is whether similar ideas would still be fruitful in an investigation of fully relativistic amplitudes fulfilling the requirements of Lorentz invariance and crossing symmetry.

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Neutron-Proton Mass Difference According to the Bound-State Model

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The proton must be heavier than the neutron according to any correct calculation treating them as pion-nucleon bound states whose masses are shifted because of one-photon-exchange corrections to the binding forces, and which takes into account no particles other than pions and nucleons. The reason is simply that only the neutron can contain two charged constituents, p and π^-, and that both the electric and the magnetic forces between these are attractive, thus binding the neutron more tightly. Dashen's previous calculation of this effect was based on an unreliable variant of the Dashen-Frautschi method for eliminating infrared divergences; the sources of the mistakes in that calculation are pointed out. On the basis of the pion-nucleon bound-state picture, we give a simple and physically well-based estimate of the Coulomb contribution to the mass splitting in terms of the pion and nucleon form factors; as compared with experiment it has the right order of magnitude but necessarily the wrong sign. The magnetic energy is more difficult to estimate, apart from its sign; but it is probably much smaller. We conclude that a consistent calculation, if it is to be successful, must include other baryons and mesons. As a by-product we obtain a simple dynamical interpretation of the fact that the neutron's charge form factor is very small.

1. INTRODUCTION

IN a remarkable paper, Dashen and Frautschi[1] (DF in the following) have applied the N/D method to calculate bound-state energy shifts due to small changes in the binding forces. They consider the problem of long-range perturbations, and in particular those due to photon exchange; the latter are important because it is universally presumed that in one form or another they dominate deviations from charge independence. The basic problem is that in an approximate calculation of the energy shift there appear infrared-divergent contributions which are known to be absent from the exact answer. Such contributions will be called IR parts in the following. DF develop several ways to eliminate this difficulty, claiming that they are all at least roughly equivalent to each other. In a second paper, Dashen[2] uses one of these methods (in first approximation) to calculate the proton-neutron mass difference. In the spirit of the N/D method he assumes that the nucleon is a bound state of the pion-nucleon system, i.e., a pole, due to a zero of the D function, of the $I=\frac{1}{2}$, $P_{1/2}$ partial wave.

Schematically, the $I_3=\pm\frac{1}{2}$ states can be written as

$$\begin{aligned}|+\tfrac{1}{2}\rangle&=-(\tfrac{1}{3})^{1/2}|p\pi^0\rangle+(\tfrac{2}{3})^{1/2}|n\pi^+\rangle,\\ |-\tfrac{1}{2}\rangle&=-(\tfrac{2}{3})^{1/2}|p\pi^-\rangle+(\tfrac{1}{3})^{1/2}|n\pi^0\rangle.\end{aligned}\tag{1.1}$$

The proton (neutron) are poles in the $|\pm\frac{1}{2}\rangle$ scattering amplitudes, respectively; they would be degenerate in the absence of electromagnetic effects. Basically, Dashen uses as his dominant perturbation the forces due to photon exchange. For the purpose in hand the anomalous Pauli moments of the nucleons can be ignored[2,3]; then a photon can be exchanged only between the particles in the $|p\pi^-\rangle$ component of the $|-\frac{1}{2}\rangle$ state, so that only the neutron mass is shifted. The mass splittings of the particles on the right of (1.1) must also be taken into account; being an isotensor, the $\pi^\pm-\pi^0$ mass difference has no effect on the isovector quantity

$$\delta M\equiv M_p-M_n,\tag{1.2}$$

but the neutron-proton mass difference itself evidently provides a "damping term," in the sense that by taking it into account on the right of (1.1) we decrease by a factor $\frac{3}{4}$ the result that would be obtained otherwise. For simplicity we shall ignore the damping term to begin with, though we shall allow for it in our final estimate in Sec. 4. Dashen's theoretical result for δM has the experimentally correct magnitude and negative sign.

In the present paper we argue that his answer is a mistake resulting from a method for eliminating IR parts that may be plausible at first sight but is inadequate in these circumstances. If his basic assumptions and input, as outlined above, are handled cor-

[1] R. F. Dashen and S. C. Frautschi, Phys. Rev. 135, B1190 (1964).
[2] R. F. Dashen, Phys. Rev. 135, B1196 (1964).
[3] Only the isoscalar magnetic moments contribute to δM, and the anomalous isoscalar moment is negligibly small.

rectly, then the calculation yields a positive value for δM. In other words, to obtain the observed negative result correctly one must introduce as input physical effects not considered by Dashen.

In the remainder of this introduction we shall dispose of the erroneous *a posteriori* arguments given by Dashen[2] to make his result qualitatively plausible on physical grounds. In Sec. 2 we shall consider the various methods of eliminating IR parts from approximate calculations such as are likely to be performed in practice. In particular we shall use potential scattering as a model in order to find out how the adequacy and reliability of such methods is limited by the approximations adopted for the unperturbed N and D functions (it is only in principle that they are known exactly). It will emerge that, at a given level of approximation, the adequacy of the formulas for P states is quite different from that of the corresponding ones for S states. We shall also give an example where a combination of approximations like Dashen's leads to an answer with the wrong sign. In Sec. 3, we consider the pion-nucleon system with a static nucleon, and estimate (the Coulomb contribution to) δM using a somewhat better authenticated approach to the IR parts, though still working with a very crude parametrization of N and D. The result of this calculation is positive. For a (hypothetical) S bound state it has a form not unlike what one would expect all along on a simple but basically correct physical picture. For the physically relevant case of a P bound state, the result has a wrong dependence on the range of the perturbation, and is adequate only for very long range; but the sign of the shift is of course unambiguously positive. All these features are common to the nonrelativistic and to the relativistic calculation. The rough method employed could be improved in detail only at the expense of great elaboration; the labor involved would not be justified, because both the general argument and the leading approximation show that the input does not include whatever physical effects are responsible for the observed sign of δM. We shall conclude with some general comments in Sec. 4.

We return now to the preliminary qualitative arguments. The dominant part of Dashen's expression for δM, deduced from photon exchange, is

$$\delta M = -(5/9)\alpha\frac{1}{f^2}\frac{\mu^2}{M}\{\ln(em/2\mu)-\tfrac{1}{2}\}\,. \qquad (1.3)$$

Here, $\alpha=1/137$ is the fine-structure constant, f the pion-nucleon coupling constant, ($f^2=0.08$), μ the pion mass, M the nucleon mass, e the base of natural logarithms; the mass m enters through the pion and the isoscalar nucleon Dirac form factor, both of which for simplicity we have taken as

$$F(t)=m^2/(m^2-t)\,, \qquad (1.4)$$

with m roughly equal to the ρ-meson mass, say $m\approx 5.5\mu$.

