Dispersion Relations for Three-Particle Scattering Amplitudes. I*

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We consider the scattering of three nonrelativistic spinless particles interacting via two-body Yukawa potentials. The on-energy-shell T-matrix element is studied as a function of the total center-of-mass kinetic energy E for fixed physical values of the vectors $\mathbf{y}_i=\mathbf{k}_i(2m_iE)^{-1/2}$, $\mathbf{y}_i'=\mathbf{k}_i'(2m_iE)^{-1/2}$; $i=1, 2, 3$, where $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ and $\mathbf{k}_1'$, $\mathbf{k}_2'$, $\mathbf{k}_3'$ are the initial and final momenta of the particles, respectively, and m_1, m_2, m_3 are their masses. We show that $T(E)$ [defined as a real analytic function: $T(E)=T^*(E^*)$] has no complex singularities in the E plane. Along the real E axis, apart from the expected unitarity branch cuts and the "potential" or left-hand cuts, we find three kinds of anomalous singularities. The first kind arises from the kinematical possibility of the particles undergoing a finite number (depending on the mass ratios) of successive binary collisions ("rescatterings") at arbitrarily large spatial separations. The other two kinds are associated with the existence of two-particle bound states. We show that the discontinuities of $T(E)$ across the anomalous cuts can be explicitly expressed in terms of on-shell physical amplitudes. Accordingly, we formulate N/D equations for the determination of the amplitude. The connection between the rescattering singularities and the convergence of the partial-wave expansion of the amplitude is briefly discussed.

I. INTRODUCTION

MOST of the recent work on the quantum-mechanical three-particle problem has dealt with integral equations for the off-energy-shell amplitude (T matrix) such as the Faddeev equations[1]: A set of interparticle potentials is given and the T matrix is obtained as the solution of a system of coupled linear integral equations. Insofar as phenomenological or empirical potentials can be used to describe the interparticle forces, this approach may be applied in principle to a large number of cases of practical interest such as binding energies of light nuclei, scattering and break up of nuclei, etc.

In the domain of high-energy particle reactions the potential approach is not available. However one may still hope for an extension of the S-matrix methods employed so far with some success for relativistic two-particle amplitudes. In fact, according to current views, the study of many-particle scattering amplitudes is essential for the completion of the S-matrix program because they are coupled through unitarity relations to the two-body amplitudes. A step in this direction has been recently taken by Mandelstam[2] who formulated ND^{-1} equations for the three-particle scattering amplitude. He used the squared center-of-mass energy as the dispersion variable and assumed that all the singularities and the corresponding discontinuities of the amplitude are given except those in the physical region which are directly related to unitarity. Clearly, for a realization of such a program a knowledge of the analytic properties of the amplitude in the dispersion variable is essential. However, from what is known about the analytic properties of even the simplest Feynman integrals the problem cannot be expected to have a simple answer. Specifically, the existence of the familiar anomalous threshold singularities leads one to believe that the analytic properties of many-particle scattering amplitudes are far more complicated than those of the two-particle amplitudes.

This paper was motivated by the thought that some insight might be obtained from a study of analytic properties in potential scattering. In particular, we have studied the scattering of three scalar particles interacting via two-particle potentials of the Yukawa type. We considered the energy-shell T-matrix element (scattering amplitude) as a function of the total center-of-mass kinetic energy E and the vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$, $\mathbf{y}_1'$, $\mathbf{y}_2'$, $\mathbf{y}_3'$ defined by

$$\mathbf{y}_i=\mathbf{k}_i(2m_iE)^{-1/2},\quad \mathbf{y}_i'=\mathbf{k}_i'(2m_iE)^{-1/2};\quad i=1,2,3,\qquad \text{(I.1)}$$

where m_1, m_2, m_3 are the particle masses and $\mathbf{k}_i$ and $\mathbf{k}_i'$ the initial and final momenta. Because of energy and momentum conservation the y's and y′'s are not independent; they satisfy the relations ($y_i=|\mathbf{y}_i|$, etc.)

$$y_1^2+y_2^2+y_3^2=y_1'^2+y_2'^2+y_3'^2=1,$$
$$\mathbf{y}_1(2m_1)^{1/2}+\mathbf{y}_2(2m_2)^{1/2}+\mathbf{y}_3(2m_3)^{1/2}=\mathbf{y}_1'(2m_1)^{1/2}+\mathbf{y}_2'(2m_2)^{1/2}+\mathbf{y}_3'(2m_3)^{1/2}=0.\qquad \text{(I.2)}$$

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[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

[2] S. Mandelstam, Phys. Rev. **140**, B375 (1965).

We have studied the analyticity properties of $\langle \mathbf{y}_i' | T(E) | \mathbf{y}_i \rangle$ in the complex E plane for fixed physical values of the $\mathbf{y}_i$ and $\mathbf{y}_i'$ vectors. These analyticity properties are shared by all partial waves because the partial-wave amplitudes as defined, e.g., by Omnes[3] are obtained by multiplying $\langle \mathbf{y}_i' | T(E) | \mathbf{y}_i \rangle$ by some continuous function of $\mathbf{y}_i$ and $\mathbf{y}_i'$ (Jacobi polynomials, etc.) and integrating over a compact domain.

Our proof of analyticity is carried out in several steps. In Secs. II and III we examine the scattering amplitude to all orders of perturbation theory for the case of three particles of equal mass. The amplitude turns out to have the expected unitarity cut from $E=0$ to $+\infty$ as well as left-hand branch cuts along the negative real E axis which are associated with "pinching" singularities among potentials, in complete analogy to the two-body problem. In addition, however, in the three-body case, we find branch cuts along the negative real axis associated with the fact that it is kinematically possible for three point particles to undergo several [at most three in the equal-mass case] successive binary contact collisions. What is important for dynamical calculations based on analyticity and unitarity is that the discontinuity across these "rescattering" cuts can be explicitly expressed in terms of the on-shell two-body T matrices.

In Sec. IV we discuss the general case of arbitrary masses. We demonstrate the connection between the "rescattering" singularities and the classical problem of successive binary collisions of three particles. We establish an upper bound on the number of such collisions which are kinematically possible for given mass ratios.

In Sec. V we consider the full amplitude as a solution to the Faddeev equations in terms of the two-body T matrices. We find that if two-body bound states are present, then in addition to the expected unitarity branch point at $E=-$(two-particle binding energy) we have two kinds of "anomalous" thresholds accompanying each two-particle bound state. Again we find that the associated discontinuities can be expressed in terms of physical on-shell amplitudes for (i) two-body scattering (ii) bound-state breakup and pickup reactions.

According to the results of Secs. II, III, IV, and V the three-particle amplitude is free of singularities in the upper half plane ($\mathrm{Im}E>0$). By defining it in the lower half plane ($\mathrm{Im}E<0$) via the "reality" relation

$$T(E)=T^*(E^*),$$

we introduce discontinuities along the real E axis associated with various singularities. In Sec. VI we give a complete list of these singularities and write down formulas for the discontinuities [except for the "potential" left-hand cuts which are not treated explicitly]. We then formulate N/D equations for the determination of the amplitude in terms of the two-body on-shell amplitudes and the discontinuity across the potential cuts which represent the "forces." Finally we discuss briefly the significance of the "rescattering" cuts in connection with the convergence of the partial-wave series.

[3] R. Omnes, Phys. Rev. **134**, B1358 (1964).

II. PERTURBATION THEORY: EQUAL MASSES, NO ANOMALOUS SINGULARITIES

We begin our discussion by considering the scattering of three spinless particles of equal mass m interacting via two-particle Yukawa potentials of the form $e^{-\mu r}r^{-1}$, where r is the interparticle distance. For simplicity of exposition we prefer not to work with general superpositions of Yukawa potentials. However, all our results would still remain valid in that case. One would only have to interpret μ wherever it appears as the smallest inverse range of the superpositions.

Let $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ and $\mathbf{k}_1'$, $\mathbf{k}_2'$, $\mathbf{k}_3'$ be the initial and final center-of-mass momenta of the three particles and let $E=k^2$ be the total center-of-mass kinetic energy (we are using units in which $\hbar=2m=1$). Then

$$\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}_1'+\mathbf{k}_2'+\mathbf{k}_3', \tag{II.1}$$

and

$$k_1^2+k_2^2+k_3^2=k_1'^2+k_2'^2+k_3'^2=k^2. \tag{II.2}$$

We also introduce the vectors

$$\mathbf{y}_i=\mathbf{k}_i/k \quad \text{and} \quad \mathbf{y}_i'=\mathbf{k}_i'/k; \quad i=1,2,3. \tag{II.3}$$

We note that Eqs. (II.1), (II.2), and (II.3) imply the inequalities[4]

$$y_i^2\leq\tfrac{2}{3} \quad \text{and} \quad y_i'^2\leq\tfrac{2}{3}. \tag{II.4}$$

The on-the-energy-shell three-particle scattering amplitude[5] may be considered as a function of k^2 and a set of independent y variables. In fact, the rotational invariance of the amplitude restricts the number of independent y variables to seven, e.g., y_1^2, y_2^2, $y_1'^2$, $y_2'^2$, $(\mathbf{y}_1-\mathbf{y}_1')^2$, $(\mathbf{y}_2-\mathbf{y}_2')^2$, and $(\mathbf{y}_3-\mathbf{y}_3')^2$. In what follows, however, we shall deliberately refrain from specifying a particular set of independent y variables because we shall only study the analytic properties of the amplitude as a function of k^2 for fixed physical values of the y variables. It appears that this choice of variables results in simple analytic properties which, furthermore, are shared by all partial-wave amplitudes as mentioned in the Introduction.

Setting aside questions of convergence for the moment we will discuss the analytic properties of the general term in the perturbation expansion for the scattering amplitude T. Symbolically, the expansion reads

$$T=V+VG_0V+VG_0VG_0V+\cdots,$$

where V is the sum of the three interparticle potentials

$$V=V_{12}+V_{23}+V_{31} \tag{II.5}$$

[4] Note that $3y_1^2+(\mathbf{y}_2-\mathbf{y}_3)^2=2$, etc.

[5] See, for instance, M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).

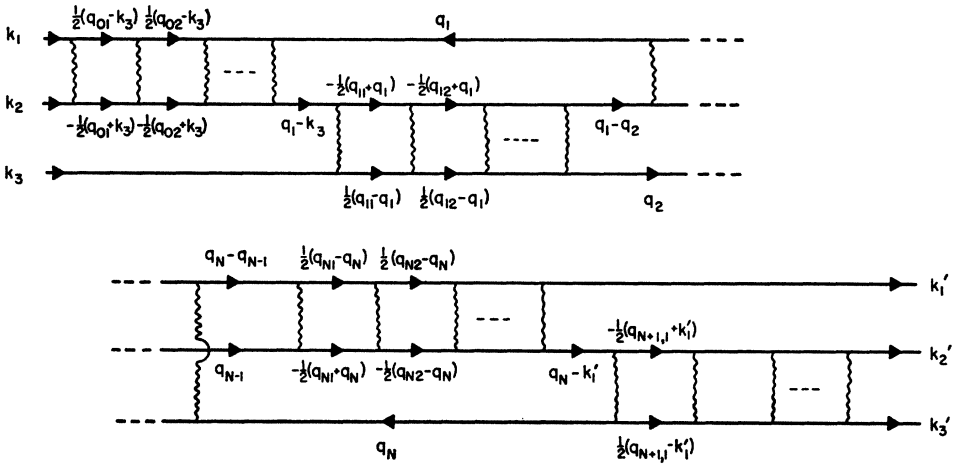

FIG. 1. A diagram representing a general term of the perturbation expansion of the three-particle scattering amplitude.

and G_0 is the free Green's function for outgoing waves

$$G_0=(k^2-H_0+i\epsilon)^{-1};\quad H_0=-\nabla_1{}^2-\nabla_2{}^2-\nabla_3{}^2. \quad \text{(II.6)}$$

The momentum space matrix elements of the potentials and the Green's function are of the form (we omit here numerical factors for simplicity):

$$\langle \mathbf{k}_1'',\mathbf{k}_2'',\mathbf{k}_3''|V_{12}|\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3'\rangle=[\mu^2+(\mathbf{k}_1''-\mathbf{k}_1')^2]^{-1}\times\delta^3(\mathbf{k}_3''-\mathbf{k}_3')\delta^3(\mathbf{k}_1''+\mathbf{k}_2''-\mathbf{k}_1'-\mathbf{k}_2'),$$

$$\langle \mathbf{k}_1'',\mathbf{k}_2'',\mathbf{k}_3''|G_0|\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3'\rangle=[k^2-k_1'^2-k_2'^2-k_3'^2+i\epsilon]^{-1}\times\delta^3(\mathbf{k}_1''-\mathbf{k}_1')\delta^3(\mathbf{k}_2''-\mathbf{k}_2')\delta^3(\mathbf{k}_3''-\mathbf{k}_3').$$

A general term in the perturbation expansion in momentum space is represented by the diagram in Fig. 1. The wavy vertical lines correspond to two-particle potentials. Each set of three horizontal lines between potentials corresponds to a free Green's function. Conservation of momentum is manifest at each vertex, i.e., the diagram is associated with the integrand obtained after performing the trivial delta-function integrations. The remaining integration momenta are denoted[6] by $\mathbf{q}_{\alpha\beta}$ or $\mathbf{q}_j$.

Our methods for studying the integral[7] are elementary. We first apply the Feynman identity to the product of all those Green's functions which are located between potentials that act on *different* pairs of particles. If we denote the Feynman parameters by $x_1, x_2, \cdots, x_{N+1}$, the resulting denominator is of the form[8]

$$Q(\mathbf{q};\mathbf{k}_3,\mathbf{k}_1')-k^2-i\epsilon, \quad \text{(II.7)}$$

where

$$\begin{aligned}Q=x_1[\mathbf{k}_3{}^2+(\mathbf{k}_3-\mathbf{q}_1)^2+\mathbf{q}_1{}^2]\\+x_2[\mathbf{q}_1{}^2+(\mathbf{q}_1-\mathbf{q}_2)^2+\mathbf{q}_2{}^2]+\cdots\\+x_N[\mathbf{q}_{N-1}{}^2+(\mathbf{q}_{N-1}-\mathbf{q}_N)^2+\mathbf{q}_N{}^2]\\+x_{N+1}[\mathbf{q}_N{}^2+(\mathbf{q}_N-\mathbf{k}_1')^2+\mathbf{k}_1'^2]. \quad \text{(II.8)}\end{aligned}$$

For reasons that will soon become clear we restrict the discussion of this section to diagrams with $N>1$. A moment's reflection will convince the reader that we thus include all but the second ($N=0$) and third ($N=1$) order terms of the iteration expansion of the Faddeev equation.[9]

The expression Q is a non-negative quadratic form in the variables $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_N$. By performing a change of variables

$$\mathbf{q}_i=\mathbf{q}_i'+\bar{\mathbf{q}}_i, \quad \text{(II.9)}$$

we can eliminate the terms that are linear in the $\mathbf{q}$'s

$$Q(\mathbf{q}_i'+\bar{\mathbf{q}}_i;\mathbf{k}_3,k_1')=\sum_{i,j}A_{ij}\mathbf{q}_i'\mathbf{q}_j'+g. \quad \text{(II.10)}$$

Clearly, $\bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2, \cdots$ is that set of values of the $\mathbf{q}$'s that minimize $Q(\mathbf{q}_i)$ and, therefore, the quantity g is the minimum value of Q under variation of the $\mathbf{q}$'s. The conditions $\partial Q/\partial\mathbf{q}_i=0$ ($i=1, 2, \cdots, N$) provide us with a system of N linear vector equations for the $\bar{q}$'s. The coefficients are simple linear combinations of the Feynman parameters. The solution to this system is unique and of the form

$$\bar{\mathbf{q}}_i=a_i\mathbf{k}_3+b_i\mathbf{k}_1', \quad \text{(II.11)}$$

where a_i and b_i are rational homogeneous functions of the Feynman parameters of order zero.

