Algebra of Currents and Form Factors at Finite Momentum Transfer*

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By using equal-time commutation relations of chiral $SU(3)\times SU(3)$ current components, and the partially conserved-axial-vector-current hypothesis, we derive several relationships among the axial-vector and electromagnetic form factors and the pion electroproduction amplitude at nonzero momentum transfer. In the approximation of retaining only the (3,3) resonance contribution to the pion electroproduction amplitude, we derive an approximate identity of the isovector Dirac charge form factor and the axial-vector form factor. Further sum rules for the isovector and isoscalar Pauli magnetic-moment form factors are obtained which are fairly well satisfied. The form factor for the induced pseudoscalar coupling is discussed.

1. INTRODUCTION

A POWERFUL method has been derived by Fubini and Furlan[1] to extract physical information from the current algebra proposed by Gell-mann.[2] This method together with the notion of a partially conserved axial-vector current[3] (PCAC) has been used by Adler[4] and Weisberger[5] to obtain the renormalization of the β-decay axial-vector coupling constant in impressive agreement with experiment. The method has since then been extended to various other decays with encouraging results.[6] All the applications of the method so far, however, have been discussed at zero momentum transfer in connection with the decay constants and not with the form factors. In this paper we apply this approach to calculate the axial-vector and the electromagnetic form factors. Thus we derive several sum rules among the axial-vector and electromagnetic form factors and the pion electroproduction amplitude at nonzero momentum transfer. At zero momentum transfer, two of the sum rules reduce to the expressions relating the isovector and isoscalar Pauli magnetic-moment form factors to the pion photoproduction amplitude, the relations derived previously by Fubini, Furlan, and Rossetti.[7] In particular, in the approximation of retaining only the (3,3) resonance contribution to the pion electroproduction amplitude, we obtain

$$G_A(t)/G_A\approx F_1{}^V(t)\,, \tag{1.1}$$

$$\tfrac{1}{2}F_2{}^V(t)\approx(8/9)\tfrac{1}{2}G_M{}^V(t)\,, \tag{1.2}$$

where $G_A(t)$ is the axial-vector form factor, $F_1{}^V(t)$ and $F_2{}^V(t)$ are the isovector Dirac and Pauli form factors, and $G_M{}^V(t)$ is the isovector Sach's magnetic-moment form factor. The form factors are normalized such that $G_A(0)=G_A$, $F_1{}^V(0)=1$, $F_2{}^V(0)=\mu_V'$, $G_M{}^V(0)=\mu_V$, where μ_V' and μ_V are respectively the isovector anomalous and total magnetic moments of a nucleon; $\mu_V=4.7$, $\mu_V'=3.7$. We also discuss the form factor for the induced pseudoscalar coupling and show that besides the usual contribution from the pion pole, this form factor has also a small second-order contribution in our approach.

At zero momentum transfer the left-hand side of Eq. (1.2) is 1.85 while the right-hand side is about 2. The small discrepancy is discussed in the last section. Equation (1.2) also implies the form equality of the form factors $F_2{}^V(t)$ and $G_M{}^V(t)$ and in turn that of $F_1{}^V(t)$ and $F_2{}^V(t)$. Such a form equality is consistent with the existing experimental data on electron-nucleon scattering. The relation (1.1) can be tested in high-energy neutrino experiments and in fact these experiments[8] do indicate that the relation (1.1) is consistent with the data. A similar approach to ours has been suggested by Fubini.[9]

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[9] S. Fubini, Nuovo Cimento (to be published).

2. FORMULATION

We start with the identity[10]

$$iq^\mu M_{\mu\nu}{}^\alpha = i\int d^4x\,\theta(x_0)\langle p'|[\partial_\mu A_\mu{}^\alpha(x), j_\nu{}^{\rm el}(0)]|p\rangle e^{-iq\cdot x} + i\int d^4x\,e^{-iq\cdot x}\delta(x_0)\langle p'|[A_0{}^\alpha(x), j_\nu{}^{\rm el}(0)]|p\rangle, \quad (2.1)$$

where $A_\mu{}^\alpha(x)$ is the axial-vector current of the isospin α, and $j_\nu{}^{\rm el} = V_\nu{}^3 + (1/\sqrt{3})V_\nu{}^8$ is the electromagnetic current. The matrix elements are taken between the nucleon states of momenta p and p'. $M_{\mu\nu}{}^\alpha$ is defined by

$$M_{\mu\nu}{}^\alpha = i\int d^4x\,e^{-iq\cdot x}\theta(x_0)\langle p'|[A_\mu{}^\alpha(x), j_\nu{}^{\rm el}(0)]|p\rangle. \quad (2.2)$$

According to the PCAC hypothesis, we have

$$\partial_\mu A_\mu{}^\alpha(x) = -(f_\pi/\sqrt{2})\mu^2\phi_\pi{}^\alpha(x), \quad (2.3)$$

where $\phi_\pi{}^\alpha$ denotes the pion field of isospin α and of mass μ. The Goldberger-Treiman relation[11] is

$$f_\pi/\sqrt{2} = -(G_A/G)[m/g_rK(0)], \quad (2.4)$$

where G_A is the axial-vector coupling constant, $G_A \simeq -1.18G$, g_r is the pion-nucleon coupling constant, $g_r^2 \simeq (4\pi)\times 14.5$, m is the nucleon mass, and $K(t)$ is the pion-nucleon vertex form factor normalized to unity at $t=\mu^2$, t being the invariant momentum transfer squared.