At least two features of (1.3) call for comment: the over-all sign and the appearance of M in the denominator.

By the basic assumption, the nucleons are bound states; the binding energy must increase if the binding forces become more attractive. As explained above, photon-exchange affects only (one component of) the neutron. The resultant Coulomb force between p and π^- is attractive; on its own it would lead to a tighter binding of the neutron, i.e., to a neutron less massive than the proton. Hence (1.3) could be correct only if photon exchange were to result in another force as well, which would need to be repulsive and to overcompensate the attractive Coulomb force. Dashen argues that the Coulomb forces are suppressed by some supposed relativistic effects, and that the magnetic interaction dominates; and that this also explains the factor $1/M$, which enters, supposedly, through the (Dirac) moment $\alpha^{1/2}/2M$ of the proton. This magnetic force, analogous to the atomic hyperfine-structure interaction,[4] results from the coupling of the proton's magnetic moment to the magnetic field of the orbiting π^-.

But in fact the magnetic interaction is also attractive, so that photon exchange leads to nothing but attractive forces and, on its own, if treated correctly, must lead to a neutron lighter than the proton. To see that the magnetic force is attractive, note first that in the $P_{1/2}$ state the orbital angular momentum $\mathbf{L}$ is antiparallel to the proton spin and therefore to the proton-magnetic moment $\mathbf{m}$. Now for a positively charged orbiting particle, the magnetic field $\mathbf{H}$ near the center of the orbit is parallel to $\mathbf{L}$; for a negatively charged orbiting particle, as here, $\mathbf{H}$ is antiparallel to $\mathbf{L}$ and therefore parallel to $\mathbf{m}$. Finally, the magnetic interaction Hamiltonian is $(-\mathbf{H}\cdot\mathbf{m})$. Therefore, in the $P_{1/2}(\pi^-,p)$ state it is negative, i.e., attractive. This result, in a similar context, was stated some time ago by Holladay.[5]

Hence, the result (1.3), which is supposed to represent the effects of photon exchange, has the wrong sign irrespective of whether or not the magnetic effects outweigh the Coulomb ones. In fact, not only is there no reason to think that the latter are suppressed, but the relativistic calculation in Sec. 3 shows explicitly that they are not.

As regards the magnetic effects, they do of course involve the factor $1/M$, but need not be small purely on this account. The estimate in Appendix B suggests that their order of magnitude relative to the Coulomb effects is measured by $m^2/3M\bar{V}_0$, where $\bar{V}_0$ is an average value of the strong short-range binding potential. We shall not consider them further in the body of this paper, but shall try to allow for them in Sec. 4 in reaching our final estimate of the effect of one-photon exchange on δM.

[4] See, for instance, H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer-Verlag, Berlin, 1957), paragraph 22. See also Appendix B.

[5] W. G. Holladay, Phys. Rev. **101**, 1202 (1956).

We proceed to indicate the mistake in Dashen's argument which was intended to show that the Coulomb effects should vanish for a pion obeying the Klein-Gordon equation, a static nucleon, a very short-range strong (binding) interaction between them, and in the limit where the total-pion energy E vanishes as it does here.

On this picture, the Coulomb contribution $-\delta M_c$ to δM is given by

$$\delta M_c = +(\text{neutron mass shift}) = -\tfrac{2}{3}\alpha\int d^3r\,\rho(\mathbf{r})\delta V(\mathbf{r})\,, \qquad (1.5)$$

where δV is the electrostatic potential due to the proton (in units of $\alpha^{1/2}$), and ρ the charge density of the π^- (in units of $-\alpha^{1/2}$); $\frac{2}{3}$ is the isotopic factor stemming from (1.1). Notice that with these conventions ρ and δV are positive. For comparison with later work, we take

$$\alpha\delta V = \alpha\{[(e^{-\lambda r}-e^{-mr})/r]-[(m^2-\lambda^2)/2m]e^{-mr}\}\,. \qquad (1.6)$$

As was pointed out by DF, $\alpha\delta V$ corresponds to the photon-exchange diagram in the usual sense that in the static limit for the nucleon the diagram by itself leads to the same scattering amplitude as does $\alpha\delta V$ in first Born approximation. (See also Appendix B.) λ is a photon mass which in this section we take at once to be zero. For simplicity, we discuss first the Coulomb shift not of a P but of an S state.

The expression for ρ in terms of the wave function ϕ of the pion depends on whether the strong potential is a Lorentz scalar or (the time component of) a four vector. In the scalar case,

$$\rho = (i/2\mu)\{\phi^*(\partial\phi/\partial t)-(\partial\phi^*/\partial t)\phi\}\,. \qquad (1.7)$$

As E tends to zero, we have

$$\phi = Ne^{-iEt}e^{-\mu r}/r\,, \qquad (1.8)$$

where N is a normalization constant fixed by

$$\int \rho(\mathbf{r})d^3r = 1\,. \qquad (1.9)$$

The range a of the binding potential is assumed small enough for the region $r<a$ to contribute negligibly to the integrals in (1.5) and (1.9). Then,

$$\delta M_c = \lim_{E\to 0} -\tfrac{2}{3}\alpha\int d^3r\,(e^{-2\mu r}/r^2)\delta V(E/\mu)\bigg/\int d^3r \times(e^{-2\mu r}/r^2)(E/\mu)$$
$$= -\tfrac{2}{3}\alpha(2\mu)\{\ln[(m+2\mu)/2\mu]-[m/(2m+4\mu)]\}\,. \qquad (1.10)$$

If $m/\mu \ll 1$, i.e., if δV varies slowly within one pion Compton wavelength of the origin, then (1.10) reduces to

$$\delta M_c \approx -\tfrac{2}{3}\alpha m/2 = -\tfrac{2}{3}\alpha\delta V(0)\,, \qquad (1.11)$$

as expected in view of the fact that such a $\delta V(r)$ may be replaced by $\delta V(0)$ and taken outside the integral in (1.5). By contrast, if $m/\mu \gg 1$ (but still subject to the range a being small enough for $am\ll 1$), (1.10) yields

$$\delta M_c \approx -\tfrac{2}{3}\alpha\mu\ln(m^2/4\mu^2 e)\,. \qquad (1.12)$$

On the other hand, if the strong potential V_0 is the time component of a vector, (1.7) must be replaced by

$$\rho = (i/2\mu)\{\phi^*[(\partial/\partial t)-iV_0]\phi - [[(\partial/\partial t)+iV_0]\phi^*]\phi\}\,. \qquad (1.13)$$

The wave function again has the form (1.8) in the asymptotic region $r>a$; but now, as $E\to 0$, the dominant contribution to the integrals in (1.5) and (1.9) comes from the interior region $r<a$. In other words the charge (though not the wave function) is contracted into the strong interaction range; in general δM_c cannot be calculated exactly without some model for this region. But as long as a is small enough to satisfy $am\ll 1$, we can still take δV outside the integral and recover the estimate (1.11).