[6] The choice of integration momenta is, of course, not unique. For our purpose, however, it is convenient to make the choice indicated in Fig. 1. Note that integration momenta occurring in more than one Green's function are specified by *one* index: $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_N$, whereas those occurring in only one Green's function have *two* indices.

[7] In what follows it will be understood that the over-all delta function factor expressing total momentum conservation is dropped from the amplitude (and also from any perturbation series term that we will consider) according to the definition [see Ref. 5].

$$\langle f|S|i\rangle=\delta_{fi}-2\pi i\delta(E_f-E_i)\delta^3(\mathbf{P}_f-\mathbf{P}_i)\langle f|T|i\rangle.$$

[8] With no loss of generality, we will often discuss a general class of diagrams by making, for concreteness, a specific choice for the pair of particles that interact first or last.

[9] See Eqs. (V.1) and (V.3).

In the following we shall need upper bounds on the quantities a_i, b_i, and g. We start by obtaining a bound on g. In doing so we may regard the variables q and k as real numbers. Since g is the minimum value of the quadratic form Q under variation of the q's we note that for $N>2$

$$\begin{aligned} g&=\min Q(q_1,q_2,\cdots,q_N)\leq Q(\tfrac{1}{2}k_3,0,0,\cdots,\tfrac{1}{2}k_1')\\ &=(\tfrac{3}{2}x_1+\tfrac{1}{2}x_2)k_3{}^2+(\tfrac{1}{2}x_N+\tfrac{3}{2}x_{N+1})k_1'^2\\ &\leq\tfrac{3}{2}(x_1+x_2)y_3{}^2k^2+\tfrac{3}{2}(x_N+x_{N+1})y_1'^2k^2\\ &\leq\tfrac{3}{2}(x_1+x_2)\tfrac{2}{3}k^2+\tfrac{3}{2}(x_N+x_{N+1})\tfrac{2}{3}k^2\leq k^2 \end{aligned}\quad\text{(II.12)}$$

and for $N=2$

$$\begin{aligned} g&=\min Q(q_1,q_2)\leq Q(\tfrac{1}{2}k_3,\tfrac{1}{2}k_1')\\ &=\tfrac{3}{2}x_1k_3{}^2+\tfrac{1}{4}x_2[k_3{}^2+k_1'^2+(k_3-k_1')^2]+\tfrac{3}{2}x_3k_1'^2\\ &\leq\tfrac{3}{2}x_1y_3{}^2k^2+\tfrac{1}{2}x_2[y_3{}^2+y_1'^2+y_3\cdot y_1']k^2+\tfrac{3}{2}x_3y_1'^2k^2\\ &\leq\tfrac{3}{2}(x_1+x_2+x_3)\tfrac{2}{3}k^2\leq k^2. \end{aligned}\quad\text{(II.13)}$$

In order to obtain a bound on b_i we set $k_3=0$ and take $k_i'>0$. Then the $\bar{q}_i=b_ik_1'$ minimize the form

$$2x_1q_1{}^2+x_2[q_1{}^2+(q_1-q_2)^2+q_2{}^2]+\cdots+x_N[q_{N-1}{}^2+(q_{N-1}-q_N)^2+q_N{}^2]+x_{N+1}[q_N{}^2+(q_N-k_1')^2+k_1'^2].$$

By minimizing *separately* with respect to q_1, q_2, $\cdots$, q_N we obtain the following inequalities for $\bar{q}_1$, $\bar{q}_2$, $\cdots$, $\bar{q}_N$:

$$\begin{gathered} 0\leq 2\bar{q}_1\leq\bar{q}_2,\\ 2\bar{q}_2\leq\max(\bar{q}_1,\bar{q}_2),\\ 2\bar{q}_3\leq\max(\bar{q}_2,\bar{q}_4),\\ \vdots\\ 2\bar{q}_N\leq\max(\bar{q}_{N-1},k_1'), \end{gathered}$$

which imply that

$$0\leq\bar{q}_1\leq\tfrac{1}{2}\bar{q}_2\leq\frac{1}{2^2}\bar{q}_3\leq\cdots\leq\frac{1}{2^{N-1}}\bar{q}_N\leq\frac{1}{2^N}k_1'.\quad\text{(II.14)}$$

We conclude that

$$0\leq b_i\leq 2^{-N+i-1}.\quad\text{(II.15)}$$

Similarly, by taking $k_1'=0$ and $k_3>0$ we find

$$0\leq a_i\leq 2^{-i}.\quad\text{(II.16)}$$

It then follows that

$$|\bar{\mathbf{q}}_i|=|a_i\mathbf{k}_3+b_i\mathbf{k}_1'|\leq(\tfrac{1}{2}+\tfrac{1}{4})(\tfrac{2}{3})^{1/2}k=(\tfrac{1}{4}\sqrt{6})k.\quad\text{(II.17)}$$

If, for later convenience we introduce

$$\bar{\mathbf{q}}_i=\mathbf{r}_i(x;\,\mathbf{y}_3,\mathbf{y}_1')k,\quad\text{(II.18)}$$

and

$$-g+k^2=a(x;\,\mathbf{y}_3,\mathbf{y}_1')k^2\quad\text{(II.19)}$$

the bounds imply that

$$|\mathbf{r}_i|\leq\tfrac{1}{4}\sqrt{6}\quad\text{and}\quad a\geq 0.\quad\text{(II.20)}$$

We are now prepared to discuss the analyticity properties of an arbitrary diagram of the class $N>1$. After performing the change of variables indicated in (II.9) we have the integral

$$\begin{aligned} I(k)=\int d^3q_1d^3q_2\cdots d^3q_N\,dx_1dx_2\cdots dx_{N+1}\\ \times\delta(x_1+x_2+\cdots-1)\cdots d^3q_{\alpha\beta}\cdots\\ \times H(k^2+i\epsilon;\,\mathbf{q}_{01},\cdots,\mathbf{q}_{\alpha\beta},\cdots;\,\mathbf{q}_1+\mathbf{r}_1k,\cdots)\\ \times[\sum_{i,j}A_{ij}(x)\mathbf{q}_i\cdot\mathbf{q}_j-ak^2-i\epsilon]^{-N-1}. \end{aligned}\quad\text{(II.21)}$$

According to the standard prescription of scattering theory the limit of this integral as $\epsilon\to 0$ is the contribution to the physical amplitude. In the spirit of dispersion theory we now try to define an analytic function of k, for fixed physical values of the y's, such that its boundary value on approaching the positive real k axis from above reproduces the physical amplitude originally defined by the $\epsilon\to 0$ process. For convenience, we shall use in the following the variable k instead of k^2 with the understanding that the complex k^2 plane is mapped on the upper half k plane.

We first note that the Feynman denominator

$$\sum_{i,j}A_{ij}\mathbf{q}_i\cdot\mathbf{q}_j-ak^2\quad\text{(II.22)}$$

does not vanish for k in the upper half plane since $\sum A_{ij}\mathbf{q}_i\cdot\mathbf{q}_j$ is non-negative. Also, approaching the positive real k axis gives the same limit as with $\epsilon\to 0$ because $a\geq 0$.

Looking next at the potential denominators in H, we see that they cannot vanish in the strip

$$-\mu<(\mathrm{Im}k)2\max_i(|\mathbf{r}_i|)<\mu\quad\text{(II.23)}$$

and, therefore, from (II.20), in the strip

$$|\mathrm{Im}k|<(\tfrac{2}{3})^{1/2}\mu.\quad\text{(II.24)}$$

Those factors in H that come from Green's functions are of two kinds:

(1) Those at the beginning and end of the diagram. Their denominators are of the form

$$\tfrac{1}{2}q_{0,\beta}{}^2+\tfrac{3}{2}k_3{}^2-k^2-i\epsilon,\quad\text{(II.25)}$$

and

$$\tfrac{1}{2}q_{N+1,\beta}{}^2+\tfrac{3}{2}k_1'^2-k^2-i\epsilon,$$

respectively. Since $y_3{}^2$, $y_1'^2\leq\frac{2}{3}$, the same remarks apply as for (II.22).

(2) The Green's functions in the middle of the diagram. Their denominators are of the form

$$\tfrac{1}{2}q_{\alpha\beta}{}^2+\tfrac{3}{2}(\mathbf{q}_\alpha+\mathbf{r}_\alpha k)^2-k^2-i\epsilon,\quad\text{(II.26)}$$

which can be written as

$$\tfrac{3}{2}(q_\alpha+zr_\alpha k)^2-k^2\sigma_\alpha{}^2+\tfrac{1}{2}q_{\alpha\beta}{}^2-i\epsilon,\quad\text{(II.27)}$$

where

$$\begin{aligned} q_\alpha&=|\mathbf{q}_\alpha|, & r_\alpha&=|\mathbf{r}_\alpha|,\\ z&=\cos(\mathbf{q}_\alpha,\mathbf{r}_\alpha), & \sigma_\alpha{}^2&=1-\tfrac{3}{2}r_\alpha{}^2(1-z^2). \end{aligned}\quad\text{(II.28)}$$

The corresponding poles in the q_α plane are given by

$$q_\alpha = -zr_\alpha k \pm (\tfrac{2}{3}\sigma_\alpha^2 k^2 - \tfrac{1}{3}q_{\alpha\beta}^2 + i\epsilon)^{1/2}. \quad \text{(II.29)}$$

We note that for $\mathrm{Im}k>0$

$$\mathrm{Im}[\tfrac{2}{3}\sigma_\alpha^2 k^2 - \tfrac{1}{3}q_{\alpha\beta}^2]^{1/2} \geq (\mathrm{Im}k)(\tfrac{2}{3})^{1/2}\sigma_\alpha \geq (\mathrm{Im}k) r_\alpha |z|. \quad \text{(II.30)}$$

It then follows that the poles given by (II.29) are never on the real q_α path for $\mathrm{Im}k>0$.

The result of the above discussion is that $I(k)$ is an analytic function of k in the strip

$$0<\mathrm{Im}k<(\tfrac{2}{3})^{1/2}\mu. \quad \text{(II.31)}$$

In this strip, $I(k)$ is a "real" analytic function in the sense

$$I(k)=I^*(-k^*). \quad \text{(II.32)}$$

In fact, it is easy to show that $I(k)$ *is analytic in the entire upper-half k plane with the possible exception of the imaginary axis above* $i\mu(\frac{2}{3})^{1/2}$. We simply note that the integral (II.21) defines an analytic function of k in the strip

$$0<\mathrm{Im}(e^{-i\psi}k)<(\tfrac{2}{3})^{1/2}\mu\cos\psi, \quad (0<\psi<\pi/2), \quad \text{(II.33)}$$

if the q variables are everywhere formally multiplied by $e^{i\psi}$ [this follows, after some straightforward algebra, by looking at the potential denominators]. It is evident then that, starting with $\psi=0$ and k in the strip (II.31), we can continue $I(k)$ in the entire first quadrant of the k plane by rotating the integration paths of the q variables in their respective complex planes so that (II.33) is satisfied at every stage. The analyticity in the second quadrant follows trivially from Eq. (II.32).

These analyticity properties may be expressed in the form of a dispersion relation[10] for $J(k^2)=I(k)$

$$J(k^2)=\frac{1}{\pi}\left[\int_{-\infty}^{-2\mu^2/3}+\int_0^\infty\right]dk'^2\frac{\mathrm{Im}J(k'^2)}{k'^2-k^2}. \quad \text{(II.34)}$$

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A typical diagram for the case $N=1$ is shown in Fig. 2. Using the notation of the previous section we have

$$Q=x_1[\mathbf{k}_3^2+(\mathbf{k}_3-\mathbf{q})^2+\mathbf{q}^2]+x_2[\mathbf{q}^2+(\mathbf{q}-\mathbf{k}_3')^2+\mathbf{k}_3'^2]$$

$$Q-k^2=2[\mathbf{q}-\tfrac{1}{2}(x_1\mathbf{k}_3+x_2\mathbf{k}_3')]^2 +\tfrac{1}{2}x_1x_2\mathbf{\Delta}^2-\tfrac{1}{2}(x_1\mathbf{k}_{12}^2+x_2\mathbf{k}_{12}'^2), \quad \text{(III.1)}$$

where

$$\mathbf{k}_{12}=\mathbf{k}_1-\mathbf{k}_2=\mathbf{y}_{12}k,$$
$$\mathbf{k}_{12}'=\mathbf{k}_1'-\mathbf{k}_2'=\mathbf{y}_{12}'k, \quad \text{(III.2)}$$
$$\mathbf{\Delta}=\mathbf{k}_3-\mathbf{k}_3'=\mathbf{y}_\Delta k.$$

[10] If, as in the case of two-particle scattering by a Yukawa potential, the amplitude tends to the Born approximation at large energies, then no subtractions are required in writing down dispersion relations like (II.34).

We now have

$$\bar{\mathbf{q}}=\tfrac{1}{2}(x_1\mathbf{k}_3+x_2\mathbf{k}_3'),$$
$$\mathbf{r}=\tfrac{1}{2}(x_1\mathbf{y}_3+x_2\mathbf{y}_3'), \quad \text{(III.3)}$$
$$a=-\tfrac{1}{2}x_1x_2y_\Delta^2+\tfrac{1}{2}(x_1y_{12}^2+x_2y_{12}'^2).$$

As one can verify from the above expression the quantity a can become negative for a certain range of values of x_1 and x_2, if[11]

$$y_\Delta>y_{12}+y_{12}'. \quad \text{(III.4)}$$

This is, in fact, what makes the case $N=1$ exceptional, because here the argument given after Eq. (II.22) no longer applies. By applying the Landau rules to the denominator function $Q-k^2-i\epsilon$, it is easy to see that the amplitude has a branch point at $y_\Delta^2=(y_{12}+y_{12}')^2$ (in the physical region) corresponding to a "pinch" between the two Green's functions. This singularity is independent of k only because we have chosen to keep the y's rather than the two-particle subenergies fixed.

If the amplitude were continued through the imaginary k axis, this singularity would prevent it from being real in the sense of Eq. (II.32). This suggests that the proper definition of the amplitude in the second quadrant of the k plane is through Eq. (II.32). It will be shown that with this definition the amplitude has a cut in the k^2 plane from $-\infty$ to 0. If the amplitude were continued through this cut, it would develop complex singularities.

It is instructive to look first at what would happen if we did continue through the imaginary k axis. Clearly, if $y_\Delta\leq y_{12}+y_{12}'$, then $a\geq0$ for all values of the x's and the arguments and results of Sec. II are still valid. However if $y_\Delta>y_{12}+y_{12}'$, then a can be negative for certain values of the x's.