With the use of Eq. (2.3), Eq. (2.1) is written

$$i\int d^4x\,e^{-iq\cdot x}\delta(x_0)\langle p'|[A_0{}^\alpha(x), j_\nu{}^{\rm el}(0)]|p\rangle = iq^\mu M_{\mu\nu}{}^\alpha + \frac{f_\pi}{\sqrt{2}}\frac{\mu^2}{q^2+\mu^2}R_\nu(\nu, \nu_B, -q^2, -k^2), \quad (2.5)$$

where

$$R_\nu{}^\alpha(\nu, \nu_B, -q^2, -k^2) = i\int d^4x\,e^{-iq\cdot x}(\mu^2 - \Box^2) \times \langle p'|\theta(x_0)[\phi_\pi{}^\alpha(x), j_\nu{}^{\rm el}(0)]|p\rangle. \quad (2.6)$$

The quantities ν, ν_B, and k are defined by

$$m\nu = -P\cdot k,\quad 2m\nu_B = q\cdot k,\quad k = p'+q-p,\ \text{and}\ P = \tfrac{1}{2}(p+p'). \quad (2.7)$$

Consider the limit $q_\mu \to 0$ of Eq. (2.5). The left-hand side reduces to[12]

$$i\langle p'|[I_\alpha{}^5(0), j_\nu{}^{\rm el}(0)]|p\rangle = i^2\epsilon_{\alpha3\beta}\langle p'|A_\nu{}^\beta(0)|p\rangle, \quad (2.8)$$

where $I_\alpha{}^5(t) = \int d^3x\,A_0{}^\alpha(\mathbf{x},t)$, and we have assumed the chiral $SU(3)\times SU(3)$ algebra of current components. Thus

$$i^2\epsilon_{\alpha3\beta}\langle p'|A_\nu{}^\beta(0)|p\rangle = \lim_{q_\mu\to 0}\left\{iq^\mu M_{\mu\nu}{}^\alpha + \frac{f_\pi}{\sqrt{2}}\frac{\mu^2}{q^2+\mu^2}R_\nu{}^\alpha\right\}. \quad (2.9)$$

To evaluate the right-hand side of Eq. (2.9), we write

$$i^2\epsilon_{\alpha3\beta}\langle p'|A_\nu{}^\beta(0)|p\rangle = \lim_{q_\mu\to 0}\left\{iq^\mu M_{\mu\nu}{}^\alpha + \frac{f_\pi}{\sqrt{2}}R_\nu{}^{\alpha(\rm Born)}\right\} + \lim_{q_\mu\to 0}\left\{\frac{f_\pi}{\sqrt{2}}\frac{\mu^2}{q^2+\mu^2}(R_\nu{}^\alpha - R_\nu{}^{\alpha(\rm Born)})\right\}, \quad (2.10)$$

where we *define*

$$R_\nu{}^{\alpha(\rm Born)} = \bar{u}(p')\Bigg\{g_rK(-q^2)\frac{\gamma_5 iq}{2m(\nu_B-\nu)}\tau_\alpha F_\nu(k) - g_rK(-q^2)F_\nu(k)\tau_\alpha\frac{iq\gamma_5}{2m(\nu_B+\nu)} - \tfrac{1}{2}g_r[\tau_\alpha,\tau_3]i\gamma_5\left[\frac{K(-\Delta^2)F_\pi(-k^2)(k-2q)_\nu}{k^2-2k\cdot q+q^2+\mu^2} + \frac{K(-q^2)F_1{}^V(-k^2) - F_\pi(-k^2)K(-\Delta^2)}{k^2}k_\nu\right]\Bigg\}u(p), \quad (2.11)$$

$$\Delta = p'-p.$$

The quantities K and F_ν are given by

$$\langle p'|\phi_\pi{}^\alpha|p\rangle = \frac{1}{\mu^2+(p-p')^2}g_rK[-(p-p')^2]\bar{u}(p')i\gamma_5\tau_\alpha u(p),$$
$$\langle p'|j_\nu{}^{\rm el}|p\rangle = i\bar{u}(p')F_\nu(p'-p)u(p), \quad (2.12)$$
$$F_\nu(k) = \left[\frac{F_1{}^S(-k^2)+\tau_3F_1{}^V(-k^2)}{2}\gamma_\nu - \sigma_{\nu\mu}\frac{k^\mu}{2m}\frac{F_2{}^S(-k^2)+\tau_3F_2{}^V(-k^2)}{2}\right].$$

[10] V. Alessandrini, M. Bég, and L. Brown, Phys. Rev. **144**, 1136 (1966). See also S. Okubo, Nuovo Cimento **41A**, 586 (1966). With Alessandrini, Bég, and Brown, we maintain that the surface term is absent if $q+p'$ does not lie on a mass hyperbola. In our case, we must keep q_μ not equal to zero, but small. The limit $q_\mu \to 0$ must be taken at the very end.

