Spontaneously Broken Symmetries and the Question of Massless Particles in a Model Field Theory*

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An example is given showing that the condition of broken symmetry can be formally satisfied without the presence of interacting massless particles. The model used is an extension of Zachariasen's model and is a relativistic generalization of the high-density electron gas. The model is defined by a sum of perturbation-theory diagrams and also by an equivalent Lagrangian. In this nodel, when the physical vacuum is not invariant under a certain continuous symmetry of the Lagrangian, massless particles must be present. If, however, a Yukawa interaction of infinite range is present, then these massless mesons decouple from all physical amplitudes.

I. INTRODUCTION

IN this paper we review some consequences of broken symmetry in a field theory that go under the name of "Goldstone theorem." The idea of broken symmetry[1,2] (or hidden symmetry or long-range order) is that, in a theory whose Lagrangian is invariant under the action of a continuous group, the ground state or vacuum may, for reasons of dynamical instability (into which we do not enter), be unsymmetric in a way to be discussed. The one-particle and other states built on this vacuum will then hide the symmetry of the original Lagrangian. The "theorem" in question claims[3–5] that, if the continuous symmetry is broken by the vacuum, then there must exist massless excitations of a definite symmetry character determined by the broken symmetry.

The merits of the hidden-symmetry approach are that while maintaining the symmetry of the Lagrangian, states which do not transform covariantly under the group are admitted, and that the existence of massless particles is apparently associated with an invariance principle. The Goldstone theorem would be remarkable, if true, since it would constitute an exact statement about the physical spectrum of real field theories, based on symmetry alone. We will use an example to show that, while the theorem is formally true, no physical conclusions can be drawn directly from the broken symmetry alone. This is not to deny that massless particles may very well appear in broken-symmetry theories as a consequence of a dynamical calculation or approximation.

There exist by now several more or less equivalent "proofs"[3–5] of the Goldstone theorem. In Sec. II we define the way in which the symmetry is broken and the sense in which the vacuum is degenerate. Following the original method of Goldstone, Salam, and Weinberg (G-S-W),[4] we emphasize the interplay of integral and differential conservation laws. We are careful to distinguish between states $|g\rangle$ of zero mass $p_g{}^2=0$, and spurious states $|0'\rangle$ which, even if they existed at all in a relativistic theory, would be uninteresting because their energy-momentum would be identically zero $p_\mu=0$. We distinguish two ways by which the symmetry may be broken, and mention more or less trivial examples in which the G-S-W condition is formally satisfied, yet the result is practically devoid of physical content.

In Sec. III we exhibit a less trivial example in which the broken-symmetry condition leads to physical particles of zero mass, except in the presence of infinite-range forces, when these massless particles decouple. Our model is an extension of that of Zachariasen[6,7] and of Nambu and Jona-Lasinio,[2] and is a relativistic generalization of the many-body problem of the high-density electron gas. Section III contains a simple diagrammatic formulation of this model. Another formulation, in terms of an equivalent Lagrangian, is given in the Appendix.

II. BROKEN SYMMETRY AND THE G-S-W CONDITION

A. Meaning of Broken-Symmetry Condition

We consider a continuous group, represented in Hilbert space by a set of unitary operators $U=e^{i\alpha Q}$ under which the fields transform infinitesimally as

$$[Q,\phi(y)]=\delta\phi\,, \tag{2.1}$$

and take vacuum expectation values

$$\langle 0|[Q,\phi(y)]|0\rangle=\langle 0|\delta\phi|0\rangle. \tag{2.2}$$

In a normal theory, $\langle\delta\phi\rangle=0$. Broken symmetry means

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that $\langle\delta\phi\rangle\neq 0$, or $\langle 0|\phi(y)|0\rangle\neq\langle 0|\phi'(y)|0\rangle$. If $|0_\alpha\rangle$ is used to designate $U|0\rangle$,

$$|0_\alpha\rangle\neq|0\rangle. \qquad (2.3)$$

Because U is a symmetry transformation, Q only has matrix elements between states of the same energy. Thus in broken-symmetry theories,

$$\langle 0|[Q,\phi(y)]|0\rangle = 2i\,\mathrm{Im}\sum_\alpha \langle 0|Q|0_\alpha\rangle\langle 0_\alpha|\phi(y)|0\rangle\neq 0, \qquad (2.4)$$

and $\phi(y)$ must have matrix elements between the different vacua $|0\rangle$ and $|0_\alpha\rangle$.

The broken-symmetry condition merely asserts the existence of a ground state which instead of being an eigenstate of Q is a coherent superposition of such eigenstates. This is a common situation in the nonrelativistic many-body problem.

Q generates uniform transformations on $\phi(x)$, and the states $|0_\alpha\rangle$ are, like $|0\rangle$, states of zero four-momentum. Unless they are the limit of a continuous excitation branch as $\mathbf{p}\to 0$, the states $|0_\alpha\rangle$ do not refer to an internal excitation of the system, but to a trivial and dynamically uninteresting "spurious state" which is degenerate with $|0\rangle$ because it is obtained from $|0\rangle$ by going to another system. Technically speaking, the Hilbert spaces built on these different vacua are inequivalent because the transformation U is not a proper transformation when the quantization volume $V\to\infty$. This is so, because the only transformations that can be actively realized in any localized experiment are those over large but finite V.

Our universe is realized on a particular Hilbert space. We have gone through this discussion in order to emphasize the formal nature of the global transformations generated by Q, and the formal nature of the vacuum degeneracy. In order to say something about localized excitations in our particular universe, we have to look at the differential conservation law for a localized current $j_\mu(x)$ or its three-dimensional Fourier transform $\tilde{j}_\mu(\mathbf{q})$ and study the limit $\mathbf{q}\to 0$ but $\mathbf{q}\neq 0$. The above considerations regarding

$$Q=\int d^3x\, j_0(x)=\tilde{j}_0(0) \qquad (2.5)$$

are relevant only if there is continuity at the limit $\mathbf{q}\to 0$, i.e., provided the forces are of finite range so that there is nothing singular about the limit $V\to\infty$. The validity of the Goldstone theorem is equivalent to such continuity at $\mathbf{q}\to 0$.

