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Finite Range of Strong Interactions and Analyticity Properties in Momentum Transfer

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We investigate how the finite range of strong interactions can be stated in terms of an experiment. It is found that it is equivalent to the fact that the probability of any process generated by strong interactions should decrease exponentially as a function of the impact parameter a. This impact parameter is defined by a translation of the initial wave packet in a direction normal to its mean velocity in the center-of-mass system. Because of the spreading of wave packets with time, it is necessary to consider wave-packets whose width in configuration space increases like $\sqrt{a}$. It is then shown that this property is equivalent to the analyticity of all absorptive parts due to different channels as functions of the momentum transfer inside an ellipse. Such analyticity properties are also valid for the amplitude of a two-body channel. The ellipse does not shrink to the physical region when the energy tends to infinity.

1. INTRODUCTION

THIS paper is part of a series in which we try to investigate what properties of the S matrix can be stated from considerations of measurement theory. In a preceding paper, we have shown that the S matrix exists, at least below the threshold for three-particle production.[1] In the present paper we want to concentrate upon a most fundamental property of strong interactions, namely, their finite-range character.

As long as one considers the Born approximation for the scattering of a particle by a potential of finite range, there is a very simple way of stating this property of finite range as the result of a measurement. Let us consider a wave packet which decreases more rapidly than any exponential in configuration space at time 0 (for instance a Gaussian wave packet), and let us translate it by a distance a (the impact parameter) in a direction normal to its mean velocity. The Born approximation to the probability of scattering decreases exponentially with a, more precisely like $e^{-2\mu a}$ if μ is the range of the potential. It could then be suggested that it is equivalent in nonrelativistic theory to assume a potential of finite range or to assume that the probability of scattering decreases like an exponential with the impact parameter.

Unfortunately, this proposition is not tenable. In Sec. 2, we show that the spreading of wave packets is such that as long as one considers a wave packet of fixed size, the probability cannot decrease exponentially. The situation in this respect is essentially the same for a relativistic or a nonrelativistic wave packet. However, the analysis of the spreading suggests that, by taking a wave packet the width of which in configuration space increases like $\sqrt{a}$, the probability decreases exponentially. Furthermore, this is the only possible form of a wavepacket which can allow such a strong decrease.

In Sec. 3, we transform the suggestion into a theorem for the nonrelativistic Schrödinger equation. In other words, we show that there is a statement of measurement theory, which we call property P, which is equivalent to the finiteness of the potential range. This property is that the probability of any reaction decreases exponentially with the impact parameter a defined by a translation of the wave packet normally to its mean velocity, if the width of the packet in configuration space increases like $\sqrt{a}$. This statement corresponds to a gedanken experiment (or, if necessary, an actual experiment) where one shoots bunches of particles with an energy of increasing precision farther and farther from the target.

It is then natural to take property P as a starting hypothesis in the relativistic case. This we do in Sec. 4 where we recall also some refinements in the notion of the position of a particle which are needed in the relativistic problem.

A few sections are then devoted to a very straightforward proof of the fact that property P is strictly equivalent to the analyticity of the absorptive parts as a function of momentum transfer inside an ellipse. This ellipse does not shrink when the energy tends to infinity.

In the last section, a comparison is made of this result with a recent paper by Martin where the same conclusions are obtained from quantum field theory. This leads to interesting consequences concerning the respective roles of spectral conditions and causality in the analyticity properties of the scattering amplitude. It is also pointed out that, as a result of Martin's work and the present one, property P is satisfied by axiomatic quantum field theory. In this framework, it appears as a very strong property of the cluster type.

2. FINITE RANGE AND THE SPREADING OF WAVE PACKETS

A nonrelativistic Gaussian wave packet is given by

$$\varphi(\mathbf{x},t)=\frac{1}{[(\sqrt{\pi})b(t)]^{3/2}}\exp\left\{-\left[\frac{\mathbf{x}-\mathbf{x}_0(t)}{2b^2(t)}\right]^2\right\}, \quad (2.1)$$

where $\mathbf{x}_0(t)$, the center of the wave packet, is related to its position $\mathbf{a}$ at time zero and to the mean momentum $\mathbf{k}$ by

$$\mathbf{x}_0(t)=\mathbf{a}+\mathbf{k}t/m \quad (2.2)$$

[1] R. Omnes, Phys. Rev. 140, B1474 (1965).