The Feynman denominator for $a<0$ can be written as $2q^2+|a|k^2$ and needs no $i\epsilon$ prescription for k physical. In fact, it does not vanish in the entire half plane $\mathrm{Re}k>0$. For the remaining denominators the discussion given for the case $N>1$ still holds, so we again have analyticity in the first quadrant, i.e., for

$$\mathrm{Re}k>0, \quad \mathrm{Im}k>0.$$

In order to continue past the imaginary axis [below $i\mu(\frac{2}{3})^{1/2}$] we can simply add the contribution from the pole at ($q=|\mathbf{q}|$)

$$q=-ik(-a/2)^{1/2}. \quad \text{(III.5)}$$

Thus, for $\mathrm{Re}k>0$, the amplitude is given by the integral

$$f_1(k)=\int_0^\infty dq\frac{F(k,q)}{[2q^2+|a|k^2]^2}, \quad \text{(III.6)}$$

whereas for $\mathrm{Re}k<0$ the continuation will consist of

[11] We will frequently denote $|\mathbf{k}_i|$, $|\mathbf{k}_i-\mathbf{k}_j|$, $|\mathbf{y}_i|$, $|\mathbf{y}_i-\mathbf{y}_j|$, etc. by simply k_i, k_{ij}, y_i, y_{ij}, $\cdots$, respectively.

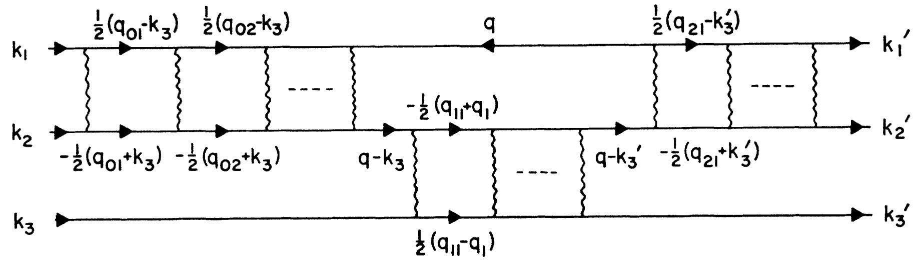

FIG. 2. A typical one-loop ($N=1$) diagram.

two terms

$$f_1(k)=\int_0^\infty dq\frac{F(k,q)}{[2q^2+|a|k^2]^2}+h(k), \qquad \text{(III.7)}$$

where the pole term $h(k)$ is given by

$$h(k)=\frac{\pi i}{2}\frac{d}{dq}\left\{\frac{F(k,q)}{q+i|a|k/2}\right\}_{q=i|a|k/2}. \qquad \text{(III.8)}$$

The first term on the right-hand side of Eq. (III.7) has the same form as the integral in Eq. (III.6) and is clearly analytic in the second quadrant.

The real difficulty arises from the pole term $h(k)$. This term is still an integral over $\mathbf{q}_{01}$, $\mathbf{q}_{02}$, $\cdots$, $\mathbf{q}_{11}$, $\cdots$ and the angles of $\mathbf{q}$. Let us start by seeking a region of the k plane in which none of denominators involved in the integrand vanish. It is straightforward to show that none of the potential denominators vanish in the region where the following inequalities hold simultaneously:

$$\begin{aligned}|\mathrm{Re}k||\mathbf{y}_3-\tfrac{1}{2}(x_1\mathbf{y}_3+x_2\mathbf{y}_3')|+|\mathrm{Im}k||a/8|^{1/2}<\mu,\\ |\mathrm{Re}k||\mathbf{y}_3'-\tfrac{1}{2}(x_1\mathbf{y}_3+x_2\mathbf{y}_3')|+|\mathrm{Im}k||a/8|^{1/2}<\mu,\end{aligned} \qquad \text{(III.9)}$$

or, for simplicity, in the smaller region

$$|\mathrm{Re}k|+\tfrac{1}{6}|\mathrm{Im}k|<(\tfrac{2}{3})^{1/2}\mu. \qquad \text{(III.10)}$$

A typical Green's-function denominator in the middle part of the diagram (the ones at the ends are similar to those in Sec. II) is

$$\tfrac{1}{2}q_{11}^2+\tfrac{3}{2}[\tfrac{1}{2}(x_1\mathbf{y}_3+x_2\mathbf{y}_3')-i\mathbf{n}|a/2|^{1/2}]^2k^2-k^2,$$

where $\mathbf{n}$ is the unit vector in the direction of $\mathbf{q}$. To avoid the zero of such denominators we may have to distort the q_{11}^2 contour in the strip $|\mathrm{Im}q_{11}^2|<\frac{1}{2}|k|^2$. The distortion modifies (III.10) into

$$|\mathrm{Re}k|+\left(\frac{1}{6}+\frac{1}{\sqrt{2}}\right)|\mathrm{Im}k|<(\tfrac{2}{3})^{1/2}\mu. \qquad \text{(III.11)}$$

Thus we have proven analyticity in a convex region of the upper half k plane which includes the origin and pieces of the real and imaginary axis. We cannot, however, use the method employed at the end of Sec. II to extend this region into the entire second quadrant.

Thus we cannot eliminate the possibility of singularities in the second quadrant. Furthermore, we note that the pole term $h(k)$ violates the "reality" of the amplitude. In fact it is clear that

$$h(k)=f_1(k)-f_1^*(-k^*). \qquad \text{(III.12)}$$

In order to obtain an explicit expression for $h(k)$ it is convenient to follow a somewhat different approach and not use the Feynman identity at all. We note that any perturbation term of the class $N=1$ can be represented in the form

$$\begin{aligned}f_1(k)&=\lim_{\epsilon\to 0}\int d^3q\frac{F(k+i\epsilon;\mathbf{q})}{[\mathbf{q}^2-\mathbf{k}_{12}^2-i\epsilon][(\mathbf{q}-\boldsymbol{\Delta})^2-\mathbf{k}_{12}'^2-i\epsilon]}\\ &=\lim_{\epsilon\to 0}\frac{1}{2}\int_{-\infty}^{\infty}dq\int_{-1}^{1}d\cos\theta\int_0^{2\pi}d\varphi\\ &\quad\times\frac{q^2F(k+i\epsilon;q,\cos\theta,\varphi)}{[\mathbf{q}^2-\mathbf{k}_{12}^2-i\epsilon][(\mathbf{q}-\boldsymbol{\Delta})^2-\mathbf{k}_{12}'^2-i\epsilon]}.\end{aligned} \qquad \text{(III.13)}$$

Our choice of variables is indicated in Fig. 2. In introducing spherical coordinates we have made use of the invariance of the integral under the change $\mathbf{q}\to-\mathbf{q}$ in the integrand, to extend the q integration from $-\infty$ to $+\infty$. We take the z axis to lie along $\boldsymbol{\Delta}$ so that $\cos\theta=\boldsymbol{\Delta}\cdot\mathbf{q}/\Delta q$.

We have already seen that $f_1(k)$ is the boundary value of a function which is analytic in the first quadrant of the k plane. It follows that $\bar{f}_1(k)=f_1^*(-k^*)$ is an analytic function of k in the *second* quadrant the boundary value of which, on approaching the *negative* real axis from above is given by

$$\begin{aligned}\bar{f}_1(k)&=\lim_{\epsilon\to 0}\frac{1}{2}\int_{-\infty}^{\infty}dq\int_{-1}^{1}d\cos\theta\int_0^{2\pi}d\varphi\\ &\quad\times\frac{q^2F(k+i\epsilon;q,\cos\theta,\varphi)}{[\mathbf{q}^2-\mathbf{k}_{12}^2+i\epsilon][(\mathbf{q}-\boldsymbol{\Delta})^2-\mathbf{k}_{12}'^2+i\epsilon]}.\end{aligned} \qquad \text{(III.14)}$$

We see from Eq. (III.12) that we can obtain an explicit expression for $h(k)$ by continuing $f_1(k)$ and $\bar{f}_1(k)=f_1^*(-k^*)$ onto the imaginary axis approaching from their respective quadrants of analyticity.

The denominators $q^2-k_{12}^2-i\epsilon$ and $(\mathbf{q}-\boldsymbol{\Delta})^2-k_{12}'^2-i\epsilon$ that have been explicitly displayed in (III.13) have zeros at

$$q_\pm=\pm(k_{12}+i\epsilon),$$

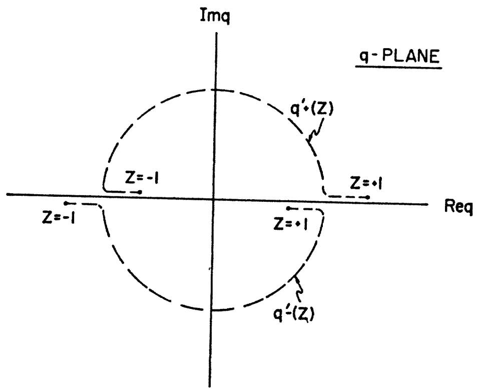

FIG. 3. The paths of the poles $q_\pm'$ of the integrand in Eq. (III.14) as z varies from -1 to $+1$ [for k real and positive].

and

$$q_\pm'=\Delta\cos\theta\pm[k_{12}'^2-\Delta^2\sin^2\theta+i\epsilon]^{1/2}. \qquad \text{(III.15)}$$

Similarly the denominators in $\bar{f}_1(k)$ have zeros at $-q_\pm^*$ and $-q_\pm'^*$. The paths in the q plane traced out by $q_\pm'$ as $\cos\theta$ varies from -1 to $+1$ are shown in Fig. 3 for k real and positive. As k moves off the real axis into the first quadrant the paths shown in Fig. 3 rotate about the origin in a counterclockwise direction through an angle equal to the phase of k. Thus in order to continue $f_1(k)$ into the first quadrant, we must add to the $(-\infty,+\infty)$ integral over q the contributions from small contours around q_+' and q_-' for those values of θ such that $\mathrm{Im}q_+'<0$ and $\mathrm{Im}q_-'>0$. For k on the imaginary axis we then have

$$f_1(k)=\left\{\int_{-\infty}^{\infty}dq\int_{-1}^{1}d\cos\theta+\oint_{(q_+')}dq\int_{-1}^{0}d\cos\theta -\oint_{(q_-')}dq\int_0^1 d\cos\theta\right\}\int_0^{2\pi}d\varphi \times\frac{q^2}{2}\frac{F(k;\,q,\cos\theta,\,\varphi)}{(q^2-k_{12}^2)(q-q_+')(q-q_-')}, \qquad \text{(III.16)}$$

where $\oint_{(q_\pm')}$ denotes a contour integral around $q_\pm'$ in the counter-clockwise direction. Similarly for $\bar{f}_1(k)$ on the imaginary axis (note that for real k the paths of $-q_\pm'^*$ are those of Fig. 3 reflected about the imaginary q axis and that on continuing into the second quadrant of the k plane the paths now rotate *clockwise*) we have

$$f_1^*(-k^*)=\left\{\int_{-\infty}^{\infty}dq\int_{-1}^{1}d\cos\theta+\oint_{(-q_+'^*)}dq\int_{-1}^{0}d\cos\theta -\oint_{(-q_-'^*)}dq\int_0^1 d\cos\theta\right\}\int_0^{2\pi}d\varphi \times\frac{q^2}{2}\frac{F(-k^*;\,q,\cos\theta,\,\varphi)}{(q^2-k_{12}^{*2})(q+q_+'^*)(q+q_-'^*)}. \qquad \text{(III.17)}$$

We can perform the q integration in the last two terms of Eqs. (III.16) and (III.17) by the residue method. If we subsequently make a change of variable from $\cos\theta$ to q_+', q_-', $q_+'^*$, and $q_-'^*$, respectively, in these four terms we find that

$$h(k)=f_1(k)-f_1^*(-k^*) =-\frac{\pi i}{2\Delta}\oint p\,dp\int_0^{2\pi}d\varphi\frac{F(k;\,p,z(p),\varphi)}{p^2-k_{12}^2}. \qquad \text{(III.18)}$$

The p integration is carried over the contour shown in Fig. 4 and

$$z(p)=(p^2+\Delta^2-k_{12}'^2)/2p\Delta. \qquad \text{(III.19)}$$

From Fig. 4 it is clear that $h(k)=0$ if $y_{12}+y_{12}'\geq y_\Delta$, in agreement with our previous remarks. Moreover, if $y_{12}+y_{12}'<y_\Delta$, we have, by the residue method,

$$h(k)=\frac{\pi^2}{\Delta}\int_0^{2\pi}d\varphi\,F(k;\,k_{12},z(k_{12}),\varphi). \qquad \text{(III.20)}$$

This result can be written *symbolically* in the form

$$h(k)=-(2\pi i)^2\int d^3q\,F(k;\,\mathbf{q}) \times\delta_+(q^2-k_{12}^2)\delta_+((\mathbf{q}-\boldsymbol{\Delta})^2-k_{12}'^2), \qquad \text{(III.21)}$$

which is, of course, reminiscent of Cutkosky's rule. Formula (III.21) is only symbolic since for $y_{12}+y_{12}'<y_\Delta$ the arguments of the two delta functions cannot vanish simultaneously. Therefore, a special prescription must supplement (III.21): one is to change variables to q, $\cos\theta$, and φ, where θ is the angle between $\mathbf{q}$ and $\boldsymbol{\Delta}$. Then the range of the $z=\cos\theta$ integration must be extended to include the value

$$z=(k_{12}^2+\Delta^2-k_{12}'^2)/2k_{12}\Delta. \qquad \text{(III.22)}$$

We have arready seen that $h(k)$ is free of singularities in the region defined in (III.11). We now try to continue $h(k)$ as given by the right-hand side of (III.20) into the

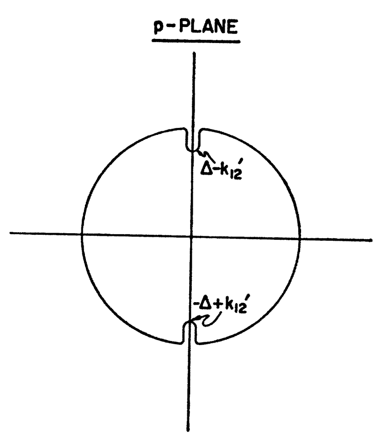

FIG. 4. The contour of the p integration in Eq. (III.18).

entire second quadrant. In order to see that $h(k)$ does indeed have singularities there (which are complex singularities in the k^2 plane) let us consider the simple third-order diagram shown in Fig. 5. In order to be able to perform the integration in (III.20) explicitly we will replace two of the potentials by constants. The corresponding contribution to the amplitude is (apart from unimportant factors)

$$f_1(k)=\int d^3q[\tfrac{1}{4}(\mathbf{q}-\mathbf{k}_1+\mathbf{k}_2)^2+\mu^2]^{-1} \times[q^2-k_{12}{}^2]^{-1}[(\mathbf{q}-\mathbf{\Delta})^2-k_{12}'^2]^{-1},$$

so that the pole term given by (III.20) is

$$h(k)=\frac{\pi^2}{\Delta}\int_0^{2\pi}d\varphi \times\{\mu^2+\tfrac{1}{2}k_{12}{}^2[1-z\cos\psi-(1-z^2)^{1/2}\sin\psi\cos\varphi]\}^{-1},$$

where ψ is the angle between $\mathbf{y}_\Delta$ and $\mathbf{y}_{12}$ and

$$z=(y_{12}{}^2+y_\Delta{}^2-y_{12}'^2)/2y_{12}y_\Delta.$$

Performing the φ integration, we obtain

$$h(k)=\frac{\pi^3}{y_\Delta k}\left\{\left(1-z\cos\psi+\frac{4\mu^2}{2y_{12}{}^2k^2}\right)^2+(z^2-1)\sin^2\psi\right\}^{-1/2}. \tag{III.23}$$

Since $z>1$ for $y_{12}+y_{12}'<y_\Delta$, it is clear that $h(k)$ and therefore $f_1(k)$ has a branch point at a complex value of k^2.