B. Local Current Conservation and the Goldstone Theorem

Goldstone, Salam, and Weinberg[4] consider

$$\langle 0|[j_\mu(x),\phi(y)]|0\rangle=\int \rho_\mu(p)e^{ip\cdot(x-y)}\,d^4p, \qquad (2.6)$$

where, in terms of a complete set of intermediate states $|g\rangle$,

$$\rho_\mu(p)=\sum_g \delta^4(p-p_g)\theta(p_g)\times 2\,\mathrm{Im}\langle 0|j_\mu(0)|g\rangle\langle g|\phi(0)|0\rangle. \qquad (2.7)$$

The intermediate states $|g\rangle$ are not states of $p_{\mu g}=0$. We will show that they are states of zero mass, $p_g{}^2=0$. In the following, we always consider $\mathbf{p}\neq 0$ but $\lim \mathbf{p}\to 0$.

Local conservation, $\partial^\mu j_\mu=0$, requires, in a manifestly covariant theory,

$$\rho_\mu(p)=p_\mu\delta(p^2)[C\epsilon(p)+D], \qquad (2.8)$$

which already shows

$$\lim_{\mathbf{p}\to 0}\rho_\mu(p)=0 \quad \text{for} \quad p_0\neq 0, \qquad (2.9)$$

so that the only states $|g\rangle$ that can contribute are, at most, those for which $p_{g0}\to 0$ as $\mathbf{p}_g\to 0$.

Now the integral conservation law that $Q=\int j_0(x)d^3x$ be independent of time, means

$$\langle\delta\phi\rangle=(2\pi)^3\int \rho_0(p^2)|_{\mathbf{p}=0}\,dp_0, \qquad (2.10)$$

so that

$$C=\langle\delta\phi\rangle. \qquad (2.11)$$

Thus

$$\langle[j_\mu(x),\phi(y)]\rangle=\langle\delta\phi\rangle\partial_\mu D(x-y) + \text{possible nonlocal term}. \qquad (2.12)$$

The role of the broken symmetry is to assure that there is excited by $\phi(0)$ a branch for which $p_{g0}\to 0$ as $p_g\to 0$.

Note that the G-S-W condition (2.12) asserts the appearance of zero-mass states as intermediate states in the rather unphysical vacuum expectation value $\langle[j_\mu(x),\phi(y)]\rangle$, and carries no implication that these states appear in S-matrix elements. ϕ can excite zero-mass particles, but the proof does not show that the spectrum of ϕ contains anything else, nor does it say anything about processes in which these particles should appear nor about the strength with which these massless particles are coupled. Thus the theorem may apparently be satisfied by the presence of free massless particles that are uninteresting physically.

C. Some Trivial Examples

The current $j_\mu(x)$ acts like a field operator in possessing matrix elements between the vacuum $|0\rangle$ and one-particle states $|g\rangle$. There are rather trivial examples in which $j_0(x)$ is practically the field canonically conjugate to $\phi(x)$, and the G-S-W condition (2.12) can be formally satisfied by the presence of free massless particles. It is useful to distinguish two kinds of field transformation:

(1) Field translation: $\delta\phi=c$ number$\equiv\eta$,

(2) Field rotation: $\delta\phi_i=T_{ij}\phi_j$, with some $\langle\phi_j\rangle\neq 0$.

The corresponding symmetry generator,

$$Q=\int d\sigma^{\mu}\, j_{\mu}(x)\,, \tag{2.13}$$

is respectively linear or bilinear in the fields

$$j_{\mu}(x)=\eta^{\dagger}\partial_{\mu}\phi\,, \tag{2.14a}$$

or

$$\mathbf{j}_{\mu}(x)=i\phi^{\dagger}\overleftrightarrow{\partial}_{\mu}\mathbf{T}\phi\,. \tag{2.14b}$$

Whenever the symmetry transformation is that of field translation, "current conservation" $\partial^{\mu}j_{\mu}=0$ is no more than the free field equation $\partial^2\phi=0$. These cases are essentially trivial, as Guralnik and Hagen[8] have pointed out, because the presence of free massless particles is asserted directly by the symmetry principle, and the self-consistency condition

$$\langle\delta\phi\rangle=\eta\neq 0 \tag{2.15}$$

does not introduce into the theory any new parameters characterizing a new ordered or condensed state; indeed η is a "gauge variable" on which nothing depends. Examples where the broken symmetry is that of field translations occur in the acoustic excitations in a homogeneous medium[9] and in the Gupta-Bleuler formulation of Lorentz gauge quantum electrodynamics.[8–10] In both these cases, the G-S-W condition is satisfied by the presence of massless mesons which, however, are not real interacting particles.

The case of field rotations is equally uninteresting when the fields are generalized free fields, i.e., fields whose commutator

$$i[\phi^{\dagger}(x),\phi(y)]=i\int da\,\sigma(a)\Delta(x-y,\,a) \tag{2.16}$$

is a c number. Then

$$[\mathbf{j}_{\mu}(x),\phi(y)]=\int da\,\Delta(x-y,\,a)\mathbf{T}\overleftrightarrow{\partial}_{\mu}\phi(x)\,, \tag{2.17}$$

and current conservation gives

$$\langle 0|[\mathbf{j}_{\mu}(x),\phi(y)]|0\rangle=\langle\phi\rangle\mathbf{T}\partial_{\mu}\Delta(x-y)\,. \tag{2.18}$$

The current operator $\mathbf{j}_{\mu}(x)$ acts like its linearized approximation $i[\langle\phi^{\dagger}\rangle\mathbf{T}\partial_{\mu}\phi-\partial_{\mu}\phi\mathbf{T}^{\dagger}\langle\phi\rangle]$, so that this example practically reduces to the previous case of field translations. All these examples point the way in which the G-S-W condition (2.12) can be satisfied and still be devoid of useful dynamical content.

