_d\omega_d)]^{-1}$$
$$+W^2/\beta\phi(z_s-1)^2. \tag{2.9}$$

In zero field a critical temperature $T_c>0$ exists, providing the integral in (2.9) converges at $z_s=1$. If this condition is satisfied, a normal saddle point z_s, as determined by (2.9), can be found only for $T>T_c$. When $T<T_c$ $(H=0)$, the saddle point "sticks" at $z_s=1$, and the partition function is given by Eq. (2.7), with $W=0$ and $z_s=1$. A normal saddle point exists for all temperatures, if no transition occurs, or if the external field is nonzero, in which case Eqs. (2.7) and (2.9) can always be used.

We now apply the general expressions[21] (2.7) and (2.9) to a ferromagnetic interaction which varies with the distance between the ith and jth spins r_{ij} as $1/r_{ij}^{d+\sigma}$ (where $\sigma>0$). It is convenient, when comparing the properties of models with different σ, to normalize the exchange energy J_{ij} so that the energy per spin at $T=0$ is constant for a fixed magnetic field. This can be achieved by defining

$$J_{ij}=J_0r_{ij}^{-(d+\sigma)}/\sum_j{}'r_{ij}^{-(d+\sigma)}, \tag{2.10}$$

[21] M. Lax has used the spherical model to determine the partition function of a "classical" spin system, in Phys. Rev. **97**, 629 (1955), and a dipole lattice system, in J. Chem. Phys. **20**, 1351 (1952). These results, when simply reinterpreted, are exactly similar to the expressions (2.7) and (2.9). If the summations in the integrands of (2.7) and (2.9) are carried out over nearest neighbors only, the results of Berlin and Kac [see Ref. 18, Eq. (51)] are obtained.

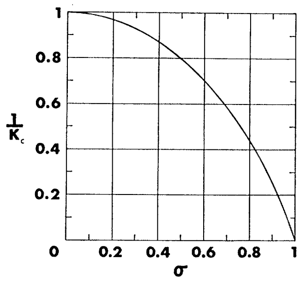

FIG. 1. Variation of Curie temperature K_c^{-1} with σ for one-dimensional spherical model.

TABLE I. Curie temperatures K_c^{-1} for the one-dimensional spherical model.

σ	K_c^{-1}	σ	K_c^{-1}
0.1	0.9918	0.6	0.7020
0.2	0.9676	0.7	0.5851
0.3	0.9272	0.8	0.4391
0.4	0.8703	0.9	0.2522
0.5	0.7959	1	0

and maintaining J_0 constant. The ground-state energy per spin then becomes

$$E_0=-\tfrac{1}{2}J_0-\mu H. \qquad (2.11)$$

For a d-dimensional "cubic" lattice we immediately find that

$$J_{l_1\cdots l_d}=J_0|\mathbf{l}|^{-(d+\sigma)}/\sum_{\mathbf{l}}{}'|\mathbf{l}|^{-(d+\sigma)}, \qquad (2.12)$$

where $|\mathbf{l}|^2=l_1^2+\cdots+l_d^2$. The substitution of (2.12) in (2.7) and (2.9), with $\phi=J_0$ and $K=\beta\phi$, gives

$$\ln Q=-\tfrac{1}{2}-\tfrac{1}{2}\ln K+\tfrac{1}{2}Kz_s+W^2/2K(z_s-1)$$
$$-\frac{1}{2(2\pi)^d}\int d\boldsymbol{\omega}\ln[z_s-S_{d,\sigma}^{-1}S_{d,\sigma}(\boldsymbol{\omega})] \qquad (2.13)$$

for the logarithm of the partition function per spin, and for the saddle-point equation

$$K=W^2/K(z_s-1)^2$$
$$+\frac{1}{(2\pi)^d}\int d\boldsymbol{\omega}[z_s-S_{d,\sigma}^{-1}S_{d,\sigma}(\boldsymbol{\omega})]^{-1}, \qquad (2.14)$$

with

$$S_{d,\sigma}(\boldsymbol{\omega})=\sum_{\mathbf{l}}{}'|\mathbf{l}|^{-(d+\sigma)}\cos(\mathbf{l}\cdot\boldsymbol{\omega}), \qquad (2.15)$$

and

$$S_{d,\sigma}(\mathbf{0})\equiv S_{d,\sigma}.$$

The limits of integration are 0 to 2π for all components of the d-dimensional vector $\boldsymbol{\omega}$, and $\mathbf{l}\cdot\boldsymbol{\omega}=l_1\omega_1+\cdots+l_d\omega_d$.

We now use the basic Eqs. (2.13) and (2.14) to determine the thermodynamic properties of the model.

3. CRITICAL TEMPERATURES

The spherical model, as is well known, exhibits a transition for $d>2$ even for nearest-neighbor interactions, while in one and two dimensions the model shows no ferromagnetic behavior with finite-range interactions. We therefore investigate the one- and two-dimensional systems, to see if a transition can be induced by allowing long-range interactions.