These complex singularities occur only in those terms of the perturbation expansion for which $N=1$ (provided the masses are equal). If we sum all these diagrams we obtain the third-order terms in the iteration expansion of the Faddeev equation.[9] Since there are only nine such terms one could in principle calculate the discontinuities across the cuts associated with the complex branch points just as one would have to do for the cuts lying along the imaginary k axis which are associated with "pinches" among potential denominators. However, it is not necessary to deal with complex singularities at all. We have seen that $f_1(k)$ is analytic in the first quadrant (i.e., upper half k^2 plane). We can *define* it in the second quadrant (i.e., lower have k^2 plane) by the relation

$$f_1(k)=f_1{}^*(-k^*). \tag{III.24}$$

Of course now there will be a cut on the imaginary axis running from 0 to $i\infty$ (i.e., a cut along the entire negative real k^2 axis). However, the discontinuity across this cut will be given explicitly by formula (III.21) which involves only two-particle T matrices *on the energy shell.* Let us denote by $\langle\mathbf{p}'|t_{ij}(E)|\mathbf{p}\rangle$ the T-matrix element for the scattering of particles i and j, where p and p' are the initial and final momentum of particle i with respect to the center of mass of the pair and E their

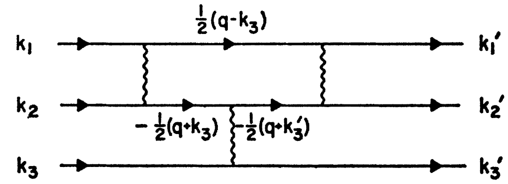

FIG. 5. A simple one-loop ($N=1$) diagram.

center-of-mass energy. Then a typical contribution to the rescattering discontinuity d_R is given by

$$-(2\pi i)^2\int d^3q\langle\tfrac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)|t_{12}(\tfrac{1}{2}k_{12}{}^2)|\tfrac{1}{2}\mathbf{q}\rangle\delta(q^2-k_{12}{}^2) \times\langle\tfrac{3}{4}\mathbf{k}_3+\tfrac{1}{4}\mathbf{q}|t_{23}[k^2-\tfrac{3}{8}(\mathbf{q}-\mathbf{k}_3)^2]|\tfrac{3}{4}\mathbf{k}_3'+\tfrac{1}{4}(\mathbf{q}-\mathbf{\Delta})\rangle \times\delta[(\mathbf{q}-\mathbf{\Delta})^2-k_{12}'^2]\langle\tfrac{1}{2}(\mathbf{q}-\mathbf{\Delta})|t_{12}(\tfrac{1}{2}k_{12}'^2)|\tfrac{1}{2}(\mathbf{k}_1'-\mathbf{k}_2')\rangle. \tag{III.25}$$

We note that after integrating over q and $\cos\theta$ with the aid of the delta functions, all three t matrices will be on the energy shell. However, in order to perform the φ integration one needs the values of the t matrices for unphysical values of the energy and momentum transfer variables.

We may express these analyticity properties in a dispersion relation for $\bar{f}_1(k^2)=f_1(k)$:

$$\bar{f}_1(k^2)=\frac{1}{\pi}\int_{-\infty}^{\infty}dk'^2\frac{\mathrm{Im}\bar{f}_1(k'^2)}{k'^2-k^2}, \tag{III.26}$$

where $\mathrm{Im}\bar{f}_1$, at least in the interval $(-2\mu^2/3, 0)$, is given by the rescattering contributions $d_R/2i$ of the type (III.25).

Special consideration is also required for the six *second*-order terms of the Faddeev expansion for the following reason. In the typical term[9] $t_{12}G_0t_{23}$, where t_{12} and t_{23} are two-body matrices, the free Green's function G_0 appears in the form

$$G_0(k^2)=k^{-2}[y_1'^2+y_3{}^2+(\mathbf{y}_1'+\mathbf{y}_3)^2-1-i\epsilon]^{-1}. \tag{III.27}$$

The singularity in this Green's function which would be expected to contribute to the unitarity relation is a pole in the y variables rather than in k^2. At the same time this pole in the y variables, because of its delta function contribution to the imaginary part, would prevent this second-order term from being a "real" analytic function of k^2. Indeed, for k^2 real and negative $G_0(k^2)$ has a nonvanishing imaginary part

$$\mathrm{Im}G_0(k^2)=\pi k^{-2}\delta[y_1'^2+y_3{}^2+(\mathbf{y}_1'+\mathbf{y}_3)^2-1]. \tag{III.28}$$

In order to avoid the complications arising from these formal difficulties we shall define $t_{12}G_0t_{23}$ in the lower half k^2 plane (i.e., second k quadrant) by the relation (III.24). With this definition the second-order term will have a "rescattering" cut along the negative k^2 axis whose discontinuity is given by

$$\mathrm{disc}[t_{12}G_0t_{23}] = 2\pi ik^{-2}t_{12}\delta[y_1'^2+y_3^2+(y_1'+y_3)^2-1]t_{23}. \quad \text{(III.29)}$$

In this expression the delta function ensures that the two-body matrices t_{12} and t_{23} are on the energy shell. Furthermore, with this definition, the second-order terms now have a *right*-hand cut in k^2 whose discontinuity is just what is expected from unitarity.

A "kinematical" branch point of the square-root type appears in the first-order Faddeev terms, namely, the ones representing the sum of the "disconnected" diagrams. It simply arises from the delta function factors. For example [in the notation of (V.2)]

$$\hat{t}_{12}=t_{12}\delta(\mathbf{k}_3-\mathbf{k}_3')=t_{12}(k^2)^{-3/2}\delta(\mathbf{y}_3-\mathbf{y}_3'). \quad \text{(III.30)}$$

We shall take the corresponding cut along the negative real axis in order to preserve (III.24). Its discontinuity is proportional to t_{12} on the energy shell.

We shall conclude this section by establishing the connection of the $N=1$ "rescattering singularities" with the familiar anomalous thresholds of relativistic scattering amplitudes and vertex functions. We consider the one-loop Feynman diagram for scattering of three particles of equal mass m shown in Fig. 6. Let p_1, p_2, p_3, and p_1', p_2', p_3' be the initial and final four-momenta and introduce the Lorentz-invariant variables

$$s_{12}=(p_1+p_2)^2,\ s_{12}'=(p_1'+p_2')^2,\ t=(p_3'-p_3)^2. \quad \text{(III.31)}$$

The location t_A of the anomalous branch point in the complex t plane is given by[12]

$$1-t_A/2m^2=(1-s_{12}/2m^2)(1-s_{12}'/2m^2) +\{(1-s_{12}/2m^2)^2(1-s_{12}'/2m^2)^2-(1-s_{12}/2m^2)^2 -(1-s_{12}/2m^2)^2+1\}^{1/2}. \quad \text{(III.32)}$$

If $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ and $\mathbf{k}_1'$, $\mathbf{k}_2'$, $\mathbf{k}_3'$ are the initial and final three-momenta of the particles, then in the nonrelativistic limit

$$k_i^2,\ k_i'^2 \ll m^2,$$

we have

$$s_{12}\approx 4m^2+(\mathbf{k}_1-\mathbf{k}_2)^2,$$
$$s_{12}'\approx 4m^2+(\mathbf{k}_1'-\mathbf{k}_2')^2, \quad \text{(III.33)}$$
$$t\approx -(\mathbf{k}_3-\mathbf{k}_3')^2=-\Delta^2.$$

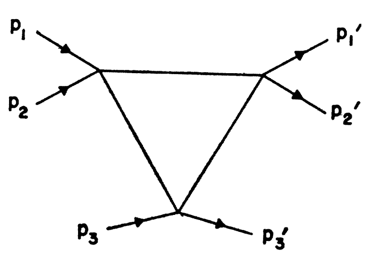

FIG. 6. The nonrelativistic limit of the anomalous threshold of this Feynman diagram (in the momentum transfer variable) is related to the "rescattering" singularities of the nonrelativistic amplitude.

Working out the nonrelativistic limit of Eq. (III.32) we obtain

$$|\mathbf{\Delta}_A|=|\mathbf{k}_1-\mathbf{k}_2|+|\mathbf{k}_1'-\mathbf{k}_2'|. \quad \text{(III.34)}$$

This is indeed the relation characteristic for the appearance of the rescattering singularities in our nonrelativistic problem. In the following they will be referred to as *anomalous singularities of type R*. Our discussion shows that they are manifestations of branch cuts in the momentum transfer variables.

IV. ANOMALOUS SINGULARITIES AND CLASSICAL COLLISIONS. GENERALIZATION TO UNEQUAL MASSES

It has been known for a long time that the anomalous thresholds associated with the triangle diagram in field theory occur at values of the external invariants compatible with all three internal momenta being on the mass shell,[12] namely, with the virtual particles being "real" and undergoing the successive collisions indicated by the Feynman diagram in real space-time. We shall now see that we have essentially the same situation in our nonrelativistic problem. In particular, the absence of anomalous singularities of type R in Faddeev terms of order higher than three corresponds to the fact that three equal-mass point particles can have at most three successive binary collisions.[13] For arbitrary masses m_1, m_2, m_3 a similar result holds. Namely, the nth order term in the iteration expansion of the Faddeev equation will have anomalous singularities of type R if and only if three classical point particles of masses m_1, m_2, and m_3 can kinematically undergo n successive binary contact collisions.

We shall study the classical collision problem by using the action principle

$$\delta\int \mathcal{L}dt=0. \quad \text{(IV.1)}$$

To be specific let us represent the entire process of $N+2$ successive collisions by the diagram in Fig. 7. The meaning of this diagram is that at time t_0 the particles have momenta $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$. The first collision occurs at time t_1 between particles 1 and 2, the second collision occurs at time t_2 between particles 2 and 3, and so on. Finally, after $N+2$ collisions the particles obtain their final momenta $\mathbf{k}_1'$, $\mathbf{k}_2'$, and $\mathbf{k}_3'$. The correspondence of this diagram with a term in the iteration expansion of the Faddeev equation is obvious.

Since the forces are of zero range, the Langrangian is merely the total kinetic energy expressed as a function of the momenta of the three particles. However, we must impose the constraint that momentum is conserved at each collision. The intermediate momenta can then

[12] A concise discussion of singularities of Feynman integrals and the Landau-Bjorken methods is given in J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Company, Inc., New York, 1965).

[13] The problem of the number of collisions of three classical hard spheres of equal mass was solved by G. Sandri, R. D. Sullivan, and P. Norem, Phys. Rev. Letters 13, 743 (1964).

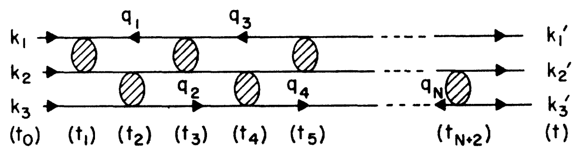

FIG. 7. Graphical representation of a classical collision process consisting of $N+2$ binary collisions at times $t_1, t_2, \cdots, t_{N+2}$.

be expressed in terms of N "loop" momenta $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_N$, whose values are to be determined from the action principle. The action integral is then given by

$$\int_{t_0}^{t} \mathcal{L}dt=(t_1-t_0)\left[\frac{\mathbf{k}_1^2}{2m_1}+\frac{\mathbf{k}_2^2}{2m_2}+\frac{\mathbf{k}_3^2}{2m_3}\right] +(t_2-t_1)\left[\frac{\mathbf{q}_1^2}{2m_1}+\frac{(\mathbf{q}_1-\mathbf{k}_3)^2}{2m_2}+\frac{\mathbf{k}_3^2}{2m_3}\right]+\cdots +(t_{N+2}-t_{N+1})\left[\frac{\mathbf{k}_1'^2}{2m_1}+\frac{(\mathbf{q}_N-\mathbf{k}_1')^2}{2m_2}+\frac{\mathbf{q}_N^2}{2m_3}\right] +(t-t_{N+2})\left[\frac{\mathbf{k}_1'^2}{2m_1}+\frac{\mathbf{k}_2'^2}{2m_2}+\frac{\mathbf{k}_3'^2}{2m_3}\right]. \quad \text{(IV.2)}$$

The action principle tells us that the values of $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_N$ of a realizable physical collision process are those for which the action integral is stationary for fixed initial and final momenta. We have also imposed the constraint that there are $N+2$ binary collisions occurring at times $t_1, t_2, \cdots, t_{N+2}$. This constraint may be incompatible with the equations of motion. If this is the case, then the $\mathbf{q}_i$'s for which $\int \mathcal{L}dt$ is an extremum will not conserve energy for all intermediate states. Setting

$$t_{i+1}-t_i=x_i(t_{N+2}-t_1); \quad i=1, 2, \cdots, N+1,$$

so that

$$x_1+x_2+\cdots+x_{N+1}=1,$$

we find that $\delta\int \mathcal{L}dt=0$ is equivalent to $\delta Q=0$ under variation of the $\mathbf{q}$'s where

$$Q=x_1\left[\frac{\mathbf{q}_1^2}{2m_1}+\frac{(\mathbf{q}_1-\mathbf{k}_3)^2}{2m_2}+\frac{\mathbf{k}_3^2}{2m_3}\right]+\cdots +x_{N+1}\left[\frac{\mathbf{k}_1'^2}{2m_1}+\frac{(\mathbf{q}_N-\mathbf{k}_1')^2}{2m_2}+\frac{\mathbf{q}_N^2}{2m_3}\right].$$

On the other hand, conservation of energy implies

$$Q=(x_1+x_2+\cdots+x_{N+1})E=E,$$

where E is the total kinetic energy. Since Q is a nonnegative quadratic form, it is stationary only at its minimum. It follows that $N+2$ collisions are *impossible* if

$$\min_{(q)} Q<E \quad \text{for all} \quad \mathbf{k}_i\mathbf{k}_i', \quad x_i>0 \quad (\textstyle\sum x_j=1).$$

This is exactly the condition for the absence of anomalous singularities of type R. Conversely, if there is a choice of values of $\mathbf{k}_i$, $\mathbf{k}_i'$ and x_i such that $\min Q>E$, then it is easily seen that there will also be a choice of x's that makes $\min Q$ equal to E so that $N+2$ classical collisions are possible and the scattering amplitude will have anomalous singularities.