Generalized free fields lead to no scattering. The model we will now discuss can be formulated as a sort of generalized free field (a field with parameter[11]) which, however, does lead to a nontrivial S matrix. We present such a formulation in the Appendix. In the next section we prefer, however, to define the model simply as a sum of selected perturbation-theory diagrams. The model will show how a broken symmetry leads to physical particles of zero mass, except in the presence of infinite-range forces, when these massless particles decouple.

III. PAIRING APPROXIMATION: THE ZACHARIASEN MODEL

A. Different Formulations

The model we now consider was first formulated by Zachariasen[6] in terms of a dispersion relation for s-wave scattering, incorporating elastic unitarity but neglecting the left cut. In this form, it appears as an approximation to a full relativistic theory in which, at least for a range of energies, the contributions of crossed channels are dominated by a single-particle pole. Because the scattering amplitude is analytic except for the right cut demanded by unitarity, crossing symmetry is violated.

The model has also been expressed by Thirring[12] in terms of selected (chain) perturbation-theory diagrams. We consider three relativistic scalar mesons A, B, C and allow vertices $A\rightleftarrows B+C$ with bare Yukawa coupling constant g_0, and $B+C\leftrightarrows B+C$ with bare Fermi coupling constant λ_0. Because we do not include the crossed diagrams $B\leftrightarrows A+\bar{C}$, $C\leftrightarrows A+\bar{B}$, or $B+\bar{C}\leftrightarrows B+\bar{C}$, the model is a generalization of the Lee model in which relativistic kinematics is used for V, N and Feynman rather than retarded propagators are employed. While crossing symmetry is still clearly violated, the Zachariasen model still has interesting physical content. Vertex corrections are present, charge renormalization is finite, and ghosts do not appear for a finite range of coupling strengths. In this form, defined as a sum of chain diagrams, the Zachariasen model is a relativistic generalization of the Gell-Mann–Brueckner high-density electron gas, and is equivalent to a linearized Hartree approximation or random-phase approximation. The A, B, and C particles correspond to phonons, electrons, and holes and the λ_0 and g_0 couplings correspond to the bare electron-electron and the bare electron-phonon interactions in a metal. The nonrelativistic limit of the Zachariasen model is therefore of definite physical interest.

Finally, the model has been treated by Thirring[13] by replacing $B(x)C(x)$ pairs by a single field $\phi(x,s)$ of

[8] G. S. Guralnik and C. R. Hagen, Imperial College Report No. ICTP/64/75 (unpublished).

[9] S. A. Bludman, in *1965 Tokyo Summer Lectures in Theoretical Physics, II. High Energy Physics*, edited by G. Takeda (Syokabō and W. A. Benjamin, Inc., New York, 1966).

[10] S. A. Bludman, in *Proceedings of Seminar on Unified Theories of Elementary Particles*, edited by H. Rechenberg (Max-Planck Institut fur Physik und Astrophysik, München, Germany, 1966).

[11] A. L. Licht, thesis, University of Maryland, 1963 (unpublished); A. S. Wightman, revised notes for lectures at the French Summer School of Theoretical Physics, Cargèse, Corsica, July, 1964 (unpublished).

[12] W. Thirring, in *Theoretical Physics*, edited by A. Salam (International Atomic Energy Agency, Vienna, 1963).

[13] W. Thirring, Phys. Rev. **126**, 1209 (1962).

continuous mass $\sqrt{s}$ equal to the sum of B and C particle energies. This is a Lagrangian field theory in which, because individual field B and C particles do not appear, asymptotic completeness is not satisfied. In this form, the Zachariasen model is a relativistic generalization of Wentzel's effective (pair-theory) Hamiltonian[14] treatment of the high-density electron gas.

These three formulations give identical results for the BC scattering amplitude $T(s)$, for the A particle propagator $\Delta(s)$, and for the ABC vertex $\Gamma(s)$. The dispersion theoretic formulation has already been given in an earlier paper.[7] In the remainder of this paper, we employ the second formulation of the Zachariasen model, as a sum of perturbation-theory chain diagrams. In the Appendix, we present the third formulation in terms of a field of continuous mass.

B. Diagram-Sum Formulation

We begin by considering three relativistic scalar mesons A, B, C with nonderivative trilinear and quadrilinear interactions,

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1, \tag{3.1}$$

where

$$\mathcal{L}_0=\partial_\mu A^\dagger\partial^\mu A-\mu_0A^\dagger A+\partial_\mu B^\dagger\partial^\mu B-m_{B0}^2B^\dagger B+\partial_\mu C\partial_\mu C-m_{C0}^2C^2, \tag{3.2}$$

$$\mathcal{L}_1=g_0AB^\dagger C+\lambda_0B^\dagger CBC+\text{H.c.} \tag{3.3}$$

The A and B particles are charged so that the current

$$j_\mu(x)=i[A^\dagger(x)\overleftrightarrow{\partial}_\mu A(x)+B^\dagger(x)\overleftrightarrow{\partial}_\mu B(x)] \tag{3.4}$$

is conserved.

We will study the s-wave scattering of B and C particles. In Born approximation, this scattering proceeds through direct BC interaction with bare coupling strength λ_0 and through the exchange of A particles of bare mass μ_0 and bare coupling g_0. The Born amplitude is

$$T_0(s)=\lambda_0+g_0^2/(s-\mu_0^2), \tag{3.5}$$

where $s=(p_B+p_c)^2$. The physical A particles, which may be elementary A particles or BC bound states, are identified as poles of the scattering amplitude $T(s)$, lying at $s=\mu^2$.