The critical temperature is determined by the equation

$$K_c=\frac{1}{(2\pi)^d}\int d\boldsymbol{\omega}[1-S_{d,\sigma}^{-1}S_{d,\sigma}(\boldsymbol{\omega})]^{-1}, \qquad (3.1)$$

providing the integral converges. It is clear, on reducing the limits of integration in (3.1) to 0 and π, that the convergence depends on the behavior of the integrand for small $|\boldsymbol{\omega}|$. This behavior may be determined, using the methods developed by Nijboer and De Wette.[22] In one dimension, we find that

$$S_{1,\sigma}(\omega_1)=\frac{1}{\Gamma(\frac{1}{2}+\frac{1}{2}\sigma)}\Bigg[\sum_{l_1=-\infty}^{\infty}{}'\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\sigma,\pi l_1^2)}{|l_1|^{1+\sigma}}\cos l_1\omega_1$$
$$-\frac{2\pi^{\frac{1}{2}(1+\sigma)}}{(1+\sigma)}+\pi^{\frac{1}{2}+\sigma}\sum_{l_1=-\infty}^{\infty}\left|l_1-\frac{\omega_1}{2\pi}\right|^{\sigma}$$
$$\times\Gamma\left\{-\frac{\sigma}{2},\pi\left(l_1-\frac{\omega_1}{2\pi}\right)^2\right\}\Bigg], \qquad (3.2)$$

where $\Gamma(n,x)$ is the incomplete gamma function and $\omega_1\neq 0$. The transformed sum (3.2), on expanding for small ω_1, becomes

$$S_{1,\sigma}(\omega_1)=2\zeta(1+\sigma)-\frac{2\pi^{\sigma+\frac{1}{2}}\Gamma(1-\frac{1}{2}\sigma)}{\sigma\Gamma(\frac{1}{2}+\frac{1}{2}\sigma)}\left|\frac{\omega_1}{2\pi}\right|^{\sigma}+O(\omega_1^2)$$
$$(\sigma<2). \qquad (3.3)$$

It follows, by substituting (3.3) into (3.1), that there is a ferromagnetic transition in one dimension, providing

$$0<\sigma<1. \qquad (3.4)$$

Since the spherical model, for nearest-neighbor interactions, gives a critical temperature below that of the corresponding Ising model, it is reasonable to conjecture that the Ising model has a transition in the same range (3.4).[23]

The detailed variation of K_c with σ has been computed by numerical integration of (3.1). The rapidly convergent series (3.2) was used to calculate the integrand, and the dominant singularity at $\omega_1=0$ was subtracted out and integrated separately by use of (3.3).

[22] B. R. A. Nijboer and F. W. De Wette, Physica 23, 309 (1957).

[23] A similar argument leading to (3.4) has been given by M. Kac (to be published); also the International Union of Pure and Applied Physics (IUPAP) Conference, Brown University, 1962 (unpublished).

The numerical values of K_c^{-1} are given in Table I, while in Fig. 1, K_c^{-1} is plotted against σ.

In the limit $\sigma \to 0$, (the mean-field limit), it can be easily shown, by expanding the integral (3.1) as

$$K_c = 1 + \frac{1}{\pi} \sum_{t=2}^{\infty} S_{1,\sigma}^{-t} \int_0^{\pi} S_{1,\sigma}^{t}(\omega_1) d\omega_1,$$

that, in one dimension,

$$K_c = 1 + (\pi^2/12)\sigma^2 + \cdots. \quad (3.5)$$

We now analyze the two-dimensional system in a similar manner. The conversion of the series $S_{2,\sigma}(\boldsymbol{\omega})$ to rapidly convergent form can be effected as before, by the methods of Nijboer and De Wette,[22] which yield

$$S_{2,\sigma}(\boldsymbol{\omega}) = \frac{1}{\Gamma(1+\frac{1}{2}\sigma)} \Bigg[\sum_{l_1 l_2=-\infty}^{\infty}{}' \frac{\Gamma\{1+\frac{1}{2}\sigma, \pi\alpha^2(l_1^2+l_2^2)\}}{(l_1^2+l_2^2)^{1+\frac{1}{2}\sigma}} \cos l_1\omega_1 \cos l_2\omega_2 - \left(\frac{2}{2+\sigma}\right)\pi^{1+\frac{1}{2}\sigma}\alpha^{2+\sigma} + \pi^{1+\sigma} \sum_{l_1 l_2=-\infty}^{\infty} \left\{\left(l_1-\frac{\omega_1}{2\pi}\right)^2 + \left(l_2-\frac{\omega_2}{2\pi}\right)^2\right\}^{\frac{1}{2}\sigma} \Gamma\left\{-\frac{1}{2}\sigma, \frac{\pi}{\alpha^2}\left[\left(l_1-\frac{\omega_1}{2\pi}\right)^2+\left(l_2-\frac{\omega_2}{2\pi}\right)^2\right]\right\}\Bigg], \quad (3.6)$$

providing $|\boldsymbol{\omega}|^2 = \omega_1^2+\omega_2^2 \neq 0$. The parameter α determines the rapidity of convergence of the two series. It follows, by expanding (3.6) for small $|\boldsymbol{\omega}|$, that

$$S_{2,\sigma}(\boldsymbol{\omega}) = S_{2,\sigma} - \frac{\pi 2^{1-\sigma}\Gamma(1-\frac{1}{2}\sigma)}{\sigma\Gamma(1+\frac{1}{2}\sigma)} |\boldsymbol{\omega}|^{\sigma} + O(|\boldsymbol{\omega}|^2), \quad (3.7)$$

providing $\sigma < 2$. When $\sigma = 2$, the expansion becomes

$$S_{2,2}(\boldsymbol{\omega}) = S_{2,2} + \tfrac{1}{4}\pi |\boldsymbol{\omega}|^2 \ln|\boldsymbol{\omega}|^2 + O(|\boldsymbol{\omega}|^2). \quad (3.8)$$

The substitution of (3.7) and (3.8) into (3.1) with a change of variable to polar coordinates leads to the conclusion that there is ferromagnetism in the range

$$0 < \sigma < 2.$$

The critical temperatures in this range have been determined by numerical integration. The integrand of (3.1) was computed via (3.6), with α^2 conveniently taken as 1/6.25. The dominant singularity $|\boldsymbol{\omega}|^{-\sigma}$ of the integrand was subtracted out, using (3.7), and integrated separately. The region of integration was subdivided into $(M \times M)$ cells and the contribution to the integral from each cell was found approximately using a cubature formula derived by Miller.[24] A set of successive values of M was taken, which yielded a sequence of approximations to the integral. Finally this sequence was extrapolated using a method suggested by Isenberg.[25] The numerical values of K_c^{-1} are presented in Table II. The methods described above may be easily extended to the three-dimensional lattice.