[11] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958); **110**, 1478 (1958).

[12] The use of this commutation relation in evaluating the axial-vector form factor was also suggested by S. Okubo, Ref. 10.

$F_\pi(-k^2)$ is the usual pion electromagnetic form factor. $R_\nu^{\alpha(\mathrm{Born})}$ is defined so that $k^\nu R_\nu^{\alpha(\mathrm{Born})}(\nu,\nu_\beta,\mu^2,-k^2)=0$. The limit of the first curly bracket on the right-hand side of Eq. (2.10) is well defined. To evaluate the second term on the right, we assume that the behavior of the term inside the second curly bracket near $q_\mu\simeq 0$ is dominated by the pole term at $-q^2=\mu^2$. Thus we assume

$$\lim_{q_\mu\to 0}\left\{\frac{\mu^2}{q^2+\mu^2}(R_\nu^\alpha-R_\nu^{\alpha(\mathrm{Born})})\right\}=\lim_{\nu,\nu_B\to 0}\{R_\nu^\alpha(\nu,\nu_B,\mu^2,-k^2)-R_\nu^{\alpha(\mathrm{Born})}(\nu,\nu_B,\mu^2,-k^2)\}. \tag{2.13}$$

3. SUM RULES

The left-hand side of Eq. (2.10) may be expressed in terms of $G_A(t)$ and $g_A(t)$:

$$\langle p'|A_\mu^\alpha|p\rangle=\bar u(p')\left[i\gamma_\mu\gamma_5\frac{G_A(t)}{G}-(p'-p)_\mu\frac{g_A(t)}{G}\gamma_5\right]\frac{\tau_\alpha}{2}u(p),$$
$$t=-\Delta^2=-(p-p')^2. \tag{3.1}$$

The term $iq_\mu M_{\mu\nu}^\alpha$ vanishes unless an intermediate state contributes which is degenerate in mass with the nucleon.[10] This is the case here. The nucleon contribution to $iq^\mu M_{\mu\nu}^\alpha$ is

$$iq^\mu M_{\mu\nu}^\alpha=\bar u(p')(1/G)\{[G_A\gamma_5+2mG_A(\gamma_5 iq/2m(\nu_B-\nu))]\tfrac12\tau_\alpha F_\nu(k)+F_\nu(k)\tfrac12\tau_\alpha$$
$$\times[G_A\gamma_5-2mG_A(iq\gamma_5/2m(\nu_B+\nu))]\}u(p)+O(|q^\mu|). \tag{3.2}$$

Hence, by virtue of (2.4), Eqs. (2.11) and (3.2) give

$$\lim_{q_\mu\to 0}\{iq^\mu M_{\mu\nu}+(f_\pi/\sqrt2)R_\nu^{(\mathrm{Born})}\}$$
$$=\bar u(p')\left\{-\frac{G_A}{G}\frac12[\tau_\alpha,\tau_3]\tfrac12F_1{}^V(t)\gamma_\nu\gamma_5-\frac{G_A}{G}m\tfrac12[\tau_\alpha,\tau_3]i\gamma_5k_\nu\left[\frac{K(t)}{K(0)}\frac{F_\pi(t)}{t-\mu^2}\right.\right.$$
$$\left.\left.+\frac{K(0)F_1{}^V(t)-F_\pi(t)K(t)}{tK(0)}\right]-\frac{G_A}{G}\left[\frac12\delta_{\alpha3}\frac{F_2{}^V(t)}{2m}+\tau_\alpha\frac12\frac{F_2{}^S(t)}{2m}\right]\gamma_5\sigma_{\nu\lambda}k^\lambda\right\}u(p). \tag{3.3}$$

The amplitude $R_\nu^\alpha(\nu,\nu_\beta,\mu^2,-k^2)$ is just that of the electroproduction of pions, and has been studied by Fubini, Nambu, and Wataghin,[13] and others.[14] In the isospin space, this can be decomposed into

$$R_\nu^\alpha=\delta_{\alpha3}T_\nu^{(+)}+\tfrac12[\tau_\alpha,\tau_3]T_\nu^{(-)}+\tau_\alpha T_\nu^{(0)} \tag{3.4}$$

and we write

$$\tilde R_\nu^\alpha(\nu,\nu_B,\mu^2,-k^2)=\{R_\nu^\alpha(\nu,\nu_B,\mu^2,-k^2)-R_\nu^{\alpha(\mathrm{Born})}(\nu,\nu_B,\mu^2,-k^2)\}. \tag{3.5}$$