We now calculate in pairing approximation, where B and C particles are always emitted and absorbed together, so that $B(x)C(x)$ is approximated by

$$B(x)^{(+)}C(x)^{(+)}+B(x)^{(-)}C(x)^{(-)}$$

with the terms $B(x)^{(+)}C(x)^{(-)}$ and $B(x)^{(-)}C(x)^{(+)}$ neglected. In this approximation the B and C particles remain undressed but the (unrenormalized) A-particle propagator is given by

$$\Delta_u^{-1}(s)=s-\mu_0^2-\pi(s), \tag{3.6}$$

where the proper self-energy is

$$\pi(s)=g_0[-B(s)]g_0\Gamma_u(s), \tag{3.7}$$

in terms of the unrenormalized proper vertex

$$\Gamma_u(s)=1/[1+\lambda_0B(s)] \tag{3.8}$$

and the irreducible A-particle polarization (BC bubble diagram)

$$B(s)=\int\frac{d^4p}{(p_B^2+m_B^2)(p_C^2+m_C^2)}=\frac{1}{\pi}\int_{(m_B+m_C)^2}^{\infty}\frac{\rho(s')\,ds'}{s'-s-i\epsilon}. \tag{3.9}$$

Here

$$\rho(s)=(1/16\pi s)\times[s^2-2(m_B^2+m_C^2)s+(m_B^2-m_C^2)^2]^{1/2} \tag{3.10}$$

is the relativistic kinematic factor for two scalar particles of physical masses m_B and m_C. Although we allow no A-particle dressing of B and C particles, m_B and m_C may differ from the bare masses m_{B0}, m_{C0} because of the possibility of BC mixing.

The A-particle self-energy is

$$\delta\mu^2=\pi(\mu^2)=-g_0^2B(\mu^2)/[1+\lambda_0B(\mu^2)], \tag{3.11}$$

and the wave-function renormalization

$$Z_3^{-1}=[d\Delta_u^{-1}/ds]_{s=\mu^2}=1+g_0^2+I(\mu^2)Z_1^{-2}, \tag{3.12}$$

where

$$I(s)=\frac{1}{s-\mu^2}[B(s)-B(\mu^2)]=\frac{1}{\pi}\times\int\frac{\rho(s')\,ds'}{(s^2-\mu^2)(s'-s-i\epsilon)}, \tag{3.13}$$

$$Z_1^{-1}=\Gamma_u(\mu^2)=[1+\lambda_0B(\mu^2)]^{-1}. \tag{3.14}$$

Note that $0\leq Z_3\leq 1$.

The BC scattering proceeds through a sum of bubble diagrams,

$$T(s)=T_0(s)+T_0(s)[-B(s)]T(s). \tag{3.15}$$

Since

$$T(s)=T_0(s)/[1+T_0(s)B(s)], \tag{3.16}$$

the A-particle poles are the zeros of $1+T_0(s)B(s)$, or

$$1+\left(\lambda_0+\frac{g_0^2}{\mu^2-\mu_0^2}\right)B(\mu^2)=0 \tag{3.17}$$

is the mass equation.

The coupling constant is given by the

$$\frac{1}{g^2}=\left[\frac{dT^{-1}}{ds}\right]_{s=\mu^2}=\frac{d}{ds}[T_0^{-1}(s)+B(s)]_{s=\mu^2}=[B(\mu)]^2\frac{g_0^2}{(\delta\mu^2)^2}+I(\mu^2)=\frac{Z_1^2}{g_0^2}+I(\mu^2). \tag{3.18}$$

[14] G. Wentzel, Phys. Rev. **108**, 1593 (1957); R. Brout, *ibid.* **108**, 515 (1957).

Since in Eq. (3.12)

$$Z_1^2/Z_3=Z_1^2+g_0^2I(\mu^2)\,, \tag{3.19}$$

we have

$$g^2=g_0^2Z_3/Z_1^2. \tag{3.20}$$

Using Eqs. (3.14) and (3.20), the mass equation (3.17) can also be rewritten

$$\begin{aligned}\delta\mu^2&=-(g_0^2/Z_1)B(\mu^2)\\&=-g^2(Z_1/Z_3)B(\mu^2).\end{aligned}$$

These two forms show that in a divergent theory $[B(\mu^2)=\infty]$ the self-energy $\delta\mu^2$ can be finite only if $g_0^2=0$ and $Z_1=0$.

Note that $Z_3=1-g^2I(\mu^2)$, so that g^2 is bounded by

$$g^2\leq g^2_{\text{crit}}\equiv I/I(\mu^2)\,, \tag{3.21}$$

and that as $g_0^2\to 0$, $Z_3\to 1$, and $g^2\to 0$.

C. Broken Symmetry in the Zachariasen Model

The broken symmetry we consider is that the physical vacuum $|0\rangle$ is such that BC pairs can disappear into it, in violation of charge conservation, i.e., that there is a nonvanishing amplitude

$$\eta=\langle 0|B(x)C(x)|0\rangle$$

for the mixing of B and C particles of different charge. We write

$$\mathcal{L}=\mathcal{L}_0'+\mathcal{L}_1'\,, \tag{3.22}$$

with

$$\mathcal{L}_0'=\mathcal{L}_0-\eta^*B(x)C(x)-\eta B^\dagger(x)C^\dagger(x)\,, \tag{3.23}$$

$$\mathcal{L}_1'=(BC)^\dagger(g_0A+\lambda_0BC+\eta)+\text{H.c.}, \tag{3.24}$$

and calculate η self-consistently by requiring that $\mathcal{L}'$ produce no BC mixing additional to that contained in $\mathcal{L}_0'$, i.e., that the physical vacuum be stable with respect to the disappearance of any further charge. Introducing a two-component charge spinor $\phi=\begin{pmatrix}B\\C^\dagger\end{pmatrix}$, a mass matrix

$$m^2=\begin{pmatrix}m_{B0}^2 & \eta\\ \eta^* & m_{C0}^2\end{pmatrix}, \tag{3.25}$$

and Feynman propagator

$$\Delta(p^2)=1/p^2-m^2+i\epsilon\,, \tag{3.26}$$

we have, in our approximation,

$$\eta\equiv\langle 0|\phi^\dagger\tau^-\phi|0\rangle=Tr\int d^4p\;T_0(0)\tau^-\Delta(p)\,, \tag{3.27}$$

or

$$\eta=T_0(0)\int d^4p\,\frac{1}{p^2-m_B^2+i\epsilon}(-\eta)\frac{1}{p^2-m_C^2+i\epsilon}, \tag{3.28}$$

where

$$\begin{aligned}m_B^2&=\tfrac{1}{2}(m_{B0}^2+m_{C0}^2)+(\Delta^2+|\eta|^2)^{1/2},\\ m_C^2&=\tfrac{1}{2}(m_{B0}^2+m_{C0}^2)-(\Delta^2+|\eta|^2)^{1/2},\\ \Delta&=\tfrac{1}{2}(m_{B0}^2-m_{C0}^2).\end{aligned} \tag{3.29}$$