It is also amusing to note that the Landau equations

$$\partial Q/\partial \mathbf{q}_i=0; \quad i=1, 2, \cdots, N$$

yield the necessary and sufficient space-time relations between classically realizable binary collisions. To see this let us consider, as an example, the three-collision process depicted in Fig. 8. We have

$$Q=x_1\left[\frac{\mathbf{q}^2}{2m_1}+\frac{(\mathbf{q}+\mathbf{k}_3)^2}{2m_2}+\frac{\mathbf{k}_3^2}{2m_3}\right] +x_2\left[\frac{\mathbf{q}^2}{2m_1}+\frac{(\mathbf{q}+\mathbf{k}_3')^2}{2m_2}+\frac{\mathbf{k}_3'^2}{2m_3}\right],$$

so that

$$\partial Q/\partial\mathbf{q}=x_1[\mathbf{q}/m_1+(\mathbf{q}+\mathbf{k}_3)/m_2]+x_2[\mathbf{q}/m_1+(\mathbf{q}+\mathbf{k}_3')/m_2] =(x_1+x_2)\mathbf{q}/m_1+x_1(\mathbf{q}+\mathbf{k}_3)/m_2+x_2(\mathbf{q}+\mathbf{k}_3')/m_2.$$

Thus the condition $\partial Q/\partial\mathbf{q}=0$ is equivalent to the statement that if particles 1 and 2 collide at time t_1, they will collide again at time t_3,

Let us now turn to the analytic properties of the quantum-mechanical amplitude for the case of unequal masses. The y variables are now defined as

$$\mathbf{y}_i=\mathbf{k}_i(2m_iE)^{-1/2}, \quad \mathbf{y}_i'=\mathbf{k}_i'(2m_iE)^{-1/2}. \quad \text{(IV.3)}$$

The inequalities (II.4) are now replaced by

$$y_i^2, y_i'^2\leq 1-\frac{m_i}{m_1+m_2+m_3}. \quad \text{(IV.4)}$$

In analogy with Eq. (II.8) we introduce the function

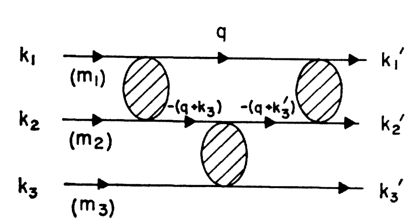

FIG. 8. Graphical representation of a process of three successive binary collisions between three particles of mass m_1, m_2, and m_3.

$Q(\mathbf{q}_1,\mathbf{q}_2,\cdots,\mathbf{q}_N)$ which for a general diagram has the form

$$Q(\mathbf{q}_1,\mathbf{q}_2,\cdots,\mathbf{q}_N)=x_1\left[\frac{\mathbf{k}_i^2}{2m_i}+\frac{(\mathbf{k}_1-\mathbf{q}_1)^2}{2(M-m_i-m^{(1)})}+\frac{\mathbf{q}_1^2}{2m^{(1)}}\right]$$
$$+x_2\left[\frac{\mathbf{q}_1^2}{2m^{(1)}}+\frac{(\mathbf{q}_1-\mathbf{q}_2)^2}{2(M-m^{(1)}-m^{(2)})}+\frac{\mathbf{q}_2^2}{2m^{(2)}}\right]+\cdots$$
$$+x_{N+1}\left[\frac{\mathbf{q}_N^2}{2m^{(N)}}+\frac{(\mathbf{q}_N-\mathbf{k}_j)^2}{2(M-m^{(N)}-m_j)}+\frac{\mathbf{k}_j^2}{2m_j}\right]. \quad \text{(IV.5)}$$

In this expression $m^{(1)}$, $m^{(2)}$, $\cdots$, $m^{(N)}$ are the masses of the particles of intermediate momenta $\mathbf{q}_1$, $\mathbf{q}_2$, $\cdots$, $\mathbf{q}_N$ and $M=m_1+m_2+m_3$. We note the restrictions

$$m_i\neq m^{(1)},\quad m^{(1)}\neq m^{(2)},\quad \vdots \quad m^{(N)}\neq m_j, \quad \text{(IV.6)}$$

arising from our choice of the Green's functions which where combined by the Feynman identity. We seek an upper bound on the quantity

$$g=\min_{(q)}Q(\mathbf{q}_1,\mathbf{q}_2,\cdots,\mathbf{q}_N). \quad \text{(IV.7)}$$

Let $N=2n-1$, where n is a positive integer. We have

$$g\leq\min_{(q)}Q(\mathbf{q}_1,\mathbf{q}_2,\cdots,\mathbf{q}_{N-1},0,\mathbf{q}_{N+1},\cdots,\mathbf{q}_{2N-1})$$
$$=\min_{(q)}L_1(\mathbf{q}_1,\mathbf{q}_2,\cdots,\mathbf{q}_{n-1})+\min_{(q)}L_2(\mathbf{q}_{n+1},\cdots,\mathbf{q}_{2n-1}), \quad \text{(IV.8)}$$

where

$$L_1=x_1\left[\frac{\mathbf{k}_i^2}{2m_i}+\frac{(\mathbf{k}_i+\mathbf{q}_1)^2}{2(M-m_i-m^{(1)})}+\frac{\mathbf{q}_1^2}{2m^{(1)}}\right]+\cdots$$
$$+x_n\mathbf{q}_{n-1}^2\left[\frac{1}{2m^{(n-1)}}+\frac{1}{2(M-m^{(n-1)}-m^{(n)})}\right], \quad \text{(IV.9)}$$

$$L_2=x_{n+1}\mathbf{q}_{n+1}^2\left[\frac{1}{2m^{(n+1)}}+\frac{1}{2(M-m^{(n)}-m^{(n+1)})}\right]+\cdots$$
$$+x_{2n}\left[\frac{\mathbf{q}_{2n-1}^2}{2m^{(2n-1)}}+\frac{(\mathbf{q}_{2n-1}-\mathbf{k}_j)^2}{2(M-m^{(2n-1)}-m_j)}+\frac{\mathbf{k}_j^2}{2m_j}\right]. \quad \text{(IV.10)}$$

Following a procedure analogous to that used to derive (II.14), we find that if $\bar{\mathbf{q}}_1$, $\bar{\mathbf{q}}_2$, $\cdots$, $\bar{\mathbf{q}}_{2n-1}$ are the values of $\mathbf{q}_1$, $\mathbf{q}_2$, $\cdots$, $\mathbf{q}_N$ which minimize L_1, then

$$|\bar{\mathbf{q}}_{n-1}|\leq\frac{m^{(n-1)}}{M-m^{(n-2)}}\frac{m^{(n-2)}}{M-m^{(n-3)}}\frac{m^{(n-3)}}{M-m^{(n-4)}}\times\cdots\frac{m^{(1)}}{M-m_i}|\mathbf{k}_i|. \quad \text{(IV.11)}$$

Maximization of $\min_{(q)}L_1$ with respect to x_1, x_2, $\cdots$, x_n for fixed $x_1+x_2+\cdots+x_n$ requires that the quantities in brackets in Eq. (IV.9) are equal. Therefore

$$\max_{(x)}\{\min_{(q)}L_1\}=\bar{\mathbf{q}}_{n-1}^2\left[\frac{1}{2m^{(n-1)}}+\frac{1}{2(M-m^{(n-1)}-m^{(n)})}\right]\times(x_1+x_2+\cdots+x_n). \quad \text{(IV.12)}$$

Assuming that $m_1\geq m_2\geq m_3$, we can use the relations (IV.12), (IV.11), (IV.4), and (IV.6) to obtain

$$\max_{(x)}\{\min_{(q)}L_1\}\leq(x_1+x_2+\cdots+x_n)\times\frac{m_1m_2}{m_3M}\left[\frac{m_1m_2}{(m_1+m_3)(m_2+m_3)}\right]^{n-2}E. \quad \text{(IV.13)}$$

In terms of the quantity

$$b=\frac{(m_1+m_3)(m_2+m_3)}{m_1m_2}, \quad \text{(IV.14)}$$

we have

$$\max_{(x)}\{\min_{(q)}L_1\}\leq(x_1+x_2+\cdots+x_n)\frac{E}{b^{n-2}(b-1)} \quad \text{(IV.15)}$$

and similarly

$$\max_{(x)}\{\min_{(q)}L_2\}\leq(x_{n+1}+\cdots+x_{2n})\frac{E}{b^{n-2}(b-1)}. \quad \text{(IV.16)}$$

Combining (IV.15) and (IV.16) with (IV.8) we have the bound

$$g\leq\frac{E}{b^{n-2}(b-1)}. \quad \text{(IV.17)}$$

It follows that if

$$b^{n-2}(b-1)>1, \quad \text{(IV.18)}$$

then $2n+1$ successive binary collisions are kinematically impossible.

In the case of five collisions, i.e., $n=2$, an explicit calculation shows that our bound (IV.18) is actually the optimum one: *Five collisions are possible if and only if* $b<2$. Also by construction of concrete examples it can be show that *if the masses are unequal* (no matter how small the deviation from equal masses is) *a fourth collision is always possible.* In general, from (IV.18) we see that for given mass ratios only a finite number of successive collisions are kinematically possible. This number, however, may be made arbitrarily large by varying the mass ratios. One need only visualize the case of a light particle bouncing back and forth between two other particles the masses of which are increased indefinitely.

The proof of analyticity now goes through just as in the equal-mass case. For those diagrams in which $g<E=k^2$ the amplitude can be continued into the entire upper-half k plane and the only singularities will be on the imaginary axis (at some distance from the origin). The diagrams with anomalous singularities will again be analytic in the first quadrant of the k plane and one can define them in the second quadrant by the relation $f(k)=f^*(-k^*)$.

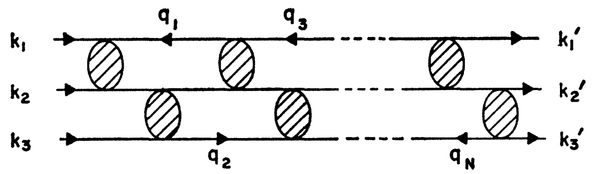

FIG. 9. Diagram representing a typical term in the iteration expansion of the Faddeev equation [see Eq. (V.3)].

Finally, Eq. (III.21) suggests that even for $N>1$ the discontinuity across the anomalous cut (drawn along the imaginary axis as explained at the end of Sec. III) can be obtained by replacing each Green's function in the corresponding Faddeev terms by $2\pi i$ times a delta function. The special prescription for integrating over the delta functions requires an individual study of the integrals involved. However, the important point is, again, that because of the delta functions, the discontinuity across the anomalous cut is given explicitly in terms of the energy-shell two-body T matrices.

V. TWO-PARTICLE BOUND STATES AND ANOMALOUS THRESHOLDS

Up to now we have assumed that both the two- and three-particle scattering amplitudes can be expanded in a perturbation series and we have shown that each term in the series for the three-particle amplitude satisfies a dispersion relation in the total energy. In this section we shall consider the possibility that the two-particle amplitudes have bound-state or resonance poles, and hence the Born series may not converge. As one might expect by looking at the off-energy-shell T matrix, we shall find that the amplitude has "unitarity" branch points at $k^2=-B_j$, where B_1, B_2, $\cdots$ are the binding energies of the two-body bound states. The branch cuts starting at these points and running to the right (i.e., towards increasing k^2) are associated with intermediate states consisting of a bound state and a free particle. In addition, however, for certain values of the y's, a new kind of anomalous branch points will emerge from the $k^2=-B_j$ unitarity branch points, and will thus extend the "right-hand cut" further to the left. Except for this phenomenon all our previous results will continue to hold.

In order to simplify the kinematics we shall only consider the case of equal-mass particles, but the generalization will be obvious. Our starting point is the Faddeev equation.[1] Denoting by f_{ij} that part of the three-body amplitude f in which the ith and jth particles interact last, we have in the usual operator notation

$$f=f_{12}+f_{23}+f_{31},\qquad f_{ij}=\hat{t}_{ij}+\hat{t}_{ij}G_0(f_{ik}+f_{ik});\quad i,\ j,\ k \text{ distinct}. \tag{V.1}$$

Here $\hat{t}_{12}(k^2)$, which is an operator on the three-particle Hilbert space, is related to the off-energy-shell two-particle scattering amplitude $\langle \mathbf{p}'|t_{12}(E)|\mathbf{p}\rangle$ by

$$\langle \mathbf{k}_1'',\mathbf{k}_2'',\mathbf{k}_3''|\hat{t}_{12}(k^2)|\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3'\rangle =\langle \tfrac{1}{2}(\mathbf{k}_1''-\mathbf{k}_2'')|t_{12}(k^2-\tfrac{3}{2}k_3'^2)|\tfrac{1}{2}(\mathbf{k}_1'-\mathbf{k}_2')\rangle \times\delta^3(\mathbf{k}_3''-\mathbf{k}_3'). \tag{V.2}$$

Similar expressions for $\hat{t}_{23}$ and $\hat{t}_{31}$ can be obtained by cyclic permutation of the indices in Eq. (V.2).

Let us first discuss the analyticity properties of the individual terms in the series obtained by iterating Eq. (V.1)

$$f_{ij}=\hat{t}_{ij}+\hat{t}_{ij}G_0(\hat{t}_{jk}+\hat{t}_{ik})+\cdots. \tag{V.3}$$

A typical term in the series is represented by a diagram in Fig. 9. The problem appears similar to the one discussed in Secs. II and III except that the potentials are now replaced by two-body t matrices and the free Green's functions are now situated only between t matrices involving different pairs of particles.

We recall that the amplitudes for two-body scattering by Yukawa potentials has a Fredholm solution[14] of the form

$$\langle \mathbf{p}'|t(E)|\mathbf{p}\rangle=V(\mathbf{p}'-\mathbf{p}) +D^{-1}(E)\int d^3p''\langle \mathbf{p}'|N(E)|\mathbf{p}''\rangle V(\mathbf{p}''-\mathbf{p}), \tag{V.4}$$

where

$$V(\mathbf{p}'-\mathbf{p})=\langle \mathbf{p}'|V|\mathbf{p}\rangle.$$

The expansions

$$D(E)=\sum_{n=0}^{\infty}D_n(E), \tag{V.5}$$

$$\langle \mathbf{p}'|N(E)|\mathbf{p}\rangle=\sum_{n=0}^{\infty}\langle \mathbf{p}'|N_n(E)|\mathbf{p}\rangle, \tag{V.6}$$

are most convenient in the Smithies[15] form:

$$D_0=1;\qquad D_n=\frac{(-1)^n}{n!}\begin{vmatrix} 0 & n-1 & 0 & \cdots & 0 & 0 & 0\\ \sigma_2 & 0 & n-2 & \cdots & 0 & 0 & 0\\ \sigma_3 & \sigma_2 & 0 & \cdots & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \sigma_{n-1} & \sigma_{n-2} & \sigma_{n-3} & \cdots & \sigma_2 & 0 & 1\\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \cdots & \sigma_3 & \sigma_2 & 0 \end{vmatrix} \tag{V.7}$$

[14] R. Jost and A. Pais, Phys. Rev. 82, 840 (1951).