Thus, neither in the scalar nor in the vector case does δM_c vanish, or show any peculiar behavior, in the limit $E\to 0$. Dashen's argument that it does seems to stem from his failure to impose the normalization condition (1.9). Again, the correct qualitative result is implicit in the work of Holladay.[5]

The same general arguments can be applied to P states bound by a strong potential of such short-range a that $am\ll 1$; the depth must be correspondingly large. It is an amusing exercise in elementary mechanics to show that the wave function is effectively trapped into the inner attractive region by the angular-momentum barrier, which pulls the bound-state wave function in just as it would push a (low-energy) scattering wave function out. Appendix C gives a sketch of the argument. In fact, the same situation obtains for P states both in the relativistic and the nonrelativistic case; we shall refer to it several times, and it should not be confused with the quite different pulling in of the charge density, in any angular momentum state, by a short-range four-vector potential. The central consequence is that for P states bound by either kind of strong potential of sufficiently small range, the wave function itself is essentially confined to $r<a$.

2. SEARCH FOR A WORKABLE FIRST APPROXIMATION

Consider S-wave potential scattering first. Let the unperturbed partial-wave amplitude be written in the usual N/D manner[2,6] as

$$\frac{e^{i\eta}\sin\eta}{k}=A(E)=N(E)/D(E),$$
$$S=e^{2i\eta}=(2ikN/D+1),\qquad(2.1)$$

where k is the wave number and $E=k^2/2\mu$. A is supposed to have a pole at $E=E_0$,

$$E_0=-q^2/2\mu<0,\qquad(2.2)$$

due to a zero of D there; the residue R is given by

$$R=N(E)/D'(E),\qquad(2.3)$$

where $D'=dD/dE$. Note that R is negative; (see Appendix A). Let δA be the change in A resulting from a small change in the potential. In particular, we shall consider the change to a potential more attractive by $-\alpha\delta V$, where δV is given by (1.6); it can be written as a superposition of exponentials:

$$-\alpha\delta V=-\alpha\left\{\int_\lambda^m d\nu\, e^{-\nu r}-[(m^2-\lambda^2)/2m]e^{-mr}\right\}.\qquad(2.4)$$

The resulting change in E_0 is δE_0, and must be negative; note that the δE_0 of this section is analogous to the $2\delta M_c/3$ of Sec. 1. Since we work to first order in α, it is possible to begin with exponential perturbations

$$\delta V_\nu(r)=-\alpha e^{-\nu r},\qquad(2.5)$$

and to construct a linear superposition at the end. IR parts, which behave like $\ln\lambda$ after taking such a superposition according to (2.4), can be identified beforehand through their proportionality to $1/\nu$.

DF show that the function

$$J(E)=D^2(E)\delta A(E)\qquad(2.6)$$

has no right-hand cut, and that it is finite at E_0; in other words δA has a double pole there. The shift δE_0 is given by

$$\delta E_0=J(E_0)/R[D'(E_0)]^2,\qquad(2.7)$$

where for $J(E)$ one writes the Cauchy integral

$$J(E_0)=(1/2\pi i)\int_{C_L}[dE/(E-E_0)]D^2(E)\delta A(E).\qquad(2.8)$$

An alternative proof of (2.7) is outlined in Appendix A. C_L is a counterclockwise (positive-direction) contour avoiding the left-hand cuts of δA and closed along the circle at infinity; it includes the pole at E_0. In realistic cases the circle at infinity makes no contribution. If E_0 lies on the left-hand cut, then the latter must be deformed away from E_0 to allow it to fall within C_L.

The expressions (2.7), (2.8) are exact, and in consequence free of IR parts if evaluated exactly. However, in practice one would invariably like to reformulate the equations so that in first approximation the input δA, which is difficult to ascertain, can be replaced by δA_B, the first Born approximation to scattering by the perturbation acting alone. The spirit of the method is to devise approximations to the integrand of (2.7) which will allow one to neglect the contributions from all distant parts of C_L and then to evaluate the remaining integral by the best means available. Provided that the approximations fulfill these conditions there is no reason why the resulting approximate integrand should retain the analytic properties of the exact integrand.

For the simple exponential perturbation (2.4) one has

$$\delta A_B{}^\nu=(4\mu\alpha/\nu)/(4k^2+\nu^2).\qquad(2.9)$$

Of course it would be nonsense simply to substitute $\delta A_B{}^\nu$ for δA^ν in (2.8); that would lead to[7]

$$\delta E_0=\frac{1}{R[D'(E_0)]^2}\frac{(-8\mu^2\alpha/\nu)}{(q^2-\nu^2/4)}D^2\!\left(-\frac{\nu^2}{8\mu}\right).$$

But R is negative; therefore this expression is not only IR divergent as $\nu\to 0$, but is positive (seeming to loosen the binding) as long as $\nu^2/4<q^2$, in spite of the fact that a purely attractive perturbation like (2.5) must tighten the binding, whatever the value of $\nu/2q$.

The whole question of successively higher order approximations has been illustrated very lucidly by Paton,[8] who uses exponentials of various ranges both for the strong potential and for δV. Here we are concerned only with the more modest problem of how to get at least qualitatively reliable results from δA_B as input, coupled with some rough parametrizations for N and D.

DF make the crucial observation that δA in the exact expression (2.8) may be replaced by

$$\delta\hat{A}=\delta A-\delta A_B S;$$

the additional integrand has no singularities within C_L and contributes nothing to $J(E_0)$. To see this note that neither D^2S nor δA_B have right-hand cuts, and that the double pole of $(E-E_0)^{-1}S(E)$ at E_0 is cancelled by the double zero of D^2, while δA_B of course "knows" nothing of the unperturbed problem and has no pole at E_0. DF point out that as a basis for approximations $\delta\hat{A}$ has several advantages over δA. First, if $\delta A\to\delta_B A$ as $E\to\infty$, then $\delta\hat{A}\to\delta A_B(1-S)$; since $S\to 1$, the new integrand vanishes faster at infinity and the result becomes less sensitive to distant singularities. Second, at least some of the IR parts are contributed by those

[6] See, for instance, S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1963), Chap. II.