The self-consistency condition can finally be written

$$\eta=\eta T_0(0)[-B(0)]. \tag{3.30}$$

This condition admits of normal symmetry-preserving solutions ($\eta=0$, $m_B=m_{B0}$, and $m_C=m_{C0}$), and of abnormal symmetry-breaking solutions ($\eta\neq 0$) given by the solutions of

$$1+T_0(0)B(0)=0. \tag{3.31}$$

This condition for broken symmetry determines $B(0)$ in terms of $1/T_0(0)$. It also determines η through $B(0)$, which depends on η through the masses for the physical B and C particles that result from the BC mixing. We have

$$\rho(s)=(1/16\pi s)[s^2-2(m_{B0}^2+m_{C0}^2)s+(m_{B0}^2+m_{C0}^2)^2+4|\eta|^2]^{1/2}, \tag{3.32}$$

and

$$(m_B+m_C)^2=m_{B0}^2+m_{C0}^2+2(m_{B0}^2m_{C0}^2-|\eta|^2)^{1/2}. \tag{3.33}$$

Note that a certain minimum strength of coupling $T_0(0)$ is necessary in order that Eq. (3.31) have any solution, and that η depends nonanalytically on the bare coupling.

Comparing Eqs. (3.17) and (3.31), we see that the abnormal ($\eta\neq 0$) self-consistent solution demands zero-mass A particles or bound states. This is the Goldstone theorem for our model.

D. Discussion of Special Cases

1. In the case of pure quadrilinear coupling, $g_0=0$, the scattering proceeds through a sum of BC chain diagrams,

$$T(s)=\lambda_0/[1+\lambda_0B(s)]. \tag{3.34}$$

The broken-symmetry condition

$$1+\lambda_0B(0)=0 \tag{3.35}$$

asserts the existence of a pole in the vertex $\Gamma_u(s)$ at $s=0$: The BC scattering proceeds through the exchange of a massless neutral bound state. This massless scalar meson is the precise analog of the massless pion in the Nambu–Jona-Lasinio theory, or of the phonon-like collective excitation in superconductivity.

2. In the case of pure Yukawa coupling

$$T(s)=g_0^2\Delta_\mu(s)\,, \tag{3.36}$$

where the A propagator

$$\Delta_u(s)=1/[s-\mu_0^2+g_0^2B(s)] \tag{3.37}$$

has a pole at μ^2. After renormalization,

$$T(s)=g^2\Delta(s)\,, \tag{3.38}$$

where

$$g^2=Z_3g_0^2. \tag{3.39}$$

The self-consistency condition

$$1-(g_0^2/\mu_0^2)B(0)=0 \tag{3.40}$$

now makes the physical A particle be of zero mass. Something interesting happens as we now consider the effect of varying the bare mass μ_0 keeping the order parameter η or $B(0)$ fixed:

(*i*) If $\mu_0 \to \infty$, then g_0 must $\to \infty$ so that g_0^2/μ_0^2 stays fixed. This elementary-particle theory is practically indistinguishable from the preceding bound-state theory. This is to be expected since as $g_0^2 \to \infty$, $Z_3 \to 0$, and $g^2 \to I/I(0)$, a number determined by the B and C particle masses.

(*ii*) More interesting is the limit $\mu_0^2 \to 0$, which makes $g_0^2 \to 0$. Then, although the $\mu^2=0$ pole remains in Δ_u, $Z_3 \to 1$, $g^2 \to 0$, and $T(s)$ vanishes. In this limit the mesons demanded by the Goldstone theorem decouple: The broken symmetry leads to a pole in the propagator, but for $\mu_0^2=0$ this pole is unphysical since it disappears from the scattering. In fact, the left side of Eq. (2.12) is independent of g_0. This example already illustrates the way in which Goldstone mesons may be unphysical.

3. In the combined theory, we obtain

$$T(s)=g_0^2\Gamma_u\Delta_u\Gamma_u+\lambda_0\Gamma_u. \tag{3.41}$$

In this expression the first term refers to scattering through the exchange of A particles, while the second term is one-particle irreducible. In terms of renormalized quantities

$$\begin{aligned}&\Delta(s)=Z_3^{-1}\Delta_u(s),\quad \Gamma(s)=Z_1\Gamma_u(s)\,,\\&T(s)=g^2\Gamma\Delta\Gamma+(\lambda/Z_3)\Gamma\,,\end{aligned} \tag{3.42}$$

where λ is defined so that $\lambda_0/Z_1=\lambda/Z_3$. Recall that $g^2=(Z_3/Z_1^2)g_0^2$ vanishes as $g_0^2 \to 0$.

Broken symmetry requires, for a fixed η, that $T_0(0) =\lambda_0-g_0^2/\mu_0^2$ remain fixed at $-1/B(0)$. If we choose to hold λ_0 fixed at some value λ_0', then as $\mu_0^2 \to 0$, $g_0 \to 0$ so that the elementary-particle term disappears from $T(s)$, leaving scattering only through the term $\lambda_0'\Gamma_u$. It may even happen that λ_0' is sufficiently strong to produce a bound state at some $\mu'^2 \neq 0$. In that case, we might speak of a bootstrap theory in which the interaction proceeds through the exchange of BC bound states whose existence itself depends on the interaction they produce. In any case, whether or not such a bound state at μ' is possible, the Goldstone theorem is satisfied by a propagator pole which disappears from the scattering amplitude when $\mu_0=0$ and leaves nontrivial scattering.

IV. CONCLUSIONS

While broken symmetry demands some states $|g\rangle$ of zero mass, these states need not appear in amplitudes of physical interest. In the preceding example, the Goldstone meson decoupled when the bare A-particle force was allowed to become of infinite range. There was still a nontrivial S matrix. This result should not be too surprising, since it parallels a similar result in the nonrelativistic many-body problem: In a superconductor, because of the infinite-range Coulomb interaction, phonon modes need not be present.