[15] F. Smithies, *Integral Equations* (Cambridge University Press, Cambridge, England, 1958).

$$N_0=VG_0;\quad N_n=\frac{(-1)^n}{n!}\begin{vmatrix} VG_0 & n & 0 & \cdots & 0 & 0 & 0 \\ (VG_0)^2 & 0 & n-1 & \cdots & 0 & 0 & 0 \\ (VG_0)^3 & \sigma_2 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots \\ (VG_0)^n & \sigma_{n-1} & \sigma_{n-2} & \cdots & \sigma_2 & 0 & 1 \\ (VG_0)^{n+1} & \sigma_n & \sigma_{n-1} & \cdots & \sigma_3 & \sigma_2 & 0 \end{vmatrix}, \tag{V.8}$$

where VG_0, $(VG_0)^2$, $\cdots$ are the kernel $V(\mathbf{p}-\mathbf{p}')\times(p'^2-E-i\epsilon)^{-1}$ and its iterates, and

$$\sigma_i=\mathrm{Tr}(VG_0)^i=\int\prod_{j=1}^{i}\frac{d^3p_j}{p_j^2-E-i\epsilon}\times V(\mathbf{p}_1-\mathbf{p}_2)V(\mathbf{p}_2-\mathbf{p}_3)\cdots V(\mathbf{p}_i-\mathbf{p}_1). \tag{V.9}$$

Since σ_i depends on E only through the Green's functions [see expression (V.9)], it is an analytic function of E with a cut on the real axis running from 0 to ∞. The series for D is uniformly convergent, so D will have the same analyticity properties as the σ_i's. Furthermore, the only zeros of D in the physical sheet correspond to bound states and lie on the negative real axis. We can therefore write the following representation for $D^{-1}(E)$:

$$D^{-1}(E)=1+\frac{C}{E+B}+\frac{1}{\pi}\int_0^\infty dE'\frac{\mathrm{Im}D^{-1}(E')}{E'-E} \tag{V.10}$$

because $D(E)\to 1$ as $|E|\to\infty$. In order to simplify the notation we have assumed that there is a single bound state with binding energy B.

Let us now go back to our three-particle problem. We first note that if there are no two-particle bound states, the results of Secs. II and III can be carried over immediately. In other words, the terms of fourth or higher order of the Faddeev expansion (V.3) will be real analytic functions of k^2 with the "unitarity" cut running from 0 to ∞ and a left-hand cut starting at some point to the left of $-\frac{2}{3}\mu^2$. The third-order terms will have the same singularities and in addition the rescattering cut running from $-\infty$ to 0 and the corresponding discontinuities will be given by expressions like (III.25).

To see this we note that according to Eq. (V.2) a typical t matrix in the diagram of Fig. 9 will be of the form

$$\langle\tfrac{1}{2}\mathbf{q}_l-\mathbf{q}_{l+1}|t_{ij}(k^2-\tfrac{3}{2}q_l^2)|\tfrac{1}{2}\mathbf{q}_l-\mathbf{q}_{l-1}\rangle. \tag{V.11}$$

The corresponding D function will satisfy

$$D_{ij}^{-1}(k^2-\tfrac{3}{2}q_l^2)=1+\frac{1}{\pi}\int_0^\infty dE'\frac{\mathrm{Im}D_{ij}^{-1}(E')}{E'-k^2+\frac{3}{2}q_l^2-i\epsilon}. \tag{V.12}$$

As a result the dispersion denominator in D_{ij}^{-1} and the denominators appearing in the σ_i's will be of the same form as the Green's-function denominators in Eqs. (II.25) and (II.26). On the other hand N_nV is just a sum of the first $n+2$ terms in the Born series for the two-particle amplitude, each term being multiplied by an appropriate product of σ's [see Eq. (V.8)]. Therefore the contribution to the Faddeev term (i.e., the diagram of Fig. 9) will be a sum of terms the analytic properties of which are those of the diagrams studied in Secs. II and III. Since the series for NV can be shown to be uniformly convergent (even after the analytic continuation of the individual terms in k^2), it possesses the analyticity properties of the individual terms.

Let us now consider the case in which there are two-particle bound states. We shall first discuss terms of fourth order or higher, i.e., those terms which do not have anomalous singularities of type R (rescattering singularities). We follow the same procedure as in Sec. II, but in the present case it is convenient to include in the Feynman identity the denominators in the D_{ij}^{-1}'s corresponding to bound-state poles [i.e., terms like $C/(E+B)$ in Eq. (V.10)] as well as the Green's functions. The resulting Feynman denominator will be typically of the form

$$Q(\mathbf{q}_i;\,\mathbf{k}_3,\mathbf{k}_1')-k^2-\alpha B-i\epsilon, \tag{V.13}$$

where

$$\begin{aligned} Q=x_1[\mathbf{k}_3^2+(\mathbf{k}_3-\mathbf{q}_1)^2+\mathbf{q}_1^2]+\cdots \\ +x_{N+1}[\mathbf{q}_N^2+(\mathbf{q}_N-\mathbf{k}_1'^2)+\mathbf{k}_1'^2]+x_{N+2}\tfrac{3}{2}\mathbf{q}_{l_1}^2 \\ +x_{N+3}\tfrac{3}{2}\mathbf{q}_{l_2}^2+\cdots+x_{N+M}\tfrac{3}{2}\mathbf{q}_{l_M}^2, \end{aligned} \tag{V.14}$$

$$N\geq M;\quad l_M>l_{M-1}>\cdots>l_1\geq 1,$$

$$\alpha=x_{N+2}+x_{N+3}+\cdots+x_{N+M}.$$

Once again we diagonalize Q by making the change of variables

$$\mathbf{q}_i=\mathbf{q}_i'+\bar{\mathbf{q}}_i, \tag{V.15}$$

such that

$$Q(\mathbf{q}_i'+\bar{\mathbf{q}}_i)=\sum A_{ij}\mathbf{q}_i'\cdot\mathbf{q}_j'+g, \tag{V.16}$$

where $g=g'k^2$ is the minimum of Q under variation of the $\mathbf{q}$'s. We note that the bounds on $\bar{\mathbf{q}}_i$ given in Eqs. (II.15), (II.16), and (II.17) still hold.

For terms in which $l_1\neq 1$ and $l_M\neq N$, i.e., for terms in which there are no bound-state poles in the first or the last loop of the diagram, we have from (II.12) and (II.13) that

$$g'<x_1+x_2+\cdots+x_{N+1}=1-\alpha,$$

so that the right-hand cut starts at $k^2=-B$. However for $l_1=1$ or $l_M=N$ we need a better bound on g. We consider the case $l_1=1$, $l_M\neq N$. Then

$$g=\min Q(\mathbf{q}_1,\mathbf{q}_2,\cdots,\mathbf{q}_N)\leq Q\left(\frac{x_1\mathbf{k}_3}{2x_1+\frac{3}{2}x_{N+2}},0,0,\cdots,\tfrac{1}{2}\mathbf{k}_1'\right)$$
$$\leq\frac{x_1k_3^2}{(2x_1+\frac{3}{2}x_{N+2})^2}[x_1^2+(x_1+\tfrac{3}{2}x_{N+2})^2+(2x_1+\tfrac{3}{2}x_{N+2})^2+2x_1x_2+\tfrac{3}{2}x_1x_{N+2}]+\tfrac{3}{4}(x_N+x_{N+1})k_1'^2$$
$$\leq(x_1+x_2+x_N+x_{N+1})k^2+\frac{1}{2}\frac{x_1x_{N+2}}{(2x_1+\frac{3}{2}x_{N+2})}k^2. \quad \text{(V.17)}$$

Inequality (V.17) yields $g'\leq 1$, but the best bound on $\alpha(1-g')^{-1}$ that one can obtain is[16]

$$\alpha/(1-g')\leq\tfrac{4}{3}. \quad \text{(V.18)}$$

This means that the right-hand cut in the k^2 plane begins at some point to the right of $k^2=-(\frac{4}{3})B$. No other singularities are found in the strip

$$0<\mathrm{Im}k<\mu/2\max(\nu_i)<(\tfrac{2}{3})^{1/2}\mu. \quad \text{(V.19)}$$

We can now proceed to prove analyticity in the entire first quadrant of the k plane. Because of the B term in Eq. (V.13) the argument used at the end of Sec. II is slightly modified. In the present case, by a rotation of the integration paths of the q variables by an angle $\psi(0<\psi<\pi/2)$, our integral is well defined and analytic in the strip

$$(\tfrac{4}{3}B)^{1/2}\cos\psi<\mathrm{Im}(e^{-i\psi}k)<(\tfrac{2}{3})^{1/2}\mu\cos\psi. \quad \text{(V.20)}$$

In order to guarantee the existence of this strip we make the natural assumption that

$$\tfrac{4}{3}B<\tfrac{2}{3}\mu^2, \quad \text{(V.21)}$$

which implies that a "gap" exists in the k^2 plane between the right-hand cuts and the left-hand ("potential") cuts. In analogy with the discussion in Sec. II, by means of successively overlapping strips like (V.20), the integral can be analytically continued in the entire first k quadrant.

We now go back to the question of whether any "anomalous" thresholds are indeed located between $k^2=-(\frac{4}{3})B$ and the normal threshold at $k^2=-B$. In order to answer this question and to locate the anomalous thresholds we shall employ the Landau-Bjorken[5] [LB] methods. First, it is important to note that, as follows from the previous discussion, no deformation of the integration paths of the Feynman parameters is necessary in order to continue analytically in the first quadrant of the k plane. Therefore, in looking for branch points located anywhere on the imaginary axis, we need consider only *real positive* values of the Feynman parameters in the LB equations. This is important because it means that for $k=iK$, K real, we can set $\mathbf{q}_i=i\mathbf{p}_i$ and thus realize the LB conditions in terms of dual diagrams in *real Euclidean p space*. By analyzing these dual diagrams it is also possible to obtain all the branch points on the imaginary axis that lie above $i\mu(\frac{2}{3})^{1/2}$. These will be considered in some detail in a future publication. The anomalous thresholds that we are looking for here arise from a "pinch" between the Green's-function denominators which were combined in (V.13). We must, therefore, consider the LB conditions in which all other denominators are "reduced," i.e., their Feynman parameters are set equal to zero, Accordingly, the dual of diagram 9 is shown in Fig. 10. The LB equations require that all vectors lie in a plane. The positivity of the Feynman parameters implies that the polygonal line $CA_1A_2A_3\cdots A_ND$ is *convex* and lies *inside* the triangle OCD. The LB equations require in addition that

$$|A_1C|=|\mathbf{p}_1-\tfrac{1}{2}K\mathbf{y}_3|=\tfrac{1}{2}Ky_{12}, \quad \text{(V.22)}$$

$$|A_ND|=|\mathbf{p}_N-\tfrac{1}{2}K\mathbf{y}_1'|=\tfrac{1}{2}Ky_{23}', \quad \text{(V.23)}$$

$$|OA_i|^2+|A_iA_{i+1}|^2+|OA_{i+1}|^2=\mathbf{p}_i^2+(\mathbf{p}_i-\mathbf{p}_{i+1})^2+\mathbf{p}_{i+1}^2=K^2;\quad i=1,2,\cdots,(N-1). \quad \text{(V.24)}$$

The possible "pinching" of a bound-state denomina-

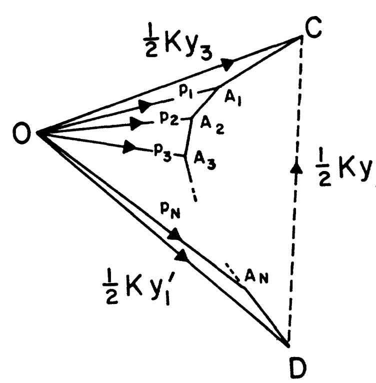

FIG. 10. The dual diagram corresponding to the diagram of Fig. 9.

[16] The same result (V.18) can be similarly derived for the cases $l_1\neq 1$, $l_M=N$ and $l_1=1$, $l_M=N$.

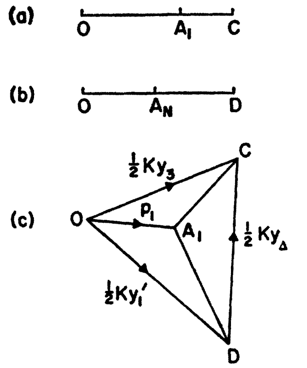

FIG. 11. The three possible dual diagrams (for equal masses) satisfying the Landau-Bjorken conditions.

tor of the form $\frac{3}{2}p_i^2-K^2+B$ would require in addition

$$|OA_i|^2=p_i^2=\tfrac{2}{3}(K^2-B)\leq\tfrac{2}{3}K^2. \qquad (V.25)$$

Reduced diagrams corresponding to, e.g., $x_i=0$ may also be possible. In such a case the polygonal line breaks in two. The points $A_1, A_2, \cdots, A_i$ must now lie on the segment OC and the points $A_{i+1}, \cdots, A_N$ must lie on the segment OD.

Given all these geometrical constraints, it is a relatively easy matter to verify that there are only three possible kinds of dual diagrams shown in Figs. 11(a), 11(b), and 11(c). Of these, 11(c) is associated with third-order Faddeev terms only.

Diagram 11(a) is the reduced diagram corresponding to a "pinch" of only two denominators: those of the first (i.e., nearest to the initial state) Green's function and a bound-state pole of the subsequent t_{23} matrix. From the diagram 9(a), the pinching condition is

$$|OA_1|+|A_1C|=|OC|. \qquad (V.26)$$

From Eqs. (V.22), (V.25), and (V.26) we obtain the position of the branch point at

$$K^2=A_{23}=\frac{B_{23}}{1-\frac{3}{8}(y_3-y_{12})^2};\quad y_3^2\geq\tfrac{1}{2}, \qquad (V.27)$$

where B_{32} is the binding energy of the (3,2) bound state. We note that for $y_3<\sqrt{2}/2$ (i.e., $y_3<y_{12}$) the point A_1 is *not* between O and C which means that one of the Feynman parameters is negative. The branch point is not on the imaginary axis yet. For $y_3=\sqrt{2}/2$ (i.e., $y_3=y_{12}$) we have $A_{32}=B_{32}$ and on further increase of y_3, the branch point A_{32} moves along the imaginary axis according to (V.27). This is the familiar case of an anomalous threshold emerging out of another Riemann sheet through a normal threshold. In fact, it is straightforward to show that A_{23} is the nonrelativistic limit of the anomalous threshold of the field-theoretic Feynman diagram shown in Fig. 12(a).

In a completely analogous fashion, diagram 11(b) corresponds to a "pinch" between the last (i.e., nearest to the final state) Green's function and a bound-state pole term of the preceding t_{12} matrix. The corresponding branch point is at

$$K^2=A_{12}'=\frac{B_{12}}{1-\frac{3}{8}(y_1'-y_{23}')^2};\quad y_1'^2>\tfrac{1}{2}, \qquad (V.28)$$

where B_{12} is the binding energy of the (1,2) bound state. This branch point emerges at the normal threshold (at $K^2=B_{12}$) and moves to the left as y_1' increases from the value $\sqrt{2}/2$. It is the nonrelativistic limit of the anomalous threshold of the Feymnan diagram shown in Fig. 12(b).