The above conclusions concerning the existence of a phase transition in the spherical model are independent of the particular form of the potential (providing all $J_{ij} > 0$), but depend only on the long-range behavior of the potential approaching $1/r^{d+\sigma}$. The short-range part of the potential alters only the values of T_c and not the existence of $T_c > 0$.

TABLE II. Curie temperatures K_c^{-1} for the two-dimensional spherical model.

σ	K_c^{-1}	σ	K_c^{-1}
0	1	1.0	0.868
0.2	0.994	1.2	0.812
0.4	0.977	1.4	0.744
0.6	0.950	1.6	0.660
0.8	0.914	2	0

4. THERMODYNAMIC PROPERTIES

To discuss the thermodynamic properties we must first analyze (2.14) in order to determine $z_s(K,W)$ explicitly in terms of K and W. The details of this analysis are given in the Appendix. The energy per spin,

$$E = k_B T^2 (\partial \ln Q/\partial T)_H,$$

may be derived from (2.13) which yields

$$-E/J_0 = \tfrac{1}{2}[z_s - K^{-1} + W^2 K^{-2}(z_s-1)^{-1}]. \quad (4.1)$$

At low temperatures $(T < T_c)$ the zero-field energy rises linearly for all σ. The high-temperature behavior $(T \gg T_c)$ may be obtained by substituting (A8) into (4.1). The variation of the critical energy E_c with σ can be calculated in one and two dimensions by use of Tables I and II.

The zero-field specific heat, derived from (4.1), is

$$C_V = \tfrac{1}{2}k_B[1 + K^2(dz_s/dK)]. \quad (4.2)$$

We see that, since $z_s = 1$ for all $T \leq T_c$, the specific heat remains constant below T_c for all σ. The behavior in the critical region, when $T > T_c$, is more interesting and may be obtained from (A6), by writing

$$dz_s/dK = (dK/d\xi)^{-1}.$$

We find that

$$\begin{aligned} C_V &\approx \tfrac{1}{2}k_B[1 - (A\sigma/(d-\sigma))(K_c-K)^{(2\sigma-d)/(d-\sigma)} + \cdots], \quad \tfrac{1}{2}d < \sigma < d \\ &\approx \tfrac{1}{2}k_B[1 + (B'/\ln(K_c-K)) + \cdots], \quad \sigma = \tfrac{1}{2}d. \end{aligned} \quad (\sigma < 2) \quad (4.3)$$

[24] J. C. P. Miller, Math. Comp. 14, 130 (1960).
[25] C. Isenberg (to be published).

In the range $0<\sigma<\frac{1}{2}d$, $(\sigma<2)$, the specific heat becomes discontinuous at T_c. The magnitude of this discontinuity, for small σ, is determined by finding $(dK/dz_s)_{K=K_c}$ from (2.14), which leads to

$$C_V^-(T_c)-C_V^+(T_c) \simeq \tfrac{1}{2}k_B\big[1-S_{d,\sigma}^{-2}\sum_{\mathbf{l}}{}'|\mathbf{l}|^{-2(d+\sigma)}+\cdots\big]. \quad (4.4)$$

The divergence of the index in (4.3), in one and two dimensions, as $\sigma\to d$, is explained by the fact that $K_c^{-1}\to 0$ as $\sigma\to d$, and the behavior of C_V, when $\sigma=d$, is nonalgebraic for large K. Consider, for example, the one-dimensional system with $\sigma=1$. In this case, the saddle-point integral (2.14) can be evaluated exactly, using

$$S_{1,1}(\omega_1)=\tfrac{1}{2}\big[(\pi-\omega_1)^2-\tfrac{1}{3}\pi^2\big], \quad (\omega_1\geq 0) \quad (4.5)$$

to give

$$K=\frac{1}{3a}\ln[(a+1)/(a-1)], \text{ where } a^2=\tfrac{1}{3}(2z_s+1), \quad (4.6)$$

and we readily find that near $T=0$

$$C_V\approx\tfrac{1}{2}k_B[1-18K^2\exp(-3K)+\cdots]. \quad (4.7)$$

It is interesting to note that this behavior is similar to that of the two-dimensional nearest-neighbor spherical model. In three dimensions with $\sigma>2$ the specific heat falls off linearly near T_c just as for the usual nearest-neighbor model.