The role of the Goldstone theorem, if it were applicable, would have been to associate symmetry breakdown with the existence of massless particles with certain quantum numbers. Our counter-example shows, however, that physical particles of zero mass need not emerge simply as a direct consequence of broken symmetry. Whether massless particles occur or not in a particular theory can be determined only after a detailed dynamical calculation. Despite its heuristic value, the G-S-W condition cannot lead directly to physical conclusions.

APPENDIX

A. Lagrangian Formulation of the Zachariasen Model

In this Appendix we generalize Thirring's treatment of the Zachariasen model[13] by including λ_0 as well as g_0 coupling and by giving B and C particles different charges and different masses. Since B and C particles are always created and annihilated together, Thirring (following a suggestion by T. D. Lee) introduces, alongside of the field $A(x)=\phi(x,s_0)$, a continuum of fields $\phi(x,s)$ which will destroy at x BC pairs of energy squared $s=(p_B+p_C)^2$. The Lagrangian density

$$\begin{aligned}\mathfrak{L}=\int_0^\infty\int_0^\infty ds\,ds'\,\phi^\dagger(x,s)\{[\overleftarrow{\partial}_\mu\overrightarrow{\partial}^\mu+sp(s)]\delta(s-s')\\+\mathfrak{M}^2(s,s')\}\phi(x,s')\,,\end{aligned} \tag{A1}$$

where

$$\begin{aligned}&p(s)=\delta(s-s_0)+\theta(s-s_t),\quad s_0=\mu^2,\quad s_t=(m_B+m_C)^2\,,\\&\mathfrak{M}^2(s,s')=g_0[\delta(s-s_0)f(s')+\delta(s'-s_0)f(s)]\\&\qquad\qquad+\lambda_0f(s)f(s')\,,\\&f^2(s)=(1/\pi)\rho(s)\theta(s-s_t)\,,\end{aligned} \tag{A2}$$

will reproduce all the results of the diagrammatic formulation of the Zachariasen model. Note that in this Lagrangian the threshold s_t and phase-space factor $\rho(s)$ have been chosen to depend on m_B and m_C rather than on any m_{B_0} and m_{C_0}. This means that any BC mixing has to be already included in our Lagrangian rather than in a choice of abnormal over normal solutions. For this reason, besides the Lagrangian, the relation (3.29) connecting the B, C masses with the mixing parameter has to be assumed.

The quadratic Lagrangian (A1) leads to the linear equations of motion

$$(\Box-s)p(s)\phi(x,s)-\int\mathfrak{M}^2(s,s')\phi(x,s')\,ds'=0, \tag{A3}$$

which can be diagonalized by the transformation

$$\phi(x,s)=\int K(s,s')\Phi(x,s')\,ds' \quad \text{(A4)}$$

which is unitary,

$$\int K^\dagger(s'',s)p(s)K(s,s')\,ds'=\delta(s''-s')p(s'), \quad \text{(A5)}$$

with weight function $p(s)$. The real symmetric matrix $\mathfrak{M}^2(s,s')$ will be diagonalized

$$\int K^\dagger(s,u)[up(u)\delta(u-u')+\mathfrak{M}^2(u,u')]K(u',t)du du' = M^2(s)\delta(s-t) \quad \text{(A6)}$$

with

$$M^2(s)=\delta(s-s_0)\mu^2+\theta(s-s_t)s \quad \text{(A7)}$$

representing a particle of discrete mass μ^2 and a continuum beginning at s_t. The secular equation (A6) has indeed two solutions. In one solution,

$$K=K_-(s,s')=\delta(s'-s_0)a(s)+\theta(s'-s_t)G_-(s,s'), \quad \text{(A8)}$$

where, for $s=s_0$,

$$K_-(s_0,s')=\delta(s'-s_0)a(s_0) +\theta(s'-s_t)g_0f(s')/(s'-s_0)D_-(s') \quad \text{(A9)}$$

and, for $s>s_t$,

$$a(s)=\frac{g_0a(s_0)}{1+\lambda_0B(\mu^2)}\frac{f(s)}{\mu^2-s}, \quad s>s_t, \quad \text{(A10)}$$

$$G_-(s,s')=\delta(s-s')+\frac{T_0(s')f(s)f(s')}{(s'-s+i\epsilon)D_-(s')}, \quad s>s_t, \quad \text{(A11)}$$

$$D_-(s)=1+T_0(s)B(s),$$

$$T_0(s)=\lambda_0+g_0^2/(s-s_0), \quad B(s)=\int\frac{f^2(s')\,ds'}{(s'-s-i\epsilon)}. \quad \text{(A12)}$$

The second solution of the secular equation, $K=K_+$, is obtained from K_- by replacing $i\epsilon$ everywhere by $-i\epsilon$.

Here $a(s_0)$ is a normalization constant to be determined from the orthogonality relations

$$\int_0^\infty |a(s)|^2p(s)\,ds=1, \quad \text{(A13)}$$

$$\int_0^\infty G^\dagger(s''s)p(s)G(s,s')ds=\delta(s''-s'), \quad s',\,s''>s_t \quad \text{(A14)}$$

that issue from Eq. (A5). The eigenvalue condition that $K_\pm$ be solutions is the vanishing of the denominator function

$$D_\pm(\mu^2)=1+[\lambda_0+g_0^2/(\mu^2-\mu_0^2)]B(\mu^2)=0. \quad \text{(A15)}$$

Although we have assumed the spectrum of $M^2(s)$ to contain one discrete point $M^2(s_0)$, solutions can also be easily obtained when Eq. (A15) has two solutions or none.