[7] With δA replaced by δA_B, the integrand has no left-hand singularities other than the ("pseudo") pole of $\delta A_B{}^\nu$. Hence the integral is best done by closing the contour around $-\nu^2/8\mu$, making it into a clockwise (negative-direction) loop around the pseudopole.

[8] J. E. Paton, Oxford University (to be published).

portions of C_L which approach the right-hand branch point (i.e., $k^2=0$) arbitrarily closely as $\lambda\to 0$. But near $k^2=0$, the left-hand discontinuities of δA and δA_B coincide, so that again we have a factor $(1-S)$ which vanishes at threshold and helps to eliminate some of the IR parts. We shall see this illustrated presently.

Accepting the *first* lead of DF, we adopt, provisionally, the following approximate three-step method for getting IR-finite answers. First, replace δA by $\delta\hat{A}$; second, ignore the far-left-hand singularities of $\delta\hat{A}$, retaining only those which coincide with the left-hand singularities of δA_B. In particular, we never accept any contributions from the contour at infinity. [Here it is crucial to observe that δV as given by (2.4) leads to a δA_B whose left-hand singularities extend no further from the origin than $-m^2/4$.] This amounts to replacing $J(E_0)$ by

$$\mathcal{J}(E_0)=(1/2\pi i)\int_{C_L}[dE/(E-E_0)]\delta A_B(1-S)D^2,$$
$$\mathcal{J}(E_0)=(1/2\pi i)\int_{C_L}[dE/(E-E_0)]\delta A_B(-2ikND). \tag{2.10}$$

The contour can now conveniently be closed in the clockwise (negative) sense at a finite distance from the origin; we shall denote by $\tilde{C}_L$ the contour closed in this way.

Third, we confine ourselves to such parametrizations of S (i.e., of N and D) as will keep $\mathcal{J}$ free of IR parts when $\lambda\to 0$. We shall see presently that this last requirement restricts us to what we call the zero-range approximation (ZRA) in which N is allowed to have no singularities at all within $\tilde{C}_L$.

Consider accordingly the perturbation (2.5), leading to the input function (2.9), and parametrize N and D as follows:

$$D=(-q-ik),\qquad N=1,$$
$$D'(E_0)=-\mu/q,\qquad R=-q/\mu. \tag{2.11}$$

This will be referred to as the "unitary ZRA." The only singularity within $\tilde{C}_L$ is the pole of $\delta A_B{}^\nu$ at $k^2=-\nu^2/4$. Hence the factor $(-2ik)=\nu$ of the integrand of (2.10) cancels the factor $1/\nu$ in the numerator of $\delta A_B{}^\nu$, leading to an IR-finite answer

$$\delta E_0{}^\nu=-\alpha[q/(q+\nu/2)], \tag{2.12}$$

whose sign moreover is correct independently of $\nu/2q$.

We have given the argument in detail because it shows that the ZRA is not only convenient, but is essential to the adequacy of the prescription. For if the integrand of $\mathcal{J}$ had additional singularities, due to N, within $\tilde{C}_L$, then these would make extra contributions to $\mathcal{J}$, in which the IR-divergent factor $1/\nu$ in the numerator of $\delta A_B{}^\nu$ would not be cancelled by the phase-space factor $(-2ik)$, and the resultant $\mathcal{J}$ would itself be IR divergent. If such additional singularities of N were to be admitted (in a departure from the ZRA), then it would be necessary to use a better approximation to δA^ν than is provided by $\delta A_B{}^\nu$ in order to eliminate all IR parts. This is illustrated in detail by the work of Paton.[8]

One can now construct the superposition analogous to (2.4) for $\delta E_0{}^\nu$ to find the shift $\delta E_0{}^S$ induced by the perturbation $-\alpha\delta V$; we let $\lambda\to 0$ and get

$$\delta E_0{}^S=-2\alpha q\left\{\ln\left[\frac{2q+m}{2q}\right]-\frac{m}{2m+q}\right\}. \tag{2.13}$$

The superfix S identifies the S-state shift. As was pointed out by DF, this exactly equals the expression $-\alpha\int d^3r|\psi|^2\delta V$ if the asymptotic form of ψ, valid outside the range a of the binding forces, is used for all r. In particular, it is satisfactory that $\delta E_0{}^S$ reduces to $-\alpha m/2=-\alpha\delta V(0)$ for small m, as it must by the general arguments of Sec. 1, and that

$$|\delta E_0{}^S|\leqslant|\alpha\delta V(0)| \tag{2.14}$$

for any m. Thus (2.13) is an adequate result as long as $am\ll 1$.

Naturally one can obtain (2.13) by substituting directly into (2.10) the δA_B which corresponds to $-\alpha\delta V$. The technical details of the integrations are the same for S and P states and for both nonrelativistic and relativistic cases; for them we refer to Sec. 3. In the remainder of this section we shall merely quote the nonrelativistic (potential theory) results for the full perturbation $-\alpha\delta V$.

As the next step we try out an even cruder parametrization for A, replacing (2.11) by

$$D=(E-E_0),\quad N=R; \tag{2.15}$$

we shall refer to this as the "nonunitary ZRA". It is analogous to that ultimately adoped by Dashen.[2] It leads straightforwardly to the estimate (with $\lambda\to 0$):

$$\delta E_0{}^S=-\alpha m/4. \tag{2.16}$$

This has the right sign and order of magnitude, but has lost all dependence on $m/2q$, and in the limit $m\to 0$ is too small by a factor of $\frac{1}{2}$, since in that limit the correct answer is $-\alpha\delta V(0)=-\alpha m/2$. By contrast, it provides an overestimate for large $m/2q$, since (2.13) shows that $|\delta E_0{}^S|$ in fact increases only logarithmically with m.

Now we come to a central point. Near the end of their paper, DF claim that the above method, which though crude is workable at least as to sign and order of magnitude, is roughly equivalent to another one, which is then actually used by Dashen[2] in his calculation of δM. This second method bypasses the use of $\delta\hat{A}$ altogether. One begins by writing down[9] δA_B with a finite photon

[9] In principle, DF distinguish between δA_B and a function δA_{infra}, the latter to contain all IR parts. In a calculation at the level of approximation of this or Dashen's (Ref. 2) paper the distinction is immaterial.

mass λ, and letting $\lambda \to 0$ puts δA_B in the form

$$\delta A_B(E) = f(E)\ \ln[\lambda^2/g(E)] + [\text{IR-finite parts}], \quad (2.17)$$

where f and g are some functions of E. Next, one replaces $J(E_0)$ by

$$K(E_0) = (1/2\pi i)\int_{C_L} [dE/(E-E_0)]D^2(E)\delta A_B(E); \quad (2.18)$$

evaluating the integral one finds an expression containing $\ln\lambda^2$. In the final step all such $\ln\lambda^2$ are replaced by $\ln(|g(E_0)|)$, resulting in

$$K(E_0) \to K'(E_0). \quad (2.19)$$

Finally, δE_0 is obtained by substituting $K'(E_0)$ for $J(E_0)$ in (2.7).