In the following we shall call the branch points associated with diagrams like 11(a) and 11(b) *anomalous thresholds of type B*. Associated with each two-particle bound state (of binding energy B say) there are two such thresholds at $k^2=-A$ and $k^2=-A'$ present in every order of the Faddeev expansion (V.3). From (V.27) and (V.28) we obtain A, $A'\leq\frac{4}{3}B$ as already anticipated from Eq. (V.18).

Having located the anomalous thresholds, we note that the discussion given in Sec. II and at the beginning of this section for the case of no bound states still holds. If $\frac{2}{3}\mu^2>\frac{4}{3}\max B_{ij}$ there will be a gap between the right- and left-hand cuts in the k^2 plane. There will also be poles at $k^2=-2B_{ij}/y_{ij}^2$ arising from the two-particle bound-state poles in the initial and final two-particle T matrices. Therefore we may write down a dispersion relation of the form

$$f(k^2)=\frac{1}{\pi}\left\{\int_{-\infty}^{-2\mu^2/3}+\int_{-A}^{\infty}\right\}dk'^2\frac{\operatorname{Im}f(k'^2)}{k^{2\prime}-k^2}$$
$$+(\text{pole terms at } k^2=-2B_{ij}/y_{ij}^2) \qquad (V.29)$$

for every term of fourth order or higher in (V.3). In Eq. (V.26), A is the anomalous threshold located farthest to the left ($A\leq\frac{4}{3}\max B_{ij}<2\mu^2/3$).

The imaginary part along the unitarity cut is determined by the unitarity relation in terms of on-the-energy-shell amplitudes for three-particle scattering and bound-state breakup and pickup. On the other hand, the discontinuity across one of the anomalous cuts is formally given by a Cutkosky prescription, namely, by replacing the relevant Green's function and the bound-

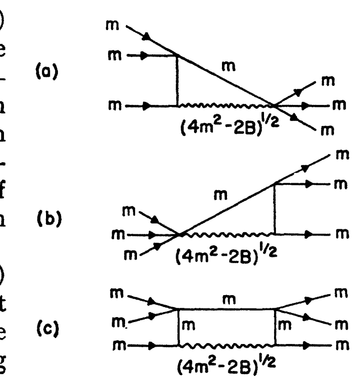

FIG. 12. In the nonrelativistic limit the leading Landau singularities of these Feynman diagrams reduce to those associated with the dual diagram of Fig. 11. The bound state of binding energy B is taken relativistically as a particle of mass $(4m^2-2B)^{1/2}$.

state pole term by delta functions. The result appears again expressible (because of the delta functions) in terms of on-the-energy-shell amplitudes evaluated for unphysical values of the energy. It is clear, therefore, that a study of the analyticity properties of bound-state breakup, etc. amplitudes is required in order to make the prescription meaningful. We shall postpone the study of these amplitudes to a future publication. Here, we only state, without proof, that the required analytic continuation is possible.

Finally, we consider the third-order terms in the Faddeev expansion (V.3). To be specific we may, without loss of generality, consider the term $t_{12}G_0t_{23}G_0t_{12}$ represented by the diagram of Fig. 8.

Let us assume that there is a bound state of the pair (2,3) of binding energy B_{23}. Then, according to the previous discussion, $t_{12}G_0t_{23}G_0t_{12}$ will have two anomalous thresholds at

$$k^2=-A_{23}=-\frac{B_{23}}{1-\frac{3}{8}(y_3-y_{12})^2}\quad\text{for}\quad y_3{}^2\geq\tfrac{1}{2}\qquad\text{(V.30)}$$

and

$$k^2=-A_{23}'=-\frac{B_{23}}{1-\frac{3}{8}(y_3'-y_{12}')^2}\quad\text{for}\quad y_3{}^2\geq\tfrac{1}{2}.$$

In addition, however, there will be a branch point associated with the dual diagram 11(c) and arising from a pinch between both of the Green's functions and the pole term in t_{23}. From the LB equations we have

$$\begin{aligned}|OA_1|^2&=\tfrac{2}{3}(K^2-B_{23}),\\|A_1C|&=\tfrac{1}{2}Ky_{12},\\|A_1D|&=\tfrac{1}{2}Ky_{12}'.\end{aligned}\qquad\text{(V.31)}$$

From the geometry of the dual diagram, it follows that the necessary and sufficient conditions for the appearance of this branch point on the imaginary k axis are the simultaneous inequalities.

$$\begin{aligned}&1>\frac{y_{12}{}^2+y_\Delta{}^2-y_{12}'^2}{2y_{12}y_\Delta}>\frac{y_3{}^2+y_\Delta{}^2-y_3'^2}{2y_3y_\Delta}>-1,\\&1>\frac{y_{12}'+y_\Delta{}^2-y_{12}{}^2}{2y_{12}'y_\Delta}>\frac{y_3'^2+y_\Delta{}^2-y_3{}^2}{2y_3'y_\Delta}>-1,\end{aligned}\qquad\text{(V.32)}$$

which ensure that the triangle A_1CD is geometrically possible and that the point A_1 is inside the triangle OCD.

If inequalities (V.32) are satisfied, then the branch point is located at

$$k^2=-A_{23}''=-B_{23}/(1-3x/8),\qquad\text{(V.33)}$$

where x is the smaller root of the equation

$$\begin{aligned}&y_\Delta{}^2x^2-[(y_\Delta{}^2-y_3{}^2-y_3'^2)(y_\Delta{}^2-y_{12}{}^2-y_{12}'^2)\\&\quad-2y_3{}^2y_{12}'^2-2y_3'^2y_{12}{}^2]x\\&\quad+(y_3{}^2-y_{12}{}^2)^2y_3'^2+(y_3'^2-y_{12}'^2)^2y_3{}^2\\&\quad-(y_\Delta{}^2-y_3{}^2-y_3'^2)(y_3{}^2-y_{12}{}^2)(y_3'^2-y_{12}'^2)=0.\end{aligned}\qquad\text{(V.34)}$$

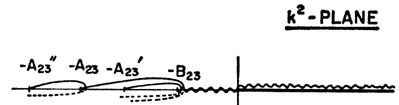

FIG. 13. Anomalous thresholds associated with a bound state of binding energy B_{23}. The figure shows the location and mutual dependence of the various cuts.

It follows from Fig. 9(c) and Eqs. (V.28) that

$$\tfrac{4}{3}B_{23}\geq A_{23}''\geq A_{23},\ A_{23}'\geq B_{23}\qquad\text{(V.35)}$$

and that, under variation of the y's, this branch point can disappear in one of the following three ways:

(i) It can move to the left and disappear into another Riemann sheet through the anomalous threshold of type B at $k^2=-A_{23}$. This corresponds to the point A crossing side OC of the triangle OCD.

(ii) Similarly it can disappear through the threshold at $k^2=-A_{23}'$ as A crosses side OD.

(iii) It can cross side CD. This happens when $y_{12}+y_{12}'=y_\Delta$, which is exactly when the rescattering cut appears (anomalous singularity of type R) and our branch point disappears through it. We shall refer to this branch cut (i.e., the one associated with the dual diagram of Fig. 11(c) as an *anomalous singularity of type BR*. It is straightforward to show that it is the nonrelativistic limit of the leading anomalous cut of the Feynman graph shown in 12(c).

The positions of the various thresholds in the k^2 plane associated with a bound state of binding energy B_{23} in t_{23} are illustrated in Fig. 13. The discussion in Sec. III can now be carried over to show that $t_{12}G_0t_{23}G_0t_{12}$ is free of singularities in the first quadrant of the k plane. For $y_\Delta<y_{12}+y_{12}'$ we can continue through the imaginary k axis [*above* the anomalous threshold at $i(A_{23}'')^{1/2}$ and *below* $i(\frac{2}{3})^{1/2}\mu$] into the second quadrant which is also free of singularities and we have $f(k)=f^*(-k^*)$. For $y_\Delta>y_{12}+y_{12}'$, the rescattering cut appears and the anomalous threshold of type BR is no longer there. There still remain, however, the anomalous cuts of type B and the unitarity cuts.

Having studied the analyticity of the individual terms in the iteration series (V.3), we would like to extend our results to the full three-particle amplitude. We shall give here only a brief outline of the procedure. If the Faddeev equation in matrix form,

$$f=\hat{t}+Kf,\qquad\text{(V.36)}$$

where

$$f=\begin{pmatrix}f_{12}\\f_{23}\\f_{31}\end{pmatrix},\quad\hat{t}=\begin{pmatrix}\hat{t}_{12}\\\hat{t}_{23}\\\hat{t}_{31}\end{pmatrix},\quad K=\begin{pmatrix}0&\hat{t}_{12}G_0&\hat{t}_{12}G_0\\\hat{t}_{23}G_0&0&\hat{t}_{23}G_0\\\hat{t}_{31}G_0&\hat{t}_{31}G_0&0\end{pmatrix},$$

is iterated once, one obtains

$$f'=K\hat{t}+K^2\hat{t}+K^2f',\qquad\text{(V.37)}$$

where

$$f'=f-\hat{t}.\qquad\text{(V.38)}$$

For values of the energy k^2 off the positive real axis, K^2 is a Schmidt kernel. Therefore, the off-the-energy-shell quantity

$$\langle \mathbf{q}_1', \mathbf{q}_2', \mathbf{q}_3' | (1-K^2)^{-1} | \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \rangle \qquad \text{(V.39)}$$

for k^2 off the positive real axis, is given by an absolutely and uniformly convergent Fredholm formula. The operator f' can be written formally as

$$\begin{aligned} f'(k^2) &= [1+K^2(1-K^2)^{-1}K^2] \sum_{i=1}^{4} K^i \hat{t} \\ &= A + B(1-K^2)^{-1}C. \end{aligned} \qquad \text{(V.40)}$$

If one takes matrix elements of f' between plane wave states, all of the dependence on the external momenta will be in A, B, and C so that only the off-the-energy-shell matrix elements of $(1-K^2)^{-1}$ are required. By applying the Feynman identity to the free Green's functions and bound state poles in B and C and simultaneously rotating the paths of integration as described in connection with Eq. (V.29), one can show that the on-energy-shell matrix elements of f' are given by the ratio of two absolutely and uniformly convergent series.[17]

Having established a uniformly and absolutely convergent expression for the three-particle amplitude we see that the only singularities of the full three-particle amplitude which are not present in the iteration expansion of the Faddeev equation (V.3) are poles corresponding to zeros of the Fredholm denominator D. On the physical sheet these poles will, of course, correspond to the three-particle bound states. Any zeros lying on a different Riemann sheet reached through the three-particle unitarity cut in D will correspond to three-particle resonances.

VI. SUMMARY OF RESULTS AND DYNAMICAL CONSIDERATIONS

We have shown that the energy-shell T matrix element for three-particle scattering

$$\langle \mathbf{y}_1', \mathbf{y}_2', \mathbf{y}_3' | T(E) | \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \rangle$$

or, for brevity, $\langle y' | T(E) | y \rangle$ is analytic in the upper-half E plane for fixed physical values of the y variables defined by Eq. (I.1). The physical region is reached on approaching the positive real E axis from above. We then *defined* $\langle y' | T(E) | y \rangle$ in the lower half plane by the "reality" condition

$$\langle y' | T(E) | y \rangle = \langle y' | T(E^*) | y \rangle^*, \qquad \text{(VI.1)}$$

thereby introducing various cuts along the real E axis. Because of (IV.1), the sum of the discontinuities across these cuts at any point of the real axis equals $2i \operatorname{Im}\langle y' | T(E) | y \rangle$. We classify these cuts as follows.

(i) *Unitarity cuts.* They are associated with various kinds of intermediate asymptotic states appearing in the extended unitarity relations [which are also valid for the off-shell amplitude]. Thus we have

(a) A branch cut from $E=0$ to $+\infty$ corresponding to intermediate states of three free particles. The discontinuity across this cut is given by

$$-2\pi i E^2 \int [dy''] \langle y'' | T(E) | y' \rangle^* \langle y'' | T(E) | y \rangle, \qquad \text{(VI.2)}$$

where

$$\begin{aligned} [dy] = (8m_1m_2m_3)^{3/2} \delta(\mathbf{y}_1(2m_1) + \mathbf{y}_2(2m_2) + \mathbf{y}_3(2m_3)) \\ \times \delta(y_1^2+y_2^2+y_3^2-1) d^3y_1 d^3y_2 d^3y_3. \end{aligned} \qquad \text{(VI.3)}$$

(b) A cut from $E=-B_{ij}$ to $+\infty$ for each bound state of the pair (i,j) with binding energy B_{ij}. The discontinuity across this cut is given by

$$\begin{aligned} -2\pi i \int d^3q\, \delta\!\left(\frac{m_1+m_2+m_3}{2m_k(m_i+m_j)} q^2 - B_{ij} - E\right) \\ \times \langle \mathbf{k}_1', \mathbf{k}_2', \mathbf{k}_3' | T_{(ij)}' | \mathbf{q} \rangle^* \langle \mathbf{q} | T_{(ij)} | \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \rangle, \end{aligned} \qquad \text{(VI.4)}$$

where $T_{(ij)}$ and $T_{(ij)}'$ are the T matrices for the transitions $1+2+3 \rightarrow (i,j)+(k)$ and $(i,j)+(k) \rightarrow 1+2+3$, respectively. The vector $\mathbf{q}$ is the momentum of particle (k) in the intermediate state.

(ii) *Anomalous cuts.* The associated discontinuities are not related to physical amplitudes via unitarity relations. However, as we have seen, they can still be expressed in terms of energy-shell amplitudes evaluated at unphysical values of their energy and momentum transfer variables. We have investigated explicitly the equal-mass case, but the extension to the general-mass case is straightforward by the same methods. We have distinguished three kinds of anomalous singularities.

(a) *Type R.* They arise essentially from singularities in momentum transfer variables and are related to the possibility of successive binary rescatterings of the three particles. The cuts run from $E=-\infty$ to 0 and are present only for values of the y variables within a proper subdomain of their physical domain of variation. Only those terms of the Faddeev iteration expansion (V.3) of order $n \leq n_0 =$ (number of kinematically possible successive binary collisions) contribute to the discontinuity across this cut. In the equal-mass case $n_0=3$ and the discontinuity is symbolically given by

$$-\sum_{(i,j,k \text{ distinct})} \{\hat{t}_{ij}(\Delta G_0)\hat{t}_{jk} + \hat{t}_{ij}(\Delta G_0)\hat{t}_{jk}(\Delta G_0)[\hat{t}_{ij}+\hat{t}_{ik}]\}, \qquad \text{(VI.5)}$$

where ΔG_0 is a delta function times $2\pi i$ which has replaced a free Green's function according to a Cutkosky

[17] In a separate paper we will present a detailed proof of the convergence of the Fredholm series for the energy-shell scattering amplitude of 3 particles interacting via superpositions of Yukawa potentials.

prescription (see Sec. III). Analogous formulas (with n_0-1 terms) hold for $n_0>3$.