That

$$\phi(x,s)=a(s)\Phi_\pm(x,s_0)+\int_{s_t}^\infty G_\pm(s,s')\Phi_\pm(x,s')\,ds' \quad \text{(A16)}$$

is a solution of Eq. (A3) can be confirmed directly using Eq. (A15) and

$$(\Box-s)\Phi_\pm(x,s)=0, \quad \text{(A17)}$$

$$(s'-s)G_\pm(s,s')=\begin{cases} g_0f(s')/D_\pm(s'), & s=s_0, \\ T_0(s')f(s)f(s')/D_\pm(s'), & s>s_t, \end{cases} \quad \text{(A18)}$$

$$\int f(s')G_\pm(s',s)\,ds'=f(s)/D_\pm(s).$$

The formulas inverse to Eq. (A16) are

$$\Phi_\pm(x,s_0)=a^*(s_0)\phi(x,s_0)+\int_{s_t}^\infty a^*(s')\phi(x,s')\,ds',$$

$$\Phi_\pm(x,s)=G_\pm{}^\dagger(s,s_0)\phi(s_0)+\int_{s_t}^\infty G_\pm{}^\dagger(s,s')\phi(x,s')\,ds', \quad s>s_t. \quad \text{(A19)}$$

The eigensolutions Φ_+ and Φ_- are, respectively, the out and in fields to which $\phi(x,s)$ weakly converges, in the sense that

$$\lim_{t\to\pm\infty}\langle{}_{\text{in}}{}^{\text{out}}BC|\phi(x,s)|0\rangle=\langle{}_{\text{in}}{}^{\text{out}}BC|\Phi_\pm(x,s)|0\rangle, \quad \text{(A20)}$$

where BC out and in states are defined by

$$\Phi_\pm(x,s)|0\rangle=|BC_{\text{in}}{}^{\text{out}}\rangle. \quad \text{(A21)}$$

Since asymptotic states for individual B and C particles are not even contemplated, and since even when in the continuum the BC pairs are in s states only, there is no question of this theory satisfying asymptotic completeness. This leads to the failure of crossing symmetry.

The asymptotic fields obey the commutation relations

$$i[\Phi^\dagger(x,s),\Phi(y,s)]=\Delta(x-y,\,s)\delta(s-t)p(s), \quad \text{(A22)}$$

from which we obtain for the interpolating fields

$$i[\phi^\dagger(x,s),\phi(y,t)] =\int\Delta(x-y,\,u)p(u)K_\pm(t,u)K_\pm{}^*(s,u)\,du. \quad \text{(A23)}$$

The Φ fields are called "fields with a parameter" by Licht.[11] The ϕ fields that are linearly related to the Φ fields are in their equivalence class. These fields with a parameter are interesting because Φ_+ differs from Φ_-, i.e., unlike ordinary generalized free fields, a nontrivial S matrix can be defined, at least in some formal sense.

B. Green's Functions

From Eq. (A23) we can now calculate the various vacuum expectation values

$$\langle 0|\phi^\dagger(x,s)\phi(y,t)|0\rangle \tag{A24}$$

for various choices of s and t. For $s=t=s_0$, we find

$$i\langle 0|\phi^\dagger(x,s_0)\phi(y,s_0)|0\rangle=\Delta_u(x-y)\,, \tag{A25}$$

the unrenormalized propagator for the A field. For $s=t>s_t$, we find

$$\lim_{\substack{y_0\to-\infty,\\ x_0\to+\infty}} \langle 0|\phi^\dagger(x,s)\phi(y,s)|0\rangle=\langle BC\,\text{out}|BC\,\text{in}\rangle=S, \tag{A26}$$

the BC scattering matrix. For $s>s_t$, we find

$$\lim_{x_0\to\pm\infty} \langle 0|\phi^\dagger(x,s)\phi(y,s_0)|0\rangle$$
$$=\langle BC\,\text{out}|\phi(s_0)|0\rangle=\Delta_u(s)\Gamma_u(s)g_0f(s)\,, \tag{A27}$$

where $\Gamma_u(s)$ is the unrenormalized vertex function. These identifications are to be expected if we recall that, for $s>s_t$, $\phi(x,s)$ was to replace the product $B(x)C(x)$.

For $\lambda_0=0$, Eq. (A25) has been established already by Thirring.[13] Directly from Eq. (A23) we obtain

$$i\langle 0|\phi^\dagger(x,s_0)\phi(y,s_0)|0\rangle$$
$$=|a(s_0)|^2\Delta(x-y,\mu^2)+\int_{s_t}^{\infty}|G_\pm(s_0,u)|^2 \times\Delta(x-y,u)\,du\,, \tag{A28}$$

where the $\Delta(x,u)$ on the right-hand side is the causal propagator for mass squared u, if the T product is understood on the left-hand side. From the sum rule (A13), the residue of the pole at μ^2 is

$$Z_3=|a(s_0)|^2=1-|a(s_0)|^2\frac{g_0^2}{[1+\lambda_0B(\mu^2)]^2}\int\frac{ds\,f^2(s)}{(s-\mu^2)^2},$$

or, in terms of the quantities defined in Eqs. (3.13), (3.14),

$$Z_3^{-1}=1+(g_0^2/Z_1^2)I(\mu^2)\,. \tag{A29}$$

For the unrenormalized weight function in Eq. (A28), we have

$$\sigma_u(s)=|G_\pm(s_0,s)|^2=g_0^2f^2(s)/(s-s_0)^2|D_\pm(s)|^2\,. \tag{A30}$$

Now the normalized form factor $F(s)$ is defined to be equal to $1/D(s)$, the denominator function in $T(s)=T_0(s)/D_-(s)=N(s)/D(s)$. In this second form for $T(s)$, $N(s)\equiv\lambda+g^2/(s-\mu^2)$, and because

$$g^2=g_0^2Z_3/Z_1^2\,, \tag{A31}$$

$$\lambda/\lambda_0=Z_3/Z_1\,, \tag{A32}$$

$$\mu^2-\mu_0^2=(g_0^2/\lambda_0)(Z_1^{-1}-1)\,, \tag{A33}$$

we have

$$\frac{T_0(s)}{N(s)}=\frac{Z_1}{Z_3}\frac{s-\mu^2}{s-s_0}, \tag{A34}$$

so that

$$\frac{1}{D(s)}=\frac{1}{D_-(s)}\frac{Z_1}{Z_3}\frac{s-\mu^2}{s-s_0}. \tag{A35}$$

Hence, in Eq. (A30),

$$\sigma_u=g^2Z_3\frac{\rho(s)}{\pi}\left|\frac{F(s)}{s-\mu^2}\right|^2, \tag{A36}$$

which is the correct form for the Lehmann-Källén weight in case of two-particle unitarity.