The limit $\sigma\to 0$ (d fixed) and the limit $d\to\infty$ with nearest-neighbor interactions, both give rise to the same Weiss-Bragg-Williams type of behavior. We therefore briefly investigate the spherical model on a hypercubical lattice with nearest-neighbor interactions, and compare the results obtained with (4.3). The nearest-neighbor saddle-point equation, obtained from (2.14) by allowing $\sigma\to\infty$, has already been studied for $d\leq 3$.[18] For higher dimensional lattices the behavior of $z_s(K,H)$ in the critical region is most conveniently determined by straightforward generalization of the methods developed by Maradudin *et al.* for studying the properties of Green's functions.[26] From the resulting asymptotic expansions of (K_c-K) in terms of ξ, we find, using similar arguments as above, that C_V varies in the neighborhood of T_c^+, as

$$\begin{aligned}[(2/k_B)C_V-1] &\approx -A(K_c-K), && d=3\\ &\approx +D/\ln(K_c-K), && d=4\\ &\approx -E[1+F(K_c-K)^{1/2}], && d=5\\ &\approx -G[1-H(K_c-K)\ln(K_c-K)], && d=6\\ &\approx -J[1+P(K_c-K)], && d=7 \quad (4.8)\end{aligned}$$

where A, D, E, $\cdots$ are positive constants. We see that the variation of C_V with d, is qualitatively similar to the variation with σ (d fixed). It is interesting to note that the results (4.8) for $d>3$ are in reasonable agreement with the estimates obtained by Fisher and Gaunt[27] for the hypercubical Ising model by using exact series expansions.

The mean magnetic moment per spin is found from (2.13) and (2.14) to be

$$M=k_BT(\partial\ln Q/\partial H)=\mu W/K\xi. \quad (4.9)$$

Substituting (A9) in (4.9) and allowing $H\to 0$, we find that the spontaneous magnetization is

$$M=\mu[(K-K_c)/K]^{1/2}, \quad (4.10)$$

which is independent of d and σ (apart from the dependence of K_c). The magnetization as a function of field *at* $T=T_c$ is easily obtained from (A7) and (4.9), which yield for small H

$$\begin{aligned}M(K_c,H)&\sim H^{(d-\sigma)/(d+\sigma)}, && \tfrac{1}{2}d<\sigma<d\\ &\sim -H^{1/3}\ln H, && \sigma=\tfrac{1}{2}d \qquad (\sigma<2)\\ &\sim H^{1/3}, && 0<\sigma<\tfrac{1}{2}d. \quad (4.11)\end{aligned}$$

In three dimensions with nearest-neighbor interactions the index in (4.11) becomes $\frac{1}{5}$. (The same result holds for $\sigma>2$ and $d=3$.) This value is in surprising agreement with the estimate of $1/5.20\pm0.15$ obtained by Gaunt *et al.*[28] for the Ising model using exact series extrapolations.

The zero-field susceptibility for $T>T_c$ is

$$\chi_0=\mu^2/J_0\xi. \quad (4.12)$$

As $T\to T_c+$, we find, using (A6), that

$$\begin{aligned}\chi_0&\sim(K_c-K)^{-\sigma/(d-\sigma)}, && \tfrac{1}{2}d<\sigma<d\\ &\sim -(K_c-K)^{-1}\ln(K_c-K), && \sigma=\tfrac{1}{2}d \qquad (\sigma<2)\\ &\sim (K_c-K)^{-1}, && 0<\sigma<\tfrac{1}{2}d. \quad (4.13)\end{aligned}$$

The variation of the susceptibility index γ [see Eq. (1.1)] with σ in three dimensions is shown in Fig. 2. The behavior of χ_0 near $T=0$ for $\sigma\geq 1$ and $d=1$ is of interest, since it shows how the model "prepares" for the onset of the phase transition in the range $0<\sigma<1$. The case $\sigma=1$ can be studied using (4.6), which leads to

$$\chi_0\approx(\mu^2/6J_0)\exp(3K), \quad (K\to\infty). \quad (4.14)$$

For $1<\sigma<2$, ($d=1$), it can be shown, using the methods described in the Appendix, that

$$\chi_0\sim K^{\sigma/(\sigma-1)}. \quad (4.15)$$

We note that as $\sigma\to 1^{\pm}$, the susceptibility index is symmetric about $\sigma=1$. When $\sigma>2$ we find that $\chi_0\sim K^2$ as $K\to\infty$. This varied behavior for different σ suggests that a study of the one-dimensional Ising model with $1/r^{1+\sigma}$ interactions would lead to interesting results.

[26] A. A. Maradudin, E. W. Montroll, G. H. Weiss, R. Herman, and H. W. Milnes, *Green's Functions for Monoatomic Simple Cubic Lattices* (Académie Royale de Belgique, Bruxelles, 1960).

[27] M. E. Fisher and D. S. Gaunt, Phys. Rev. **133**, A224 (1964).

[28] D. S. Gaunt, M. E. Fisher, M. F. Sykes, and J. W. Essam, Phys. Rev. Letters **13**, 713 (1964).

The low-temperature susceptibility may be derived from (A9) and (4.9). When $\sigma<2$, it is found that χ_0 only exists for $0<\sigma<\frac{1}{2}d$, in which range the critical behavior is $\chi_0\sim(K-K_c)^{-1}$.

5. CORRELATION FUNCTION

The correlation Γ_{ij} between two spins ϵ_i, ϵ_j, defined as

$$\Gamma_{ij}=\langle\epsilon_i\epsilon_j\rangle/\langle\epsilon_i^2\rangle^{1/2}\langle\epsilon_j^2\rangle^{1/2}, \qquad (5.1)$$

was derived for the nearest-neighbor spherical model by Berlin and Kac.[18] Their analysis is readily extended to include further-neighbor interactions. It is found that for $T>T_c$ $(H=0)$

$$\Gamma_{0j}=\frac{(\beta\phi)^{-1}}{(2\pi)^d}\int_0^{2\pi}\cdots\int d\omega_1\cdots d\omega_d$$

$$\times\frac{\cos(\lambda_1\omega_1+\cdots+\lambda_d\omega_d)}{z_s-\phi^{-1}\sum_{l_1}\underset{-\infty}{\overset{\infty}{\cdots}}\sum_{l_d}{}' J_{l_1\cdots l_d}\cos(l_1\omega_1+\cdots+l_d\omega_d)}, \qquad (5.2)$$

where $\lambda_1 a, \cdots, \lambda_d a$ are the coordinates of the jth spin with respect to the ith spin.