(m being the reduced mass). The spreading of the wave packet with time is given by

$$b^2(t)=b^2+t^2/m^2b^2. \quad (2.3)$$

In nonrelativistic physics, an interaction is said to be of a finite range when the potential vanishes at least exponentially with the distance. The overlapping of the potential and the wave packet (2.1),

$$\int V(x)\varphi(x,t)\,d^3x, \quad (2.4)$$

decreases exponentially with the impact parameter a, which is chosen normal to k, at any finite time t. However, when t tends to infinity, the value of $\varphi(x,t)$ at the origin of space is given by

$$\lim_{t\to\infty}[b(t)]^{3/2}\varphi(0,t)=\exp[-(b^2k^2/2)], \quad (2.5)$$

and therefore the finiteness of the range will not result in any simple behavior of the scattering probability as a function of the impact parameter, due to the spreading of wave packets.

Let us now consider a wave packet which is farther and farther from the origin, i.e., let a increase. Furthermore, let the width b vary with a. According to Eq. (2.5), if we let b^2 increase linearly with a,

$$b^2=\lambda a, \quad (2.6)$$

the overlapping integral (2.4) will decrease exponentially with a uniformly for any value of t. Note that b^2 has to be linear in a, otherwise there would not be an exponential decrease of (2.4), either at finite or infinite time.

It is therefore suggested that the finite-range character of the interaction can be exhibited by using wave packets the sizes of which increase with the impact parameter as in Eq. (2.6). The criterion for finite range would be that the probability decreases exponentially with the impact parameter. That this suggestion is correct will be proved in the next section.

The essential properties of the wave packets remain true for relativistic particles. A Gaussian wave packet will then behave like

$$\varphi(x,t)=\int e^{-(\mathbf{p}-\mathbf{k})^2b^2/2}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{a})}e^{-i\omega t}\,d^3p, \quad (2.7)$$

where $\omega^2=p^2+m^2$. A straightforward computation of the asymptotic behavior shows that, when t tends to infinity

$$\lim_{t\to\infty}t^{3/2}\varphi(0,t)\approx\exp[-(k^2b^2/2)]\times\text{constant}, \quad (2.8)$$

just as in the nonrelativistic case. It shows that, when k is large, the effect of the spreading of wave packets is small and λ can be taken small.

3. FAST DECREASE OF THE PROBABILITY FOR A FINITE-RANGE POTENTIAL

We shall now prove that if a potential is everywhere finite and decreases exponentially, with the distance, i.e.,

$$|V(r)|e^{\mu r}<C, \quad (3.1)$$

where C is a constant, then the probability for scattering decreases exponentially with the impact parameter. This result will be obtained by using a Gaussian wave packet, the width of which increases with the impact parameter like $\sqrt{a}$. Our method will be a slight adaptation of a method first given by Brenig and Haag for the case of a square-well potential.[2]

Denoting the scattering matrix by T, as usual, we shall start from an inequality given by Brenig and Haag:

$$\|T\varphi\|\le\int_{-\infty}^{+\infty}\|V\varphi(t)\|\,dt. \quad (3.2)$$

In order to find a bound for the right-hand side of this inequality, we shall split the potential into two parts which are essentially a square-well potential of radius ρ smaller than a and the tail of an exponential potential:

$$V(r)=V_1(r)+V_2(r), \quad (3.3)$$

$$V_1(r)=0 \quad \text{for} \quad r<\rho, \quad (3.4)$$

$$V_2(r)=0 \quad \text{for} \quad r>\rho,$$

$$|V_1(r)|\le U_0, \quad (3.5)$$

$$|V_2(r)|\le U_1e^{-\mu r}, \quad (3.6)$$

where U_0 and U_1 are constants.

The contribution of V_2 to the integral in Eq. (3.2) is easily majorized to give

$$\int_{-\infty}^{+\infty}\|V_2\varphi(t)\|\,dt\le\text{constant }e^{-\mu\rho}. \quad (3.7)$$

On the other hand, according to Brenig and Haag, the contribution of V_1 is majorized by

$$\|V_1\varphi(t)\|\le U_0\left[\frac{\rho}{b(t)}\right]^{3/2} \times\exp\left(-\frac{1}{2b^2(t)}\right)\left[\left(a^2+\frac{k^2t^2}{m^2}\right)^{1/2}-\rho\right]^2. \quad (3.8)$$

In order to majorize the integral of $\|V_1\varphi(t)\|$ upon t, we introduce the function

$$f(t)=\frac{1}{2b^2(t)}\left[\left(a^2+\frac{k^2t^2}{m^2}\right)^{1/2}-\rho\right]^2=\frac{k^2b^2}{2}\frac{(\tau-\rho)^2}{\tau^2+\beta^2}, \quad (3.9)$$

where

$$\tau^2=a^2+k^2t^2/m^2, \quad (3.10)$$

$$\beta^2=b^2-a^2/k^2b^2. \quad (3.11)$$

[2] W. Brenig and R. Haag, Fortschr. Physik **17**, 183 (1959).