Hence from Eqs. (2.10), (2.13), (3.1), (3.3)–(3.5), we obtain

$$\bar u(p')\left[-\frac{G_A(t)}{G}\gamma_\nu\gamma_5-i(p'-p)_\nu\frac{g_A(t)}{G}\gamma_5\right]u(p)$$
$$=\bar u(p')\left\{-\frac{G_A}{G}F_1{}^V(t)\gamma_\nu\gamma_5+\left(-\frac{G_A}{G}\right)2m\left[\frac{K(t)}{K(0)}\frac{F_\pi(t)}{t-\mu^2}+\frac{K(0)F_1{}^V(t)-F_\pi(t)K(t)}{tK(0)}\right]i\gamma_5(p'-p)_\nu\right\}u(p)$$
$$+\sqrt2 f_\pi\lim_{\nu,\nu_B\to0}\tilde T_\nu^{(-)}(\nu,\nu_B,\mu^2,-k^2), \tag{3.6}$$

$$0=\left(-\frac{G_A}{G}\right)\frac12\frac{F_2{}^V(t)}{2m}\bar u(p')\gamma_5\sigma_{\nu\lambda}k^\lambda u(p)+\frac{1}{\sqrt2}f_\pi\lim_{\nu,\nu_B\to0}\tilde T_\nu^{(+)}(\nu,\nu_B,\mu^2,-k^2),$$

$$0=\left(-\frac{G_A}{G}\right)\frac12\frac{F_2{}^S(t)}{2m}\bar u(p')\gamma_5\sigma_{\nu\lambda}k^\lambda u(p)+\frac{1}{\sqrt2}f_\pi\lim_{\nu,\nu_B\to0}\tilde T_\nu^{(0)}(\nu,\nu_B,\mu^2,-k^2).$$

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[13] S. Fubini, Y. Nambu, and A. Wataghin, Phys. Rev. **111**, 329 (1958).
[14] P. Dennery, Phys. Rev. **124**, 2000 (1961). See also J. S. Ball, *ibid.* **124**, 2014 (1961).

Writing $-M_A=\gamma_5\sigma_{\nu\lambda}k^\lambda=-i\gamma_5(\gamma_\nu k-k\gamma_\nu)$, the last two equations can be put into the form

$$-\frac{G_A}{G}\frac{1}{2}\frac{F_2^V(t)}{2m}\bar{u}(p')M_Au(p)=(1/\sqrt{2})f_\pi\lim_{\nu,\nu_B\to 0}\tilde{T}_\nu^{(+)}(\nu,\nu_B,\mu^2,-k^2),\quad(3.7)$$

$$\left(-\frac{G_A}{G}\right)\frac{1}{2}\frac{F_2^S(t)}{2m}\bar{u}(p')M_Au(p)=(1/\sqrt{2})f_\pi\lim_{\nu,\nu_B\to 0}\tilde{T}_\nu^{(0)}(\nu,\nu_B,\mu^2,-k^2).\quad(3.8)$$

First we observe that since the electromagnetic current is conserved, we must have in the limit of zero momentum transfer ($k^2=0=t$), the following relation

$$-\frac{G_A(0)}{G}=-\frac{G_A}{G}F_1^V(0),\quad(3.9)$$

i.e., $F_1^V(0)=1$. Hence for $t=0$, there is no contribution from the last term of (3.6), i.e., from the continuum.

We now discuss the contributions from $\tilde{T}^-$, $\tilde{T}^+$, $\tilde{T}^0$ to Eqs. (3.6), (3.7), and (3.8). It has been shown in Ref. 13, that any of T's or $\tilde{T}$'s can be written as

$$T_\nu=\bar{u}(p')[H_AM_A+H_BM_B+H_CM_C+H_DM_D+H_EM_E+H_FM_F]u(p),\quad(3.10)$$

where

$$\begin{aligned}
M_A&=\tfrac{1}{2}i\gamma_5(\gamma_\nu k-k\gamma_\nu)=-\gamma_5\sigma_{\nu\lambda}k_\lambda,\\
M_B&=2i\gamma_5[2m\nu_BP_\nu+m\nu q_\nu],\\
M_C&=\gamma_5[2m\nu_B\gamma_\nu-kq_\nu],\\
M_D&=2\gamma_5[-m\nu\gamma_\nu-P_\nu k]-2mM_A,\\
M_E&=i\gamma_5[2m\nu_Bk_\nu-k^2q_\nu],\\
M_F&=\gamma_5[kk_\nu-k^2\gamma_\nu].
\end{aligned}\quad(3.11)$$

We assume that $H_i(\nu,\nu_B,+\mu^2,-k^2)$ satisfy the unsubtracted dispersion relations[15]

$$\tilde{H}_i(\nu,\nu_B,\mu^2,-k^2)=\frac{1}{\pi}\int_{\nu_0}^{\infty}d\nu'\,\mathrm{Im}H_i(\nu',\nu_B,\mu^2,-k^2)\times\left[\frac{1}{\nu'-\nu}\pm\frac{1}{\nu'+\nu}\right],\quad(3.12)$$

where $\tilde{H}_i=[H_i-H_i(\mathrm{Born})]$, $\nu_0=\nu_B+\mu^2+\mu^2/2m$. In the dispersion integral the plus sign holds for $H_A^{+,0}$, $H_B^{+,0}$, $H_D^{+,0}$, H_C^-, H_E^-, H_F^- and the minus sign for H_A^-, H_B^-, H_D^-, $H_C^{+,0}$, $H_E^{+,0}$, $H_F^{+,0}$. It follows, then, from (3.12) that

$$\begin{aligned}
0&=\tilde{H}_A^-(0,\nu_B,\mu^2,-k^2)=\tilde{H}_B^-(0,\nu_B,\mu^2,-k^2)\\
&=\tilde{H}_D^-(0,\nu_B,\mu^2,-k^2)=\tilde{H}_C^{+,0}(0,\nu_B,\mu^2,-k^2)\\
&=\tilde{H}_E^{+,0}(0,\nu_B,\mu^2,-k^2)=\tilde{H}_F^{+,0}(0,\nu_B,\mu^2,-k^2).
\end{aligned}\quad(3.13)$$