We now restrict our attention to long-range interactions of the form (2.10). The correlation function then becomes

$$\Gamma_{0j}=\frac{K^{-1}}{(2\pi)^d}\int d\boldsymbol{\omega}\frac{\cos(\boldsymbol{\lambda}\cdot\boldsymbol{\omega})}{z_s-S_{d,\sigma}{}^{-1}S_{d,\sigma}(\boldsymbol{\omega})}. \qquad (5.3)$$

To study the behavior of Γ_{0j} in the critical region, we rewrite (5.3) in polar coordinates, using the expansion (A1), as follows[29]:

$$\Gamma(r)\simeq DK^{-1}r^{1-\frac{1}{2}d}\int_0^\pi\frac{k^{\frac{1}{2}d}J_{\frac{1}{2}d-1}(kr)}{B\xi+k^\sigma+O(k^2)}dk, \quad (\sigma<2), \qquad (5.4)$$

where $r^2=\lambda_1{}^2+\cdots+\lambda_d{}^2$, and B, D depend only on d and σ. Since the dominant part of $\Gamma(r)$ near T_c is determined by the integration near the origin, we can allow, to a first approximation, the upper limit in (5.4) to become infinite. If the integral then diverges at the upper limit an exponential damping factor can be included to ensure convergence. We find, by putting $x=kr$, that the critical-point correlation function $(\xi=0)$ decays for large r as

$$\Gamma(r)\simeq E/r^{d-\sigma}, \quad (T=T_c). \qquad (5.5)$$

Fisher[30] has shown that, if the pair correlation function (for a fluid) $G(r)$ is $\simeq D/r^{d-2+\eta}$ at the critical point, then the direct correlation function $C(r)$ is $\simeq F/r^{d+2-\eta}$. Applying these results to the spin-spin correlation functions, with $\eta=2-\sigma$, we have

$$C(r)\simeq F/r^{d+\sigma} \quad (r\to\infty). \qquad (5.6)$$

We see that $C(r)$ falls off at T_c, in the same way as J_{ij}. It is easy to show, using (5.2) and the definition of $C(r)$ [see Ref. 30, Eq. (3.8)], that the direct correlation function for the spherical model decays in the same way as J_{ij} for *all* temperatures $(T>T_c, H=0)$.

The behavior of Γ_{0j} for $T>T_c$ in the critical region has not been found generally, but two special cases are now considered which are particularly amenable to analysis. We discuss first $d=2$, $\sigma=1$. In this case (5.4) becomes

$$\Gamma(r)\simeq DK^{-1}\int_0^\infty\frac{kJ_0(kr)}{\alpha+k}dk \quad (\alpha=B\xi). \qquad (5.7)$$

This integral can be evaluated[31] as

$$\Gamma(r)\simeq A/r-\tfrac{1}{2}A\pi\alpha[H_0(\alpha r)-Y_0(\alpha r)], \qquad (5.8)$$

where A is a constant, H_0 is Struve's function, and Y_0 is a Bessel function of the second kind. Two asymptotic formulas follow from (5.8) according to the magnitude of (αr). When α is fixed and $r\to\infty$, the asymptotic expansion of $H_0(x)$ for $x\to\infty$ can be used to derive

$$\Gamma(r)\simeq D/\alpha^2r^3. \qquad (5.9)$$

However, for r large and fixed, and $\alpha\to 0$, we find that

$$\Gamma(r)\simeq A/r+A\alpha\ln(\alpha r), \qquad (5.10)$$

where

$$\alpha\sim-(K_c-K)/\ln(K_c-K).$$

We note that the correlations for large r do not decay exponentially away from T_c as would be predicted by the usual Orstein-Zernike theory.

As a second example consider the case $d=3$, and $\sigma=1$. We find from (5.4) that

$$\Gamma(r)\simeq\frac{G'}{r}\int_0^\infty\frac{k\sin(kr)}{\alpha+k}dk. \qquad (5.11)$$

If a damping factor is introduced to ensure convergence when $\alpha=0$, the integral can be evaluated[31] to give

$$\Gamma(r)\simeq G'/r^2+(G'\alpha/r)\times[\sin(\alpha r)\,\mathrm{ci}(\alpha r)+\cos(\alpha r)\,\mathrm{si}(\alpha r)]. \qquad (5.12)$$

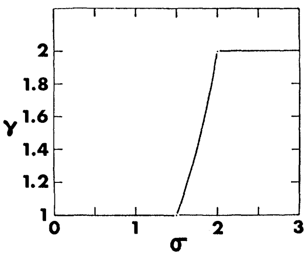

FIG. 2. Variation of the susceptibility index γ with σ for the three-dimensional spherical model. The asymptotic behavior of χ_0 when σ equals $\frac{3}{2}$ and 2 is not purely algebraic.

[29] See I. N. Sneddon, *Fourier Transforms* (McGraw-Hill Book Company, Inc., New York, 1951), p. 65.

[30] M. E. Fisher, J. Math. Phys. **5**, 944 (1964).

[31] A. Erdelyi *et al.*, *Tables of Integral Transforms 2* (McGraw-Hill Book Company, Inc., New York, 1954).