Paton[8] has already given reasons to doubt that this last procedure is indeed equivalent to the first, or that it is correct in its own right. We shall show now that it can give the wrong sign for δE_0 when used together with a parametrization of the type (2.15).

To find $g(E)$ we need only the IR-divergent part of δA_B corresponding to $-\alpha\delta V$ as given by (1.6) or (2.4); this is determined by the component $-\alpha e^{-\lambda r}/r$ alone, and leads to

$$\delta A_B = \{(\alpha\mu/2k^2)\ \ln[(\lambda^2+4k^2)/\lambda^2] + [\text{IR}-\text{finite parts}]\}. \quad (2.20)$$

Letting $\lambda \to 0$, we identify

$$g(E) = 4k^2, \quad |g(E_0)| = 4q^2. \quad (2.21)$$

Now with the nonunitary ZRA the full perturbation $-\alpha\delta V$ leads to

$$K(E_0) = \frac{1}{2\pi i}\int_{C_L} dE(E-E_0)\delta A_B(E) = -(\alpha q^2/8\mu)\{\ln[m^2/\lambda^2]-1\} = -(\alpha q^2/8\mu)\ \ln[m^2/e\lambda^2]. \quad (2.22)$$

The input $\delta A_B(E)$ is obtainable from (3.11) below. Replacing $\ln\lambda^2$ by $\ln 4q^2$, according to the prescription, we get

$$K'(E_0) = -(\alpha q^2/8\mu)\ \ln[m^2/4eq^2], \quad (2.23)$$

and[10]

$$\delta E_0^S = -(\alpha q^2/8\mu R)\ \ln[m^2/4eq^2]. \quad (2.24)$$

In general, (2.24) stands condemned by the circumstance that its sign depends on $m/2q$, whereas in truth δE_0 must be negative for all values of $m/2q$. Hence it is almost superfluous to point out that with the numbers actually of interest in the pion-nucleon system, one would have $m \approx (\rho \text{ mass})$, $q \approx (\text{pion mass})$, so that $m^2 > 4eq^2$; then the logarithm in (2.24) is positive, and since R is negative, δE_0 itself not only can but does have the wrong sign.

We conclude that, contrary to the claim made by DF, *their second method is not equivalent to their first; and that, unlike the first, it can easily give the wrong sign. Dashen's result for δM was obtained*[2] *by using the second method (in a relativistic version), which explains how it could yield the physically impossible answer that it does.*

Finally, we record the results of the analogous P-state calculations. Here we write

$$e^{i\eta}\sin\eta/k^3 = A = N/D, \quad S = e^{2i\eta} = (2ik^3N/D+1), \quad (2.25)$$

and in the "unitary ZRA" parametrize by

$$D = (-ik^3+q^3), \quad N=1, \quad D'(E_0) = 3q\mu. \quad (2.26)$$

(Because of the kinematic factor k^{-2}, the residue of A is now positive.) The input function δA_B is obtainable from (3.2) below by substituting μ for ω. With it, one finds the P analog of (2.13):

$$\delta E_0{}^P = \frac{1}{3q\mu}\frac{1}{2\pi i}\int_{C_L}\frac{dE}{E-E_0}\delta A_B(-2ik^3)D, \quad (2.27)$$

$$\delta E_0{}^P = -\alpha m\{\tfrac{1}{3}x + \tfrac{1}{6}[1/(1+x)] + (1/3x)\ \ln(1+x)\}, \quad (2.28)$$

where $x = (m/2q)$. This has the correct limit $-\alpha m/2 = -\alpha\delta V(0)$ for small m, but increases too fast (quadratically) with increasing m, whereas the true shift is limited by $|\alpha\delta V(0)|$. Thus the calculated expression is is much less reliable for the P than for the S wave.

With the "nonunitary" ZRA (2.15) for a P state, one finds (in the limit $\lambda \to 0$),

$$\delta E_0{}^P = -3\alpha m/2. \quad (2.29)$$

This exceeds the limit $-\alpha\delta V(0)$ by a factor of 3, but does again have at least the right sign.

We have gone into some detail in assessing the results of the DF method in first approximation, because their limitations show up clearly in the potential case; thus we are alerted as to their misleading features, and are in a position to identify and ignore the same features when they turn up in a relativistic or dispersion-theory calculation.

3. CALCULATION FOR STATIC NUCLEONS AND RELATIVISTIC PIONS

In this section we use the (first) DF method to estimate the photon-exchange contribution, δM_c, to the mass splitting δM defined by (1.2). For later conveniences we call $\delta\omega_p$ the shift which would result in a bound state containing only (p,π^-); then (apart from the damping factor of $\frac{3}{4}$)

$$\delta M_c = +\tfrac{2}{3}\delta\omega_p, \quad (3.1)$$

[10] The formula (2.24) is closely akin to Dashen's Eq. (10), which we quoted in our Eq. (1.3).

as explained in the introduction. Following Dashen[2] we construct the photon-exchange amplitude between p and π^- (see also Appendix B); project it on the $P_{1/2}$ partial wave; and finally we take the limit[11] $M\to\infty$. This yields

$$\delta A_B{}^P=-\frac{\alpha\omega}{k^2}\left\{\frac{1}{k^2}\frac{(2k^2+m^2)}{(4k^2+m^2)}+\frac{1}{2k^2}\ln\left[\frac{(4k^2+m^2)\lambda^2}{(4k^2+\lambda^2)m^2}\right]\right\},\quad(3.2)$$

where we have dropped terms which vanish as $\lambda\to 0$. $\delta A_B{}^P$ is so normalized as to be an approximation to the function $e^{i\eta}\sin\eta/k^3$, where η is the $P_{1/2}$ phase shift. In (3.2), ω and k are the pion energy and wave number,

$$\omega^2=k^2+\mu^2\,;\quad(3.3)$$

λ is the photon mass, and the mass m has entered through our use of approximate pion and nucleon electromagnetic form factors, as explained above (1.4). Since we have taken $M\to\infty$, the nucleon pole in this amplitude occurs at $\omega=\omega_p=0$. However, it is inconvenient for kinematical reasons to work with an amplitude having a pole at $\omega=0$; hence we shall consider a finite ω_p,

$$\omega_p{}^2=\mu^2-q^2,\quad(3.4)$$

and let $\omega_p\to 0$, i.e., $q\to\mu$ at the end of the calculation; as we shall see, there is no ambiguity in taking the limit.