(b) *Type B*. To each bound state of the pair (i,j) with binding energy B_{ij} there correspond two anomalous branch points at $E=-A_{ij}$ and $E=-A_{ij}'$ [see Eq. (V.34) for the equal-mass case]. They emerge at certain values of the y's and the y''s, respectively, into the physical sheet through the normal threshold at $E=-B_{ij}$ and and move to the left. The associated cuts lie along the intervals $(-A_{ij}, -B_{ij})$ and $(-A_{ij}', -B_{ij})$ and the discontinuities are symbolically[18]

$$(\hat{t}_{ik}+\hat{t}_{jk})(\Delta G_0)(\Delta\hat{t}_{ij})G_0(f_{ik}+f_{jk}), \qquad \text{(VI.6)}$$

and

$$(f_{ik}'+f_{jk}')G_0(\Delta\hat{t}_{ij})(\Delta G_0)(\hat{t}_{ik}+\hat{t}_{jk}). \qquad \text{(VI.7)}$$

The explicit form of $\hat{t}_{12}(\Delta G_0)(\Delta\hat{t}_{23})G_0(f_{12}+f_{13})$, for example, is

$$-(2\pi i)^2\int d^3q\left\langle\frac{m_2\mathbf{k}_1'-m_1\mathbf{k}_2'}{m_1+m_2}\right|$$
$$\times t_{12}\left(E-\frac{m_1+m_2+m_3}{2m_3(m_1+m_2)}k_3^2\right)\left|\mathbf{q}+\frac{m_1}{m_1+m_2}\mathbf{k}_3\right\rangle$$
$$\times R_{23}^{1/2}\delta\left(\frac{q^2}{2m_1}+\frac{(\mathbf{q}+\mathbf{k}_3{}^2)}{2m_2}+\frac{k_3^2}{2m_3}-E\right)$$
$$\times\delta\left(E+B_{23}-\frac{m_1+m_2+m_3}{2m_1(m_2+m_3)}q^2\right)\langle\mathbf{q}|T_{(23)}|\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\rangle, \qquad \text{(VI.8)}$$

where R_{23} is the residue of the bound-state pole in t_{23}. In complete analogy to Eq. (III.21) a prescription for evaluating expression (VI.8) is the following: (1) use spherical coordinates $|\mathbf{q}|$, θ, φ where θ is the angle between $\mathbf{q}$ and $\mathbf{k}_3$, (2) perform the $|\mathbf{q}|$ integration via the second delta function, and then (3) perform the $\cos\theta$ integration via the first delta function by ignoring the restriction $-1<\cos\theta<1$.

(c) *Type BR*. Associated with two-body bound states, there is still another kind of anomalous thresholds arising from two-dimensional dual diagrams like 11(c). They are confined to the first few terms $(2<n\leq n_0)$ of the Faddeev expansion. In the equal-mass case, which we have studied in detail, only one such diagram is possible: the one shown in Fig. 9(c). The branch point, at $E=-A_{ij}''$, is given by Eqs. (V.37) and (V.38) and the branch cut, extending over the segment $[-A_{ij}'', \max(A_{ij},A_{ij}')]$ has the discontinuity

$$(\hat{t}_{ik}+\hat{t}_{jk})(\Delta G_0)(\Delta t_{ij})(\Delta G_0)(\hat{t}_{ik}+\hat{t}_{jk})$$
$$(i,\ j,\ k\ \text{distinct}). \qquad \text{(VI.9)}$$

[18] Here $\Delta\hat{t}_{ij}$ stands for the "discontinuity" of $\hat{t}_{ij}$ at its bound-state pole, i.e., $V_{ij}|\psi_{ij}\rangle\langle\psi_{ij}|V_{ij}$ times the delta function [ψ_{ij} is the bound-state wave function]. The amplitudes f_{ij} and f_{ij}' represent the sum of terms in which V_{ij} acts last and first, respectively. Note that $\langle\psi_{ij}|V_{ij}G_0(f_{ik}+f_{jk})$ is just $T_{(ij)}$.

(iii) *Potential cuts*. They lie along the negative real axis. In the equal-mass case, for example, they lie to the left of $E=-2\mu^2/3$. In the language of perturbation theory, we can say that the corresponding branch points arise from "pinches" involving a number of potential denominators. In the two-body problem, the analogous left-hand cuts play the role of "forces" in dynamical calculations based on dispersion relations. In the three-body case, although we have not give the positions of these branch points in detail, they can be found by a straightforward application of the Landau-Bjorken method. In this connection it is important that the dual diagrams can be constructed in a real three-dimensional Euclidean space, as discussed in Sec. V.

(iv) *Kinematical cuts*. These are simply due to a "kinematical" factor of $E^{-3/2}$ in the "disconnected" parts of the amplitude [first-order terms in (V.3)]. For example,

$$\hat{t}_{12}=\hat{t}_{12}(E)\delta^3(\mathbf{k}_3'-\mathbf{k}_3)$$
$$=(2m_3E)^{-3/2}t_{12}(E)\delta^3(\mathbf{y}_3'-\mathbf{y}_3). \qquad \text{(VI.10)}$$

We take the cut of $E^{-3/2}$ to run from $-\infty$ to 0. The associated discontinuity of $\hat{t}_{12}$ is simply equal to $2i(-2m_3E)^{-3/2}\hat{t}_{12}$.

Finally, aside from branch points, the three-particle amplitude has poles on the negative real E axis corresponding, first of all, to three-particle bound states. In addition, for each bound state of the pair (i,j) with binding energy B_{ij} there are two poles at

$$E=-B_{ij}\Big/\left[1-y_k^2\left(\frac{m_1+m_2+m_3}{m_i+m_j}\right)\right],$$

and $\qquad$ (VI.11)

$$E=-B_{ij}\Big/\left[1-y_{k'}\left(\frac{m_1+m_2+m_3}{m_i+m_j}\right)\right],$$

where i, j, k are distinct. These poles are simply due to the existence of the corresponding bound-state poles in the initial and final two-body t matrices in every term of the Faddeev expansion. Their residues are proportional to $T_{(ij)}'$ and $T_{(ij)}$, respectively.

These analyticity properties of the amplitude can now be combined with the extended unitarity relations for the purpose of dynamical calculations by the N/D method. The basis for such a calculation is the knowledge of the total discontinuity across the negative real axis which we denote by

$$2i\langle y'|\alpha(E)|y\rangle. \qquad \text{(VI.12)}$$

Let us assume that there are no two-body bound states. Then $\alpha(E)$ is the sum of (i) the rescattering discontinuity which is known in terms of the two-body amplitudes, (ii) the kinematical discontinuity of the disconnected parts, and (iii) the discontinuity across the "potential" cut which, as in the two-body case, represents the "forces" and is assumed given. In practice, it will be approximated by the discontinuity of only the first few terms of the Faddeev expansion. In a more ambitious, relativistic framework one would wish to relate these forces to physical amplitudes for "crossed" reactions.

We now write our amplitude as

$$\langle y'|T(E)|y\rangle=\int\langle y''|N(E)|y'\rangle[dy'']\langle y''|D^{-1}(E)|y\rangle. \tag{VI.13}$$

In this formalism T, N, and D appear as E-dependent integral operators on functions of the y's. The measure of integration $[dy]$ is as specified by (VI.3).

Putting the unitarity cut in D and the cut across the negative real E axis, in N we have the following coupled equations for N and D:

$$\langle y'|N(E)|y\rangle=\frac{1}{\pi}\int_{-\infty}^{0}dE'\,\frac{E'}{E}\frac{\langle y'|(\alpha D)(E')|y\rangle}{E'-E}, \tag{VI.14}$$

$$\langle y'|D(E)|y\rangle=\langle y'|y\rangle+\int_{0}^{\infty}dE'\,E'^2\frac{\langle y'|N(E')|y\rangle}{E'-E}. \tag{VI.15}$$

The factor E'/E on the right-hand side of Eq. (VI.14) was introduced to ensure convergence of the integral because $\alpha\sim E^{-3/2}$ for $E\to 0$. We have assumed that N and D vanish sufficiently rapidly as $E\to\infty$, so that no subtractions are required. It should be emphasized that if subtractions were necessary, the subtraction "constants" would in principle be arbitrary functions of the y's.

From Eqs. (VI.14) and (VI.15) we obtain the following integral equation for $N(E)$:

$$N(E)=L(E)+\int_{0}^{\infty}dE'\,E'^2\frac{L(E')-L(E)}{E'-E}N(E'), \tag{VI.16}$$

where

$$L(E)=\frac{1}{\pi}\int_{-\infty}^{0}dE'\,\frac{E'}{E}\frac{\alpha(E')}{E'-E}. \tag{VI.17}$$

Equation (VI.16) is still a singular equation because $\alpha(E)$ and therefore $L(E)$ contains the delta function terms corresponding to the disconnected parts. It can, however, be reduced to a Fredholm equation in a straightforward way [see Ref. 2]. Having obtained $N(E)$ we can calculate $D(E)$ from (IV.15). The final step is to invert $D(E)$, which again amounts to calculating the resolvent of a kernel acting on a space of functions of many variables. For instance, even if one is working with states of definite total angular momentum, two of the y variables and two of y' variables will be independent, so one will have to solve a two-dimensional integral equation in order to invert D. It is perhaps worth noting that even in the so-called "determinantal approximation" in which

$$N(E)\cong L(E),$$

and

$$D(E)\approx I+\int_{0}^{\infty}dE'\,\frac{E'^2}{E'-E}L(E'),$$

one will have to face the problem of inverting D.

If there are two-particle bound states, the unitarity relations will involve the various amplitudes for scattering of one particle off a bound state of the other two. Therefore, if we wish to perform a pure S-matrix calculation, we must treat the channels for bound-state scattering, rearrangement, breakup and pickup on an equal footing with the channel for the scattering of three free particles. Thus we have to study the analyticity properties of the corresponding amplitudes in order to be able to write down multichannel ND^{-1} equations. We plan to do so at a later time. However, if one is willing to take the bound-state pickup and breakup amplitudes as given (in their respective energy planes), then one can formulate ND^{-1} equations analogous to Eqs. (VI.14) and (VI.15) and solve for the three-particle amplitude in terms of these given amplitudes.

Aside from the possibility of N/D calculations, another point of physical significance is reflected in the analytic structure of the three-particle amplitude and specifically in the existence of the rescattering singularities which are essentially singularities in momentum transfer variables in the physical region. They will not, of course, prevent us from projecting states of definite total angular momentum for the study, for example, of three-particle bound states or resonances. On the other hand, because of the rescattering singularities, the partial-wave expansion is not expected to converge uniformly. It is interesting to realize that in this respect the case of two-body scattering [via short-range forces] is unique: There the high angular momenta are suppressed because they correspond to large impact parameters. In three-particle scattering (and in fact for scattering of any number of particles greater than two) the rescattering mechanism makes important contributions to all partial waves even at low energies. If we denote by T_R the contribution of the rescattering singularities to the amplitude T, so that $T-T_R$ is free of singularities in the physical region, we can expand $T-T_R$ in partial waves and write

$$T=T_R+\sum_{J=0}^{\infty}(T-T_R)_J.$$

We can then approximate T by truncating the series for $T-T_R$. This approximation scheme is feasible because, as we have seen, T_R is given in terms of the two-body amplitudes.

In conclusion we would like to note that the elementary methods employed in this paper can be carried over to the general case of N-particle scattering in a straightforward way. However, the challenging question is whether similar ideas would still be fruitful in an investigation of fully relativistic amplitudes fulfilling the requirements of Lorentz invariance and crossing symmetry.

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Neutron-Proton Mass Difference According to the Bound-State Model

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The proton must be heavier than the neutron according to any correct calculation treating them as pion-nucleon bound states whose masses are shifted because of one-photon-exchange corrections to the binding forces, and which takes into account no particles other than pions and nucleons. The reason is simply that only the neutron can contain two charged constituents, p and π^-, and that both the electric and the magnetic forces between these are attractive, thus binding the neutron more tightly. Dashen's previous calculation of this effect was based on an unreliable variant of the Dashen-Frautschi method for eliminating infrared divergences; the sources of the mistakes in that calculation are pointed out. On the basis of the pion-nucleon bound-state picture, we give a simple and physically well-based estimate of the Coulomb contribution to the mass splitting in terms of the pion and nucleon form factors; as compared with experiment it has the right order of magnitude but necessarily the wrong sign. The magnetic energy is more difficult to estimate, apart from its sign; but it is probably much smaller. We conclude that a consistent calculation, if it is to be successful, must include other baryons and mesons. As a by-product we obtain a simple dynamical interpretation of the fact that the neutron's charge form factor is very small.

1. INTRODUCTION

In a remarkable paper, Dashen and Frautschi[1] (DF in the following) have applied the N/D method to calculate bound-state energy shifts due to small changes in the binding forces. They consider the problem of long-range perturbations, and in particular those due to photon exchange; the latter are important because it is universally presumed that in one form or another they dominate deviations from charge independence. The basic problem is that in an approximate calculation of the energy shift there appear infrared-divergent contributions which are known to be absent from the exact answer. Such contributions will be called IR parts in the following. DF develop several ways to eliminate this difficulty, claiming that they are all at least roughly equivalent to each other. In a second paper, Dashen[2] uses one of these methods (in first approximation) to calculate the proton-neutron mass difference. In the spirit of the N/D method he assumes that the nucleon is a bound state of the pion-nucleon system, i.e., a pole, due to a zero of the D function, of the $I=\frac{1}{2}$, $P_{1/2}$ partial wave.

Schematically, the $I_3=\pm\frac{1}{2}$ states can be written as

$$\begin{aligned}|+\tfrac{1}{2}\rangle&=-(\tfrac{1}{3})^{1/2}|p\pi^0\rangle+(\tfrac{2}{3})^{1/2}|n\pi^+\rangle,\\ |-\tfrac{1}{2}\rangle&=-(\tfrac{2}{3})^{1/2}|p\pi^-\rangle+(\tfrac{1}{3})^{1/2}|n\pi^0\rangle.\end{aligned}\tag{1.1}$$

The proton (neutron) are poles in the $|\pm\frac{1}{2}\rangle$ scattering amplitudes, respectively; they would be degenerate in the absence of electromagnetic effects. Basically, Dashen uses as his dominant perturbation the forces due to photon exchange. For the purpose in hand the anomalous Pauli moments of the nucleons can be ignored[2,3]; then a photon can be exchanged only between the particles in the $|p\pi^-\rangle$ component of the $|-\frac{1}{2}\rangle$ state, so that only the neutron mass is shifted. The mass splittings of the particles on the right of (1.1) must also be taken into account; being an isotensor, the $\pi^\pm-\pi^0$ mass difference has no effect on the isovector quantity

$$\delta M\equiv M_p-M_n,\tag{1.2}$$

but the neutron-proton mass difference itself evidently provides a "damping term," in the sense that by taking it into account on the right of (1.1) we decrease by a factor $\frac{3}{4}$ the result that would be obtained otherwise. For simplicity we shall ignore the damping term to begin with, though we shall allow for it in our final estimate in Sec. 4. Dashen's theoretical result for δM has the experimentally correct magnitude and negative sign.

In the present paper we argue that his answer is a mistake resulting from a method for eliminating IR parts that may be plausible at first sight but is inadequate in these circumstances. If his basic assumptions and input, as outlined above, are handled cor-

[1] R. F. Dashen and S. C. Frautschi, Phys. Rev. 135, B1190 (1964).
[2] R. F. Dashen, Phys. Rev. 135, B1196 (1964).
[3] Only the isoscalar magnetic moments contribute to δM, and the anomalous isoscalar moment is negligibly small.