[15] The only subtraction is that included in $R_\nu^{\alpha(\mathrm{Born})}$, Eq. (2.11), to make $k^\nu R_\nu^{\alpha(\mathrm{Born})}(\nu,\nu_B,\mu^2,-k^2)=0$. It is possible that, for negative k^2, corresponding to $t>0$, subtractions are necessary. To be more specific, we assume that amplitudes $\tilde{H}_i(\nu,\nu_B,\mu^2,-k^2)$ obey the unsubtracted dispersion relations (3.12) for positive k^2, or $t<0$. Equations (3.22) to (3.25) are then valid only for $t<0$. For $t>0$, these equations must be analytically continued from $t<0$.

It turns out, however, that the decomposition (3.10) is not a convenient one for our purpose, since the amplitudes H_B and H_E contain singularities at $\nu_B=0$ as we show in the Appendix. A more convenient set of amplitudes is gotten by writing[14]

$$T_\nu=\bar{u}(p')\sum_{i=1}^{6}A_iM_iu(p),\quad(3.14)$$

where

$$M_1=M_A,\quad M_2=2i\gamma_5[P_\nu(2m\nu_B-\tfrac{1}{2}k^2)+(q_\nu-\tfrac{1}{2}k_\nu)m\nu],$$
$$M_3=M_C,\quad M_4=M_D,\quad M_5=M_E,\quad M_6=M_F.$$

We then have

$$\begin{aligned}
A_2&=\frac{4m\nu_B}{4m\nu_B-k^2}H_B,\\
A_5&=H_E+\frac{2m\nu}{4m\nu_B-k^2}H_B,
\end{aligned}\quad(3.15)$$

and A_1, A_3, A_4, A_6 are equal to H_A, H_C, H_D, and H_F respectively. The amplitudes A_i are devoid of kinematical singularities as $\nu_B\to 0$ (see Appendix).

Now as $q_\mu\to 0$ ($\nu_B\to 0$, $\nu\to 0$), we have

$$M_1=2i\gamma_5P_\nu=M_A,\quad M_2=i\gamma_5P_\nu t=\tfrac{1}{2}tM_A,\quad(3.16)$$
$$M_3=M_4=M_5=0,\quad M_6=-2im\gamma_5(p'-p)_\nu+t\gamma_5\gamma_\nu,$$

and

$$A_2(\nu',0,\mu^2,t)=\frac{4m}{t}\lim_{\nu_B\to 0}\nu_BH_B(\nu',\nu_B,\mu^2,t).\quad(3.17)$$

Thus from (3.10), (3.12), (3.13), (3.15), (3.16), and (3.17), we obtain

$$\tilde{T}_\nu^{+,0}(0,0,\mu^2,t)=J^{+,0}(t)\bar{u}(p')M_Au(p),\quad(3.18)$$

$$\tilde{T}_\nu^-(0,0,\mu^2,t)=\tilde{H}_F^-(0,0,\mu^2,t)[t\bar{u}(p')\gamma_5\gamma_\nu u(p)-2im\bar{u}(p')\gamma_5(p'-p)_\nu u(p)],\quad(3.19)$$

where

$$\begin{aligned}
J^{+,0}(0,0,\mu^2,t)&=\frac{2}{\pi}\int\frac{d\nu'}{\nu'}\mathrm{Im}J^{+,0}(\nu',0,\mu^2,t),\\
\tilde{H}_F^-(0,0,\mu^2,t)&=\frac{2}{\pi}\int\frac{d\nu'}{\nu'}\mathrm{Im}H_F^-(\nu',0,\mu^2,t),
\end{aligned}\quad(3.20)$$

with

$$\mathrm{Im}J^{+,0}(\nu',0,\mu^2,t)$$
$$=\lim_{\nu_B\to 0}[\mathrm{Im}H_A{}^{+,0}(\nu',\nu_B,\mu^2,-k^2)$$
$$+2m\nu_B\,\mathrm{Im}H_B{}^{+,0}(\nu',\nu_B,\mu^2,-k^2)]. \quad (3.21)$$