We chose ω as our dispersion variable, and adopt the "unitary ZRA":

$$A(\omega)=N(\omega)/D(\omega)\,,\quad D(\omega)=-ik^3+q^3,\quad N=1\,,\quad R^{-1}=D'(\omega_p)=3q\omega_p.\quad(3.5)$$

Then the shift is given by the analog of (2.27):

$$\delta\omega_p=\frac{1}{3q\omega_p}\frac{1}{2\pi i}\int_{\tilde{C}_L}\frac{d\omega}{\omega-\omega_p}(-2ik^3)(q^3-ik^3)\delta A_B{}^P.\quad(3.6)$$

As a function of k, $\delta A_B{}^P$ has the same singularity structure as in the nonrelativistic case; in the ω plane it has cuts running between $\pm(\mu^2-\lambda^2/4)^{1/2}$ along the real axis, and between $\pm i(m^2/4-\mu^2)^{1/2}$ along the imaginary axis; we have taken the realistic case $m/2>\mu$. $\tilde{C}_L$ is a clockwise loop around these cuts, as explained in Sec. 2.

The first term on the right on (3.2) evidently contributes two poles to the integrand of (3.6) whose residues are picked up trivially. In fact, this term is decomposed as follows:

$$\frac{1}{k^2}\frac{2k^2+m^2}{4k^2+m^2}=\frac{1}{k^2}-\frac{2}{4k^2+m^2},$$

[11] Our δA_B is thus $(-\frac{2}{3})^{-1}$ times Dashen's δA_γ.

and the $1/k^2$ part is dropped, because having no singularities within $\tilde{C}_L$ it contributes nothing to $\delta\omega_p$. For the same reason, the (apparently IR-divergent) component

$$(1/2k^2)\ln(\lambda^2/m^2)$$

can be dropped from (3.2). Finally, the remaining logarithm is written as

$$\int_{\lambda^2}^{m^2}d\nu^2/(4k^2+\nu^2),$$

the contour integral being done first, since it now contains noting but poles, and the ν^2 integral afterwards. In this way, (3.6) yields

$$\delta\omega_p=(-\alpha m)\{\tfrac{1}{3}x+\tfrac{1}{6}[1/(1+x)]+[1/3x]\ln(1+x)\}\,,\quad(3.7)$$

where

$$x=m/2q.\quad(3.8)$$

Not surprisingly, this is formally the same expression as the nonrelativistic (2.28), the difference between them stemming only from the different definitions of q. Just because (3.7) depends on q alone there is no difficulty in taking the limit $\omega_p\to 0$, $q\to\mu$. The convenience of the slight detour via finite ω_p seems to be analogous to the convenience of dealing with the ratio of the two integrals in (1.10).

Of course (3.7) is suspect for the same reason as was (2.28), namely that for large $m/2\mu$ it exceeds $-\alpha m/2$. Putting in $m\approx 750$ MeV, $\mu\approx 140$ MeV, we find from (3.7)

$$\delta\omega_p\approx -10.6\text{ MeV},\quad(3.9)$$

to be compared to

$$-\alpha m/2\approx -2.74\text{ MeV}.\quad(3.10)$$

For completeness, we mention that the relativistic P-state calculation using the "nonunitary ZRA" with $D=(\omega-\omega_p)$, $N=R$, yields the same as the nonrelativistic result (2.29), and that the result for the shift in a (hypothetical) S bound state also coincides exactly with (2.13), but with q defined by $\omega_s{}^2=\mu^2-q^2$. The input function in this case is

$$\delta A_B{}^S=-\alpha\omega\left\{\frac{2}{m^2+4k^2}+\frac{1}{2k^2}\ln\left[\frac{(4k^2+m^2)}{(4k^2+\lambda^2)}\frac{\lambda^2}{m^2}\right]\right\}.\quad(3.11)$$

To obtain the nonrelativistic input functions from (3.2) and (3.11) one merely replaces the leading ω by μ and re-interprets k as $(2\mu E)^{1/2}$. In fact, (3.2) and (3.11) coincide precisely with the first Born approximation to the scattering of a Klein-Gordon particle by the electrostatic potential $-\alpha\delta V$.

4. COMMENTS AND CONCLUSIONS

We have seen that Dashen's immediate program, when implemented correctly, cannot yield the observed sign of the neutron-proton mass difference; rather naturally it gives a result of the correct order of magnitude but with the wrong sign. Of course, this does not rule out the possibility of calculating electromagnetic mass splittings on the assumption that the baryons are baryon-meson bound states. We have shown only that in order to have a chance of success such a calculation would need to include other particles in addition to nucleons and pions. Essentially the same conclusion is reached by Wojtaszek, Marshak, and Riazuddin,[12] though their approach to the dynamics is quite different.

Other particles might need to be included either in the direct channel, i.e., on the same footing as the pions and nucleons on the right of (1.1), or in the crossed channel, or in both. For instance, if the strong binding forces are partly due to the exchange of isomultiplets whose masses, (or whose couplings to pions and nucleons), are themselves split, then there will be corresponding contributions to δM. The fact that we have not considered such effects in this paper does not mean that we think them unimportant, but merely that they present problems which technically are quite different from those of photon exchange. The differences emerge clearly if one remembers that the exchange of massive particles corresponds to short-range forces; hence, to allow for the mass splittings here, one needs information equivalent to detailed knowledge of the bound-state wave function at small distances. By contrast, we have found above that one can make a good estimate of photon-exchange effects without such information. The difficulties in dealing realistically with short-range perturbations are evident from the work of Paton[8] in potential theory, and from the work of Shaw and Wong[13] in dispersion theory.