The two asymptotic formulas, obtained from (5.12), are as follows:

$$\Gamma(r)\sim 1/r^2-\pi\alpha/2r\,,\quad r\text{ large and fixed}\,,\quad \alpha\to 0$$
$$\sim 1/\alpha^2 r^4\,,\qquad \alpha\text{ small and fixed}\,,\quad r\to\infty\,,\quad (5.13)$$

where $\alpha\sim(K_c-K)$. It is reasonable to conjecture from the above discussion that, for the Ising model, $\Gamma(r)$ in the critical region does not decay exponentially for large r, when long-range interactions of the form $1/r^{d+\sigma}$ ($\sigma<2$) are present.

The behavior of the zero-field susceptibility in the critical region may be obtained, for the above two examples, by substituting (5.8) and (5.12) into the fluctuation relation

$$\chi_0=\mu^2\beta\sum_j\langle\epsilon_0\epsilon_j\rangle\,.$$

The summation can be replaced by an integral and the substitution $x=\alpha R$ leads to results in agreement with (4.13). The fluctuation relation is not valid for the spherical model when $H\neq 0$. To obtain the correct susceptibility per spin in a field, the sum of the spin-spin correlation function must be found for N spins before the limit $N\to\infty$ is taken.[32]

6. ANISOTROPIC MODELS

The derivation of exact series expansions, with long-range interactions present, requires the evaluation of multiple integrals whose integrands involve the Fourier transform of the interaction energy. The interaction (2.10) is inconvenient, since its Fourier transform cannot be expressed in simple closed-form. Some models for which the Fourier transform of the interaction energy is particularly simple will therefore be introduced.

The first model is a two-dimensional quadratic lattice system in which the interaction between a spin in the kth column and lth row, and one in the k'th column and l'th row is given by

$$\begin{aligned}J(kl,k'l')&=J_0|k-k'|^{-2}\,, && l'=l\\ &=\tau J_0\,, && l'=l\pm1\,,\quad k'=k\\ &=0\,, && \text{otherwise}\,.\end{aligned}\qquad(6.1)$$

Baker,[6] Kac and Helfand[7] have studied a similar model, except that the interaction energy along the lth row was exponential. We find on substituting (6.1) in (2.9), and using (4.5) that the saddle-point equation in zero field is

$$K=\frac{[\tau+\zeta(2)]}{\pi^2}\int_0^\pi\!\!\int\frac{d\omega_1 d\omega_2}{z_s[\tau+\zeta(2)]-\tau\cos\omega_2-\frac{1}{2}S_{1,1}(\omega_1)}\,.\qquad(6.2)$$

The evaluation of (6.2) is carried out, first, by integrating over ω_2, and then expressing the final integral as an elliptic integral, giving

$$K=(4/\pi^2)[\tau+\zeta(2)]a^{-1}F(\theta,t)\,,$$

where

$$\begin{aligned}\theta&=\sin^{-1}(1/b)\,,\quad t^2=b^2/a^2\,,\\ b^2&=(4/\pi^2)[\tau(z_s-1)+z_s\zeta(2)]+\tfrac{1}{3}\,,\\ a^2&=(4/\pi^2)[\tau(z_s+1)+z_s\zeta(2)]+\tfrac{1}{3}\,,\end{aligned}\qquad(6.3)$$

and $F(\theta,t)$ is an elliptic integral of the first kind. At the critical temperature, (6.3) reduces to

$$K_c=(4/\pi^2)[\tau+\zeta(2)]xF(\tfrac{1}{2}\pi,x)\,,\qquad(6.4)$$

where

$$x=(1+8\tau\pi^{-2})^{-1/2}.$$

To determine the critical properties of the model we expand (6.3), as follows:

$$K=K_c-B\xi^{1/2}+O(\xi)\,,\qquad(6.5)$$

and use the methods developed in Sec. 4. The critical behavior is similar to the usual three-dimensional spherical model with nearest-neighbor interactions. A preliminary analysis[16] of the exact series expansions for the Ising model, defined by (6.1), indicates that near T_c the susceptibility index γ is approximately 1.3, whereas the spherical model gives $\gamma=2$. We note that the estimated γ lies between the nearest-neighbor Ising value of 1.75 and the mean-field value 1.

The second model is also a quadratic lattice system but the interaction between the spins is defined by

$$\begin{aligned}J(kl,k'l')&=J_0|k-k'|^{-2}\,, && l'=l\\ &=J_0|l-l'|^{-2}\,, && k'=k\\ &=0\,, && \text{otherwise}\,.\end{aligned}\qquad(6.6)$$

The Fourier transform of this interaction may be written down in terms of (4.5), and the saddle-point equation becomes

$$K=\frac{2\zeta(2)}{\pi^2}\int_0^\pi\!\!\int\frac{d\omega_1 d\omega_2}{2\zeta(2)z_s-\frac{1}{2}S_{1,1}(\omega_1)-\frac{1}{2}S_{1,1}(\omega_2)}\,.\qquad(6.7)$$

Integration over ω_2 leads to

$$K=\frac{4\zeta(2)}{\pi^2}\int_0^{\sin^{-1}(1/v)}\ln\left(\frac{v\cos\theta+1}{v\cos\theta-1}\right)d\theta\,,\qquad(6.8)$$

where

$$v=[\tfrac{1}{3}(4z_s+2)]^{1/2}.$$

This integral may be evaluated at T_c to give

$$K_c=\tfrac{4}{3}G\simeq 1.221287\cdots\,,\qquad(6.9)$$

where G is Catalan's constant.