Hence by using Eqs. (3.18) to (3.21), we obtain from Eqs. (2.4), (3.6)–(3.8), the following sum rules:

$$-\frac{G_A(t)}{G}=\left(-\frac{G_A}{G}\right)\left[F_1{}^V(t)-\frac{2m}{g_rK(0)}\frac{2t}{\pi}\times\int\frac{d\nu'}{\nu'}\mathrm{Im}H_F{}^-(\nu',0,\mu',t)\right], \quad (3.22)$$

$$\frac{g_A(t)}{G}=\left(-\frac{G_A}{G}\right)2m\left[\frac{K(t)}{K(0)}\frac{F_\pi(t)}{\mu^2-t}+\frac{F_\pi(t)K(t)-F_1{}^V(t)K(0)}{tK(0)}+\frac{2m}{g_rK(0)}\frac{2}{\pi}\times\int\frac{d\nu'}{\nu'}\mathrm{Im}H_F{}^-(\nu',0,\mu^2,t)\right], \quad (3.23)$$

$$\frac{1}{2}\left(\frac{F_2{}^V(t)}{2m}\right)=\frac{m}{g_rK(0)}\frac{2}{\pi}\int\frac{d\nu'}{\nu'}\mathrm{Im}J^+(\nu',0,\mu^2,t), \quad (3.24)$$

$$\frac{1}{2}\left(\frac{F_2{}^S(t)}{2m}\right)=\frac{m}{g_rK(0)}\frac{2}{\pi}\int\frac{d\nu'}{\nu'}\mathrm{Im}J^0(\nu',0,\mu',t). \quad (3.25)$$

It should be noted that Eqs. (3.22) and (3.23) give

$$D(t)\equiv\left[-\frac{G_A(t)}{G}2m+t\frac{g_A(t)}{G}\right]=\left(-\frac{G_A}{G}\right)2m\left[\frac{K(t)}{K(0)}\frac{\mu^2F_\pi(t)}{(\mu^2-t)}\right], \quad (3.26)$$

i.e., $D(t)$ does not depend on dispersion integrals. This, in fact, follows from current conservation, since the current conservation,

$$k_\nu[R_\nu{}^\alpha(\nu,\nu_B,\mu^2,-k^2)-R_\nu{}^{\alpha(\mathrm{Born})}(\nu,\nu_B,\mu^2,-k^2)]=0,$$

and Eq. (2.10) imply that

$$i^2\epsilon_{\alpha3\beta}k_\nu\langle p'|A_\nu{}^B(0)|p\rangle=k_\nu\lim_{q_\mu\to0}\left[iq_\mu M_{\mu\nu}{}^\alpha+\frac{f_\pi}{\sqrt{2}}R_\nu{}^{\alpha(\mathrm{Born})}\right],$$

which, on using (3.3) and (3.1), gives Eq. (3.26).

Since in the production of pions by real photons ($t=0$) M_E and M_F do not appear, one can in principle extract $\mathrm{Im}J^{+,0}(\nu',0,\mu^2,t=0)$ from the experimental data on the photoproduction of pions in evaluating the sum rules (3.24) and (3.25) for the case $t=0$. For the general case of $t\neq0$ for these sum rules as well as for (3.22) and (3.23) to get any experimental information in evaluating the dispersion integrals one has to appeal to the electroproduction of pions. Such an information may be very hard to get. We, therefore, in the next section evaluate the dispersion integrals in the resonance approximation, keeping only the 3,3 resonance in the absorptive parts.

4. RESONANCE APPROXIMATION

We introduce the c.m. frame $\mathbf{p}+\mathbf{k}=0=\mathbf{p}'+\mathbf{q}$. Then the c.m. particle energies are given by

$$k_0=\frac{W^2-m^2-k^2}{2W},\quad q_0=\frac{W^2-m^2-q^2}{2W},$$
$$p_0=\frac{W^2+m^2+k^2}{2W},\quad p_0'=\frac{W^2+m^2+q^2}{2W}, \quad (4.1)$$
$$\nu-\nu_B=\frac{W^2-m^2}{2m},\quad |\mathbf{q}|=(q_0{}^2+q^2)^{1/2},$$
$$\omega=W-m,\quad |\mathbf{k}|=(k_0{}^2+k^2)^{1/2},$$

where W is the total c.m. energy. When $q_\mu\to0$, then in the c.m. frame

$$W\to m,\quad |\mathbf{q}|\to0,\quad k_0\to\frac{t}{2m},$$
$$p_0\to\frac{-t+2m^2}{2m},\quad p_0'\to m.$$

We shall evaluate the dispersion integrals in (3.22), (3.23), (3.24), and (3.25) by assuming that they are completely dominated by the 3,3 resonance. We believe it is a reasonable approximation for the following reasons: (i) Such an approximation for the sum rule (3.24) for the special case of $t=0$ has been shown by Fubini, Furlan, and Rossetti[7] to be quite good, and (ii) even for large values of momentum transfer t the final pion-nucleon state is still at low energy (in the c.m. system) and therefore is completely dominated by the (3–3) resonance. Our procedure will now be as follows[13]: First solve the Eqs. (3.12) in the static approximation keeping only the (3,3) resonance in the dispersion integrals; then insert the imaginary part of the solution thus obtained into the right-hand side of the dispersion relations (3.12) to create the real part of the whole amplitude. The procedure is reasonable because the recoil is more effective in producing new multipoles than in any large change of the resonant amplitudes since the static approximation is good for the resonant amplitudes as in the end we have to take the limit $|\mathbf{q}|\to0$. In the static approximation it has been shown in Ref. 13 that the scattering amplitude in the c.m. system is