Perhaps it is worth summarizing what we regard as the best present estimates, on this simple bound-state picture, of the effects on δM of one-photon-exchange forces. Since the strong binding force is presumably of short range a, the pion wave function will be contracted into the interaction region $r<a$, as discussed in Sec. 1 and Appendix C. Then the Coulomb energy δM_c in the $(p\pi^-)$ state will be close to

$$\delta M_c \approx -\alpha\delta V(0) = -\alpha m/2\,, \tag{4.1}$$

as discussed in Secs. 2 and 3. On the same basis, Appendix B estimates the magnetic energy as

$$\delta M_{\text{mag}} \approx -(\alpha m^3/6M|\bar{V}_0|) \approx \delta M_c(m^2/3M|\bar{V}_0|)\,, \tag{4.2}$$

where $|\bar{V}_0|$, defined by (B9), is an average magnitude of the strong-binding potential. Since the latter must continue to hold a bound state, $|\bar{V}_0|$ must diverge as $a\to 0$; in that limit δM_{mag} will become negligible. Even if given the benefit of every doubt, $|\bar{V}_0|$ could hardly fall below m, in which case

$$\delta M_{\text{mag}}/\delta M_c \approx m/3M \approx 0.27\,.$$

Therefore, with a realistic attitude to the present level of accuracy, the magnetic effect can be ignored. Then our best estimate for δM is reached by multiplying δM_c with the isotopic factor $(-\frac{2}{3})$ and the damping factor $(\frac{3}{4})$:

$$\begin{aligned}\delta M &= (M_p - M_n) = (-\tfrac{2}{3})(\tfrac{3}{4})(-\alpha m/2)\,,\\ (\delta M)_{\text{theory}} &= +\alpha m/4 \approx 1.4 \text{ MeV}.\end{aligned} \tag{4.3}$$

The experimental value is -1.3 MeV.

Finally we point out that the result $-\alpha m/2$ for the Coulomb energy between π^- and p has a very simple and fundamental physical interpretation, once we accept that the basic input of the calculation is the one-photon exchange diagram with the form factor (1.4) for both π^- and p. Insofar as these form factors can be interpreted in configuration space, they imply that the charge of each is distributed in a Yukawa cloud whose density $\rho(r)$ at a distance r from the center is given by

$$\rho(r) = \mp m^2 e^{-mr}/4\pi r\,, \tag{4.4}$$

where ρ is normalized to ∓ 1. Now in the neutron, by our argument about P states, the centers of the two charge distributions are practically superimposed; this is compatible with the observation that the neutron's charge form factor is either zero or very small. But the mutual electrostatic energy of two such coincident charge distributions is $(-\alpha m/2)$. This interpretation also shows that our theoretical value is an upper limit; it would be decreased either if the strong interaction range were to be appreciable compared to m^{-1}, or if the form factors were to lead to a charge density more diffuse than (4.4). The actual form factors can be used in the way indicated by Eq. (B4), namely by replacing (4.1) with

$$\delta M_c = U(0) = -(\alpha/\pi)\int_0^\infty dx\, F^2(-x)/\sqrt{x}\,, \tag{4.5}$$

or, allowing the charge form factors of the pion and the proton to be different,

$$\delta M_c = -(\alpha/\pi)\int_0^\infty dx\, F_\pi(-x)F_P(-x)/\sqrt{x}\,. \tag{4.6}$$

To indicate how a change in F affects δM_c, we note that with

$$F(t) = [\kappa^2/(\kappa^2+|t|)]^2\,,$$

(4.5) yields

$$\delta M_c = -\alpha\kappa\tfrac{5}{16}\,.$$

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[12] J. H. Wojtaszek, R. E. Marshak, and Riazuddin, Phys. Rev. 136, B1053 (1964).

[13] G. L. Shaw and D. Y. Wong, Phys. Rev. (to be published).

APPENDIX A

We outline a direct proof, applicable to potential scattering, of the formula

$$\delta E_0=\frac{1}{R[D'(E_0)]^2}\lim_{E\to E_0}[D^2(E)\delta A(E)], \tag{A1}$$

which is clearly equivalent to (2.6) and (2.7). Various standard expressions of formal scattering theory will be quoted; for their justification we refer to Goldberger and Watson.[14]

Let $|\psi_E^\pm\rangle$ be the projection on the appropriate angular momentum state of the exact state vectors for scattering by the unperturbed potential V, with outgoing and incoming scattered-wave boundary conditions, respectively; E denotes the energy. Let $|E\rangle$ be the corresponding projections of free states. We recall the standard results

$$R=-|\langle E|V|B\rangle|^2|_{E\to E_0}, \tag{A2}$$

$$\delta A=-\langle\psi_E^-|\delta V|\psi_E^+\rangle, \tag{A3}$$

$$|\psi_E^\pm\rangle=\left\{1-\frac{1}{H_0+V-E\mp i\epsilon}\right\}|E\rangle, \tag{A4}$$

where $|B\rangle$ is the vector for the bound state, and H_0 the free Hamiltonian:

$$H=H_0+V,\quad H|\psi_E^\pm\rangle=E|\psi_E^\pm\rangle,$$
$$H_0|E\rangle=E|E\rangle,\quad H|B\rangle=E_0|B\rangle.$$

In formulas like (A2) it is understood that $\langle E|V|B\rangle$ is to be evaluated as a function of E for physical E, and then continued to $E=E_0$. Since D^2 has a double zero at E_0, the only part of δA which contributes to (A1) is that with a double pole. Substituting (A4) into (A3), retaining only the double pole, and inserting the result into (A1), we find

$$\delta E_0=-[D'(E_0)]^2\left\{(E-E_0)^2\langle E|V\frac{1}{H_0+V-E-i\epsilon}\delta V\right.$$
$$\left.\left.\times\frac{1}{H_0+V-E-i\epsilon}V|E\rangle\right\}\right|_{E=E_0}\frac{1}{R[D'(E_0)]^2}.$$

Next, we insert into the matrix element two sums over the complete sets of eigenstates of the unperturbed Hamiltonian H. Evidently only the bound state $|B\rangle$ contributes, giving

$$\delta E_0=-R^{-1}|\langle E|V|B\rangle|^2\langle B|\delta V|B\rangle$$
$$=\langle B|\delta V|B\rangle, \tag{A5}$$

which is indeed the correct first-order result. In the last step we have used (A2).

[14] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), Chaps. 3–5.

APPENDIX B

Our object is to extract a magnetic potential from the one-photon-exchange diagram, and to use it to estimate the magnetic energy $\delta M_{\rm mag}$ of the bound (π^-p) system in the $P_{1/2}$ state.