In the critical region ($T>T_c$) we find from (6.7) that

$$(K_c-K)\sim-\xi\ln\xi\,.\qquad(6.10)$$

We deduce, therefore, that the critical properties of this spherical model are the same as those obtained for a

[32] I am grateful to J. L. Lebowitz for pointing out this anomaly.

$1/r^3$ interaction acting in *all* directions. The analogous conjecture for the Ising model has been tested by analyzing the high-temperature series expansions for the susceptibility. It is found that both the model (6.6) and the $1/r^3$ model[17] have, within the errors of extrapolation, the same susceptibility in the critical region, namely

$$\chi_0 \sim (K_c - K)^{-1.13\pm0.01} \tag{6.11}$$

as the spherical approximation would suggest. Furthermore the critical temperature (6.9) is in reasonable agreement with the value $K_c \simeq 1.183$, estimated from the exact Ising-model series expansions. Evidently these simple models with anisotropic interactions can simulate the behavior of more physically realistic isotropic models.

7. CONCLUDING REMARKS

The one-dimensional Ising model, with an exponential interaction $J\gamma \exp(-\gamma r_{ij})$ between the spins, exhibits a phase transition only in the limit $\gamma \to 0$. In this limit one obtains the mean-field result. From the discussion in Sec. 3, it appears likely but not proved, that the Ising model, with $1/r^{1+\sigma}$ interactions, has a transition in one dimension for $0<\sigma<1$. In this range we have found that the high-temperature critical properties, in the spherical-model approximation, are classical for $0<\sigma<\frac{1}{2}$ and nonclassical for $\frac{1}{2}<\sigma<1$. The existence of a nonclassical behavior for certain $1/r^{d+\sigma}$ interactions is further supported by exact series extrapolations. Thus there are indications that the long-range behavior predicted by the exponential model is not typical of that given by a $1/r^{d+\sigma}$ interaction.

The Heisenberg model has been studied by Lax,[21] using the spherical model. More recently, various authors[33,34] have used Green's-function methods to obtain closed-form approximations for the Heisenberg model, which are valid over the whole temperature range. The existence of a phase transition in both these theories is determined by the convergence or divergence of the same integral as in (2.9). We see, therefore, that the Heisenberg model, with long-range interactions of the form $1/r^{d+\sigma}$, has, in the Green's-function approximation a phase transition for $\sigma<d$, in one and two dimensions. The critical temperatures of the Heisenberg model for $\sigma<d$ are easily obtained, using the results given in Tables I and II. In this range, it is readily found that the spin-wave expansion for the deviation of the magnetization from the saturated state ΔM has a sensible first term

$$\Delta M \sim A T^{d/\sigma} \quad (\sigma<d),$$

where A depends on d and σ, and becomes infinite as $\sigma \to d$.

The relationship between the Ising model and the lattice gas[35] enables us to interpret the results of this paper as an approximate description of the liquid-gas transition with long-range interactions. For example, (4.11) becomes the critical isotherm of a lattice gas:

$$\begin{aligned}(p-p_c) &\approx B|\rho-\rho_c|^{(d+\sigma)/(d-\sigma)}\,\mathrm{sgn}\{\rho-\rho_c\}, && \tfrac{1}{2}d<\sigma<d \\ &\approx -C|\rho-\rho_c|^3\,\mathrm{sgn}\{\rho-\rho_c\}/\ln^3|\rho-\rho_c|, && \sigma=\tfrac{1}{2}d \qquad (\sigma<2) \\ &\approx D|\rho-\rho_c|^3\,\mathrm{sgn}\{\rho-\rho_c\}, && 0<\sigma<\tfrac{1}{2}d,\end{aligned}$$

where p and ρ denote pressure and density, respectively. Similarly the zero-field susceptibility ($T\geq T_c$) corresponds to the isothermal compressibility K_T at $\rho=\rho_c$.

The Fourier transform of the spin-spin correlation function $\hat{\Gamma}(k)$, which is the integrand of (5.2) may be written as

$$\hat{\Gamma}(k)=k_BT/[(\mu^2/\chi_0)+\hat{J}(0)-\hat{J}(k)], \tag{7.1}$$

where $\hat{J}(k)$ is the Fourier transform of the exchange interaction. The spin-spin correlation function in zero field corresponds to the net pair correlation function $G(\mathbf{r})$ of the lattice gas at $\rho=\rho_c$. We find, therefore, from (7.1) that

$$\hat{G}(k)=\frac{8k_BT}{\hat{\Phi}(k)-\hat{\Phi}(0)-\rho_c^{-3}(\partial p/\partial V)_T} \quad (T>T_c), \tag{7.2}$$

where $\Phi(k)$ is the Fourier transform of the interaction energy of the lattice gas. Uhlenbeck, Hemmer, and Kac[3] (UHK) have shown, using an approximate method, based on van der Waals and Orstein-Zernike theories, that the long-range part of $\hat{G}(k)$ for a continuum gas is given by

$$\hat{G}^{l\cdot r}(k)=\frac{\rho^{-2}k_BT}{\hat{\Phi}(0)+\rho^{-3}(\partial p/\partial V)_T}\times\frac{\hat{\Phi}(k)}{\hat{\Phi}(k)-\hat{\Phi}(0)-\rho^{-3}(\partial p/\partial V)_T}. \tag{7.3}$$

Near the critical point, $(\partial p/\partial V)_T$ is small, and (7.3) becomes, for small k and $\rho=\rho_c$,

$$\hat{G}^{l\cdot r}(k)\simeq\frac{\rho_c^{-2}k_BT}{\hat{\Phi}(k)-\hat{\Phi}(0)-\rho_c^{-3}(\partial p/\partial V)_T},$$

which has the same form as the spherical-model result (7.2). This agreement is perhaps not so surprising since UHK made a basic assumption that the direct correla-

[33] H. B. Callen, Phys. Rev. **130**, 890 (1963).
[34] R. A. Tahir-Kheli and D. ter Haar, Phys. Rev. **127**, 88 (1962).
[35] T. D. Lee and C. N. Yang, Phys. Rev. **87**, 410 (1952).

tion function for large distances was proportional to $\Phi(r)$ which is true for the spherical model.