$$\{\tfrac{2}{3}\delta_{\alpha 3}-\tfrac{1}{6}[\tau_\alpha,\tau_3]\}\{2\mathbf{q}\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})+i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mathbf{q}\cdot\mathbf{k}-i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}\}B(-k^2)f_{33}/|\mathbf{q}|^2, \quad (4.2)$$

where $\boldsymbol{\varepsilon}$ is an arbitrary vector and

$$f_{33}=e^{i\delta_{33}}\sin\delta_{33}/|\mathbf{q}|, \quad (4.3)$$

while

$$B(-k^2)=\frac{g_r}{2m}\frac{G_M{}^V(-k^2)/2m}{2(g_r{}^2/4\pi)(1/4m^2)}. \quad (4.4)$$

Then $\mathrm{Im}H_i$ are shown in Ref. 13 to be given by

$$\mathrm{Im}H_i{}^{33}(\nu,\nu_B,\mu^2,-k^2)=S_iB(-k^2)\,\mathrm{Im}f_{33}/|\mathbf{q}|^2, \quad (4.5)$$

where

$$\begin{aligned}
&S_i{}^0=0,\\
&S_A{}^+=-2S_A{}^-=\tfrac{2}{3}(\omega-k^2/m+6\nu_B),\\
&S_B{}^+=-2S_B{}^-=-(1-k^2/3m\nu_B)(1/m),\\
&S_C{}^+=-2S_C{}^-=-\tfrac{2}{3}(1+3k^2/4m^2),\\
&S_D{}^+=-2S_D{}^-=\tfrac{4}{3},\\
&S_E{}^+=-2S_E{}^-=2\omega/3m\nu_B,\\
&S_F{}^\pm=0,
\end{aligned} \quad (4.6)$$

and the dispersion relations (3.12) are now written as

$$\tilde{H}_i(\nu,\nu_B,\mu^2,-k^2)=\frac{P}{\pi}\int d\omega'\,\mathrm{Im}H_i{}^{33}(\omega'+\nu_B,\nu_B,\mu^2,-k^2)\times\left[\frac{1}{\omega'+\nu_B-\nu}\pm\frac{1}{\omega'+\nu_B+\nu}\right]. \quad (4.7)$$

Then Eqs. (3.20), (3.21), (4.5), (4.6), and (4.7) give

$$J^{0,33}(0,0,\mu^2,t)=0,$$

$$J^{+,33}(0,0,\mu^2,t)=\tfrac{4}{3}B(t)\frac{2}{\pi}\int d\omega'\frac{\mathrm{Im}f_{33}(\omega')}{|\mathbf{q}'|^2}, \quad (4.8)$$

$$H_F{}^{-,33}(0,0,\mu^2,t)=0.$$

In the narrow-resonance approximation[16] one finds that

$$\frac{1}{\pi}\int d\omega'\frac{\mathrm{Im}f_{33}(\omega')}{|\mathbf{q}'|^2}=\frac{g_r{}^2}{12\pi m^2}, \quad (4.9)$$

so that we obtain finally

$$J^{0,33}(0,0,\mu^2,t)=0,$$

$$J^{+,33}(0,0,\mu^2,t)=\frac{g_r}{2m}\frac{8}{9}\frac{G_M{}^V(t)}{2m}, \quad (4.10)$$

$$H_F{}^{-,33}(0,0,\mu^2,t)=0.$$

[16] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

Hence in the resonance approximation, our sum rules (3.22)–(3.25), give, respectively,

$$[G_A(t)/G_A]=F_1{}^V(t), \quad (4.11)$$

$$\frac{g_A(t)}{G}=\left(-\frac{G_A}{G}\right)\frac{2m}{K(0)}\left[\frac{K(t)F_\pi(t)}{\mu^2-t}+\frac{F_\pi(t)K(t)-F_1{}^V(t)K(0)}{t}\right], \quad (4.12)$$

$$\tfrac{1}{2}F_2{}^V(t)=(1/K(0))(4/9)G_M{}^V(t), \quad (4.13)$$

$$\tfrac{1}{2}F_2{}^S(t)=0. \quad (4.14)$$

5. DISCUSSION AND CONCLUSION

Equations (4.11) to (4.14) are our final results. These have been obtained in the resonance approximation. Equation (4.14) implies that $\frac{1}{2}\mu_s'=0$, the left side being -0.06 and so Eq. (4.14) is satisfied to a good approximation. Equation (4.13) gives, for zero momentum transfer [taking $K(0)\approx 1$],