In terms of renormalized quantities, we can write

$$\phi(x,s_0)=Z_3^{1/2}\left\{\Phi_\pm(x,s_0)+g\int\frac{f(s')\Phi_\pm(x,s')\,ds'}{(s'-\mu^2)D(s')}\right\},$$

$$\phi(x,s)=f(s)\left\{\frac{g\Phi_\pm(x,s_0)}{\mu^2-s}+\int\frac{T(s')f(s')\Phi_\pm(x,s')\,ds'}{s'-s\mp i\epsilon}\right\},\quad s>s_t,$$

and

$$\Phi_\pm(x,s)=Z_3^{1/2}\phi(x,s_0)+g\int\frac{f(s')\phi(x,s')\,ds'}{\mu^2-s'},$$

$$\Phi(x,s)=\int G_\pm^\dagger(s,s')\phi(x,s')\,ds',$$

$$G_-(s,s')=G_+^*(s,s')=\delta(s-s')+T(s')f(s)f(s')/(s'-s+i\epsilon),\quad s>s_t.$$

The vacuum expectation value in Eq. (A26) is, before we take the double limit, the U matrix between finite times y_0 and x_0. After taking the limit, we have

$$S=\langle 0|\Phi_+^\dagger\Phi_-|0\rangle=\int G_+(s',s)p(s')G_-^\dagger(s,s')\,ds'\,, \tag{A37}$$

the last equality following from Eq. (A19). Now

$$G_+(s',s)=[D_+(s)/D_-(s)]G_-(s',s)\,, \tag{A38}$$

so that, using Eq. (A14),

$$S(s)=1-2i\rho(s)T(s)=D_+(s)/D_-(s)\,. \tag{A39}$$

Thus

$$T(s)=\frac{D_-(s)-D_+(s)}{2i\rho(s)D_-(s)}=\frac{T_0(s)}{1+T_0(s)B(s)}, \tag{A40}$$

in agreement with the result (3.16) obtained by summing BC bubble diagrams.

We finally have from Eq. (A19), using the orthonormality of the $\phi(x,s)$ and Eq. (A8),

$$\lim_{x_0\to+\infty} \langle 0|\phi^\dagger(x,s)\phi(y,s_0)|0\rangle = G_-(s_0,s) = g_0 f(s)/(s-s_0)D_-(s). \quad (A41)$$

From Eqs. (A35) and (A31), this equals

$$g_0Z_3/Z_1 f(s)/(s-\mu^2)D(s) = gZ_3^{1/2}f(s)F(s)/(s-\mu^2), \quad (A42)$$

which establishes Eq. (A27).

The renormalized quantities are

$$\Delta_r = Z_3^{-1}\Delta_u, \quad \Gamma_r = Z_1\Gamma_u, \quad (A43)$$

where

$$\Delta_r\Gamma_r = F(s)/(s-\mu^2). \quad (A44)$$

From the Lagrangian (A1) and the interpretation (A21), we have thus obtained the A propagator, BC scattering amplitude, form factor and vertex function of the Zachariasen model, in agreement with the results of the dispersion-relation[6,7] and diagram-summing formulations. These last two formulations can, however, be envisaged as parts of or approximations to a complete field theory. The Lagrangian formulation, on the other hand, cannot directly be extended to include higher sectors or other partial waves.

C. Broken Symmetry

The Lagrangian (A1) admits the phase transformations

$$\phi(x,s) \to e^{i\alpha}\phi(x,s), \quad \phi^\dagger(x,s) \to e^{-i\alpha}\phi^\dagger(x,s) \quad (A45)$$

associated with the conservation of the current

$$j_\mu(x) = i\int \phi^\dagger(x,s)\overleftrightarrow{\partial}_\mu p(s)\phi(x,s)\,ds. \quad (A46)$$

We now impose the symmetry-breaking condition

$$\langle\phi(s)\rangle = \langle 0|\phi(x,s)|0\rangle \equiv \eta(s) \neq 0, \quad \langle\phi^\dagger(s)\rangle = \langle 0|\phi^\dagger(x,s)|0\rangle \equiv \eta^*(s) \neq 0, \quad (A47)$$

and obtain in the equation of motion (A3)

$$sp(s)\langle\phi(s)\rangle + \int \mathfrak{M}^2(s,s')\langle\phi(s')\rangle\,ds' = 0.$$

The condition for a solution with $\langle\phi(s)\rangle \neq 0$ is, using (A2),

$$1 + (\lambda_0 - g_0^2/\mu_0^2)\int \frac{f^2(s)\,ds}{s} = 0. \quad (A48)$$

Comparing with Eq. (A15), we see that symmetry breaking demands a solution with $\mu^2=0$.

Equations (A48) and (A15) are precisely Eqs. (3.31) and (3.17), so that the discussion of the Goldstone theorem and of the decoupling of Goldstone mesons when $\mu_0^2 \to 0$ proceeds as in the main text. From the current (A46) and the commutation relations (A23), the G-S-W condition is directly confirmed in the form (2.18). In fact, $\langle 0|[j_\mu(x),\phi(y,s)]|0\rangle$ is linear in η and independent of g_0 or g, so that Eq. (2.12) is satisfied whether or not the massless particles it calls forth are coupled. In the Lagrangian formulation, the condition (A48) for a symmetry-breaking ($\eta\neq 0$) solution does not explicitly contain η. Instead, Eq. (A48) imposes only a relation between the bare parameters λ_0, g_0, μ_0 and the masses m_B and m_C already put into the Lagrangian (A1). Only in a more complete theory contemplating individual B and C particles can these masses be themselves referred to the order parameter η, as in Eq. (3.29). This is another reminder that the diagram-summing formulation can be extended to a more complete theory than the Lagrangian formulation of the Zachariasen model can be.