We have seen that the spherical model provides an interesting approximate representation of systems with long-range interactions. Whether the critical properties of the model are typical of the Ising model with long-range interactions is not at present known definitely; but it is hoped that the results given will form a qualitative background to future work on this difficult problem.

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APPENDIX: ANALYSIS OF SADDLE-POINT EQUATION

The behavior of the saddle-point parameter $z_s(K,W)$ is here determined under several different conditions by analyzing (2.14).

We consider first the case $T>T_c$, $H=0$ in the critical region. The methods described in Sec. 3 may be used to show that

$$S_{d,\sigma}(\omega)=S_{d,\sigma}-Ak^{\sigma}+O(k^2)\,,\quad(\sigma<2)\qquad\text{(A1)}$$

where $k^2=(\omega_1{}^2+\cdots+\omega_d{}^2)$, and A depends only on d and σ. When $\sigma=2$, the expansion becomes

$$S_{d,\sigma}(\omega)=S_{d,\sigma}+Bk^2\ln k+O(k^2)\,.\qquad\text{(A2)}$$

Since there is no transition in one and two dimensions when $\sigma=2$, this special case is not discussed further. For $\sigma>2$ the leading term in the expansion is proportional to k^2, which results in critical properties similar to those given by finite-range interactions. Attention will therefore be restricted to the range $0<\sigma<2$. On substituting (A1) into (2.14), and changing to polar coordinates, we find that

$$K\simeq D\int_0^{\pi}\frac{k^{d-1}}{\xi+Ek^{\sigma}+O(k^2)}dk\,,\qquad\text{(A3)}$$

where $\xi=z_s-1$, and D, E depend only on d and σ. In the critical region ξ is small and positive, and hence the main contribution to the integral comes from integration about $k=0$. Using (A3), we write

$$(K_c-K)\simeq\frac{D}{E}\int_0^{\pi}\frac{k^{d-\sigma-1}}{1+E\xi^{-1}k^{\sigma}}dk\,.\qquad\text{(A4)}$$

A simple change of variable leads to

$$(K_c-K)\simeq\frac{D\xi^{(d-\sigma)/\sigma}}{\sigma E^{d/\sigma}}\int_0^{E\pi^{\sigma}/\xi}\frac{x^{(d-2\sigma)/\sigma}}{1+x}dx\,.\qquad\text{(A5)}$$

Depending on whether this integral converges or diverges as $\xi\to0$, three possibilities arise, as follows:

$$\begin{aligned}(K_c-K)&\sim\xi^{(d-\sigma)/\sigma}, && \tfrac{1}{2}d<\sigma<d\\ &\sim-\xi\ln\xi, && \sigma=\tfrac{1}{2}d \qquad(\sigma<2)\\ &\sim\xi, && 0<\sigma<\tfrac{1}{2}d.\end{aligned}\qquad\text{(A6)}$$

Thus (A6) determines the variation of (z_s-1) with K, for $H=0$, in the critical region $(T>T_c)$.

The behavior of (z_s-1) *at* $T=T_c$ for small H is readily obtained to first-order, by use of (A6) and (2.14), giving

$$\begin{aligned}\xi&\sim H^{2\sigma/(d+\sigma)}, && \tfrac{1}{2}d<\sigma<d\\ &\sim-H^{2/3}/\ln H, && \sigma=\tfrac{1}{2}d \qquad(\sigma<2)\\ &\sim H^{2/3}, && 0<\sigma<\tfrac{1}{2}d.\end{aligned}\qquad\text{(A7)}$$

For small H at high temperatures, one has $z_s\gg1$, and the integrand in (2.14) can be expanded as a geometric series. The resulting equation may be solved iteratively, to give

$$z_sK=1+W^2+K^2S_{d,\sigma}{}^{-2}\sum_{\mathbf{l}}{}'|\mathbf{l}|^{-2(d+\sigma)}+\cdots.\qquad\text{(A8)}$$

For $T<T_c$ and H small, the expansion (A6) can again be used, since a normal saddle point exists with (z_s-1) small. The saddle-point equation (2.14) can be solved by iteration for ξ, to yield finally

$$\xi\approx(W/K)(K/K-K_c)^{1/2}\times[1-GF(K,W)(K-K_c)^{-1}+\cdots],\qquad\text{(A9)}$$

where

$$\begin{aligned}F(K,W)&=(W/K)^{(d-\sigma)/\sigma}(K/K-K_c)^{(d-\sigma)/2\sigma}, && \tfrac{1}{2}d<\sigma<d\\ &=-(W/K)(K/K-K_c)^{1/2}\times\ln(W/K)(K/K-K_c)^{1/2}, && \sigma=\tfrac{1}{2}d\\ &=(W/K)(K/K-K_c)^{1/2}, && 0<\sigma<\tfrac{1}{2}d.\end{aligned}\qquad\text{(A10)}$$