$$\tfrac{1}{2}\mu_V'\approx(4/9)\mu_V. \quad (5.1)$$

The left side here is 1.85, while the right side is about 2. The value on the right side is consistent with that obtained by Fubini, Furlan, and Rosetti[7] who obtained it to be 1.99 by using the isobar model of Gourdin and Salin.[17] The agreement between the left and right sides of Eq. (5.1) is fair; the small discrepancy here as well as in Eq. (4.14) may be removed by the contribution from higher states to the dispersion integrals. Indeed it has been shown in Ref. 7 that the (1,3) resonance at 1515 MeV reduces the right-hand side of (5.1) to the right amount. We also note that the resonance approximation gives an overestimate. This is the case with our Eq. (5.1) and in Ref. 7. This was also the case in Adler and Weisberger's calculation[4,5] where, as shown by Adler,[4] the (3,3) resonance alone gives $-G_A/G=1.44$ which is reduced to 1.2 by the higher states. Equation (4.13) also implies the shape equality of the form factors $F_2{}^V(t)$ and $G_M{}^V(t)$ which in turn implies that the form factors $F_2{}^V(t)$ and $F_1{}^V(t)$ have the same form since $G_M{}^V(t)=F_1{}^V(t)+F_2{}^V(t)$.

Our Eq. (5.1) shows that the (3,3) resonance approximation is quite good. This may, however, not be a good approximation for the axial-vector form-factor sum rule since otherwise it will be hard to understand the relation (4.11). The point is that while $F_1{}^V(t)$ is dominated at low momentum transfer by the ρ-meson pole, no axial-vector meson has been found at about the mass of the ρ meson to dominate $G_A(t)$. Hence the higher states in the dispersion integral of the sum rule (3.22) might be important and change the relation (4.11) obtained on the resonance approximation. It will be very interesting to measure the axial-vector form

[17] M. Gourdin and P. Salin, Nuovo Cimento **27**, 193 (1963).

factor in high-energy neutrino experiments to test the relation (4.11). Relation (4.12) is interesting because it includes, besides the usual contribution from the pion pole [the first term on the right side of (4.12)], the second-order contribution. It is reasonable to take $F_\pi(t)\approx F_1^V(t)$ for low momentum transfers[18] and we know from the success of the Goldberger-Treiman relation that $K(0)\approx 1$; then the second term on the right side of (4.12) is ≈ 0 for zero momentum transfer.

Our sum rules (3.22) to (3.25) are dependent solely on the equal-time commutation relation (2.8), the PCAC hypothesis, and the dispersion relations with no subtractions [except that dictated by gauge invariance, namely the term $\{F_1^V(-k^2)-F_\pi(-k^2)K(-\Delta^2)\}/k^2$ in our expression for $R_\nu^{\alpha(\mathrm{Born})}$, Eq. (2.11)]. Actually the PCAC hypothesis is not necessary[5]; all that is needed is that

$$\tilde{R}_\nu^\alpha(\nu, \nu_B, -q^2, -k^2) =[R_\nu^\alpha(\nu, \nu_B, -q^2, -k^2)-R_\nu^{\alpha(\mathrm{Born})}],$$

where

$$R_\nu^\alpha(\nu, \nu_B, -q^2, -k^2) =i\int d^4x e^{-iq\cdot x}\,\theta(x_0)\langle p'|[\partial_\mu A_\mu^\alpha(x),j_\nu^{\mathrm{el}}(0)]|p\rangle,$$

should be dominated near $q_\mu\approx 0$ by the pion pole at $-q^2=\mu^2$. The sum rules (4.11)–(4.14) are further dependent on the resonance approximation of keeping only the (3,3) resonance in the dispersion integrals, which appears to be a good approximation at least for the relations (4.13) and (4.14).

Note added in proof. After this paper was submitted for publication, it came to our attention that the sum rule, Eq. (3.22) was derived also by S. L. Adler [Proceedings of the International Conference on Weak Interactions, Argonne National Laboratory, 1965, p. 291 (unpublished)].

It also came to our notice that the sum rules for the axial vector form factors have also been discussed by Furlan, Jengo and Remiddi (Nuovo Cimento, to be published). The treatment of the (3,3) resonance in their paper is different from ours. They explicitly put the (3,3) resonance as a particle in the intermediate state which dominates the dispersion integral, and in this way they get a contribution to $G_A(t)$ from the (3,3) resonance whereas we do not. This is probably due to different treatment of the (3,3) resonance in the two papers. Out treatment is based on the FNW method which makes use of static approximation. The (3,3) contribution to $G_A(t)$ in our method is probably a higher order effect.

[18] For experimental indication in support of this view, see C. W. Akerlof *et al.*, Phys. Rev. Letters **16**, 147 (1966).

APPENDIX

In this Appendix we show that in general the amplitudes M_B and M_E have a kinematic singularity at $\nu_B=0$. It has been shown by Dennery[14] that if one writes $T(\nu, \nu_B\mu^2, -k^2)$ as

$$T=\bar{u}(p')\sum_{i=1}^{6} A_i M_i u(p), \quad \text{(A1)}$$

where

$$M_1=M_A, \quad M_2=2i\gamma_5[P_\nu(2m\nu_B-\tfrac{1}{2}k^2)+(q_\nu-\tfrac{1}{2}k_\nu)m\nu],$$
$$M_3=M_C, \quad M_4=M_D, \quad M_5=M_E, \quad M_6=M_F, \quad \text{(A2)}$$

then A_i do not have any kinematic singularity at $\nu=0$ and $\nu_B=0$. Now one can write M_2 as