The diagram contributes to the S-matrix the element[15]

$$i(2\pi)^4\delta(p+k-p'-k')(-4\pi\alpha)F^2(t)/(t-\lambda^2)$$
$$\times(k+k')^\lambda\bar{u}(p')\gamma_\lambda u(p)[M^2/p_0p_0']^{1/2}[1/4\omega\omega']^{1/2}, \tag{B1}$$

where $p(p')$ and $k(k')$ are the initial (final) proton and π^- momenta, respectively, and $\omega=k_0$. We go to the center-of-mass frame, and let $M\to\infty$; then $\bar{u}'\gamma_0 u\to 1$, and $\bar{u}\,\boldsymbol{\gamma}u\to(i/2M)\mathbf{K}\times\boldsymbol{\sigma}$, where $\mathbf{K}=(\mathbf{k}-\mathbf{k}')$, so that $t=-\mathbf{K}^2$. In this limit (B1) leads to the following scattering amplitude $\mathcal{A}$:

$$\mathcal{A}=-\frac{\alpha F^2(t)}{t-\lambda^2}\left\{2\omega-(\mathbf{k}+\mathbf{k}')\frac{i}{2M}\mathbf{K}\times\boldsymbol{\sigma}\right\}. \tag{B2}$$

The first Born approximation to the scattering of a Klein-Gordon particle in a static external four-vector potential

$$U_\mu=(U,\mathbf{U}) \tag{B3}$$

is given by

$$\mathcal{A}_B=\alpha\left\{\frac{\omega}{2\pi}\int d^3r\,e^{i\mathbf{K}\cdot\mathbf{r}}U-[(\mathbf{k}+\mathbf{k}')/4\pi]\int d^3r\,e^{i\mathbf{K}\cdot\mathbf{r}}\mathbf{U}\right\};$$

equating $\mathcal{A}$ and $\mathcal{A}_B$ we get

$$U=(-4\pi\alpha)/(2\pi)^3\int d^3K\,e^{-i\mathbf{K}\cdot\mathbf{r}}F^2(t)/(t-\lambda^2), \tag{B4}$$

which leads to $U=-\alpha\delta V$, with δV given by (1.6); we also get

$$\mathbf{U}=-(2\alpha/2M)\mathbf{s}\times\mathbf{r}\frac{1}{r}\frac{d}{dr}\delta V, \tag{B5}$$

where $\mathbf{s}=\boldsymbol{\sigma}/2$. In turn, $\mathbf{U}$ leads to a magnetic energy of the bound state:

$$\delta M_{\rm mag}=-\int d^3r\,\mathbf{U}\cdot\frac{i\alpha}{2\mu}\phi^*\overleftrightarrow{\nabla}\phi$$
$$=-(2\alpha/2M\mu)\mathbf{s}\cdot\mathbf{L}\int d^3r\,(1/r)(d\delta V/dr)\phi^*\phi, \tag{B6}$$

where ϕ is the pion wave function and $\mathbf{L}$ the orbital angular momentum. In the $P_{1/2}$ state, $\mathbf{s}\cdot\mathbf{L}=-1$. By our standard argument for the P state, we can replace

[15] See, for instance, S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson, & Company, Evanston, Illinois, 1961), pp. 478, 483, and 262.

$r^{-1}d\delta V/dr$ by its value $-m^3/6$ at the origin and take it outside the integral:

$$\delta M_{\rm mag} = -(\alpha m^3/6M\mu)\int d^3r\, \phi^*\phi. \qquad \text{(B7)}$$

Here we evidently bog down in nonsense if the strong binding potential is a Lorentz scalar; for then the normalization conditions (1.7) and (1.9) show that the integral in (B7) diverges as $E \to 0$. This merely marks the limits where a scalar potential stops being useful as an intuitive guide to the real situation. (For such a potential not only $\delta M_{\rm mag}$, but all magnetic effects would diverge, including the orbital magnetic moment.) Hence we proceed on the basis that the strong potential V_0 is the time component of a four-vector. Then in the limit $E \to 0$ the normalization integral is

$$-\frac{1}{\mu}\int d^3r\, \phi^* V_0 \phi = 1. \qquad \text{(B8)}$$

Since the P wave function is pulled into the interaction region, we can usefully define an average value $|\bar{V}_0|$ of $|V_0|$ by

$$-\frac{1}{\mu}\int d^3r \phi^* V_0 \phi = \frac{1}{\mu}|\bar{V}_0| \int d^3r \phi^* \phi; \qquad \text{(B9)}$$

for instance for a square well $|\bar{V}_0|$ would be close to the depth of the well. From (B7) to (B9) we get finally

$$\delta M_{\rm mag} \approx -(\alpha m^3/6M|\bar{V}_0|). \qquad \text{(B10)}$$

This result is discussed further in Sec. 4.

APPENDIX C

Consider the Schrödinger equation for a P state bound by a potential which is negligible beyond $r=a$; let $a \to 0$, simultaneously increasing the strength of the potential so that the bound state energy $E=-q^2/2\mu$ remains constant and finite. We prove that in this limit the radial moments $\langle r^n \rangle$ vanish for all positive n and diverge for all negative n. The $\langle r^n \rangle$ are defined by

$$\langle r^n \rangle = \int_0^\infty R^2 r^n r^2 dr, \qquad \text{(C1)}$$

$$\int_0^\infty R^2 r^2 dr = P = 1, \qquad \text{(C2)}$$

where $R(r)$ is the normalized radial wave function. Define the inner and outer contributions to P:

$$P_1 = \int_0^a R^2 r^2 dr, \quad P_2 = \int_a^\infty R^2 r^2 dr, \quad P_1 + P_2 = 1. \qquad \text{(C3)}$$

If P_2 vanishes as $a \to 0$, there is nothing left to prove. We proceed to deal with the least favorable case where P_1 becomes negligible compared to P_2. Then

$$\langle r^n \rangle = \int_a^\infty R^2 r^{n+2} dr \Big/ \int_a^\infty R^2 r^2 dr. \qquad \text{(C4)}$$

But in the force-free region $r > a$, we known the form of the wave function:

$$R = Ne^{-qr}\{(qr)^{-1} + (qr)^{-2}\}, \qquad \text{(C5)}$$

where N is a constant. For sufficiently small a, (and constant q) the integral for P_2 is dominated by the term in R^2 which is proportional to r^{-4}, so that one has

$$1 \approx N^2 \int_a (qr)^{-4} r^2 dr \approx N^2/q^4 a. \qquad \text{(C6)}$$

By the same argument, the numerator in (C4) diverges less strongly than $1/a$ when $n > 0$, and more strongly when $n < 0$; Q.E.D.

Similarly, for any function $F(r)$ well-behaved at the origin, we have

$$\lim_{a\to 0} \int_0^\infty R^2 F(r) r^2 dr = F(0), \qquad \text{(C7)}$$

the limit being taken in the way explained above.

Notice the essential difference from the case of an S-bound state, whose wave function in the force-free region remains finite when continued to the origin.