One has $f(0)=(a-\rho)^2/2b^2$ and $f(\infty)=k^2b^2/2$ so that, once more, we shall find an exponential bound only if b^2 is of the order of a, i.e.,

$$b^2=\lambda a\,, \tag{3.12}$$

or

$$\beta^2=a(\lambda-1/\lambda k^2)\,. \tag{3.13}$$

According to Eq. (3.13) two cases have to be distinguished:

(1) $k^2\lambda^2\geq 1$. In this case β^2 is positive and $f(t)$ increases from $t=0$ to $t=\infty$, so that $f(0)$ is a lower bound of $f(t)$.

(2) $k^2\lambda^2<1$. In this case β^2 is negative and $f(\tau)$ has a minimum at $\tau=-\beta^2/\rho$. This minimum will be outside the range of variation of τ if

$$-\beta^2/\rho<a\,,\quad \text{i.e.,}\quad \rho>(1-k^2\lambda^2)/k^2\lambda\,, \tag{3.14}$$

and, once again, $f(0)$ will be a lower bound of $f(t)$.

Finally, for ρ satisfying (3.14) we have obtained a bound

$$\|T\varphi\|\leq C_1\exp\left(-\frac{(a-\rho)^2}{2\lambda a}\right)+C_2\exp[-(\mu\rho)]\,. \tag{3.15}$$

The best bound will be obtained when both exponentials have the same argument, i.e., we shall have

$$\|T\varphi\|\leq C_3\exp(-\mu\rho)\,, \tag{3.16}$$

with

$$(a-\rho)^2=2\lambda\mu\rho a\,. \tag{3.17}$$

It is clear that for $\lambda\mu$ small, ρ will differ very little from a so that we shall have

$$\|T\varphi\|\leq C_3e^{-(\mu-\epsilon)a} \tag{3.18}$$

with ϵ small.

4. THE BASIC HYPOTHESIS

We want to investigate the conditions under which the following property holds:

Property P. When the impact parameter a tends to infinity together with the width of the wave packet in x space:

$$b^2=\lambda a\,, \tag{4.1}$$

then the probability of any physical process generated by strong interactions decreases exponentially with the impact parameter.

We shall consider this property as a precise formulation of the finite-range character of strong interactions.

A few comments about definitions and notations are in order.

In relativistic physics, the states of a free particle lie within a Hilbert space. We shall, for simplicity, consider the particles to be spinless. Then we can introduce eigenstates of the momentum together with their scalar products:

$$P_\mu|p\rangle=p_\mu|p\rangle\,, \tag{4.2}$$

$$\langle p|p'\rangle=p^0\delta(\mathbf{p}-\mathbf{p}')\,,\quad p^0=(m^2+\mathbf{p}^2)^{1/2}\,. \tag{4.3}$$

We shall work with Gaussian wave packets

$$\int C(\mathbf{p})\frac{d^3p}{p^0}|p\rangle\,, \tag{4.4}$$

where

$$C(\mathbf{p})=[(\sqrt{\pi})A^2]^{-3/2}\exp\{-[(\mathbf{p}-\mathbf{k})^2/2A]\}\,. \tag{4.5}$$

Such a wave packet corresponds to a Gaussian wave packet in configuration space

$$B(\mathbf{x})=\int G(\mathbf{p})\frac{d^3p}{p^0}\langle\mathbf{x}|\mathbf{p}\rangle =[(\sqrt{\pi})A^{-2}]^{-3/2}\exp[-(A^2x^2/2)]e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{4.6}$$

if we define the operator $\mathbf{x}$ by

$$\langle\mathbf{x}|\mathbf{p}\rangle=e^{i\mathbf{p}\cdot\mathbf{x}}p^0(2\pi)^{-3/2}\,. \tag{4.7}$$

In fact, such an operator has no simple meaning. On the other hand, the Newton-Wigner position operator $\boldsymbol{\xi}$ defined by[3]

$$\langle\boldsymbol{\xi}|\mathbf{p}\rangle=e^{i\mathbf{p}\cdot\boldsymbol{\xi}}p_0^{1/2}(2\pi)^{-3/2} \tag{4.8}$$

characterizes the position of the particle at time zero. The relation of the wave packets in $\mathbf{x}$ space and in $\boldsymbol{\xi}$ space is given quite generally by

$$B(\mathbf{x})=\int d^3\xi\, A(\boldsymbol{\xi})\left(\frac{m}{|\mathbf{x}-\boldsymbol{\xi}|}\right)^{5/4}H_{5/4}^{(1)}(im|\mathbf{x}-\boldsymbol{\xi}|)\,. \tag{4.9}$$

Let us now consider a collision experiment between two particles which we shall take to be of the same mass for simplicity. We shall call $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{P}$, and $\mathbf{p}$, respectively, the momenta of the two particles, the total momentum, and the relative momentum

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2\,,\quad \mathbf{p}=\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)\,. \tag{4.10}$$

A translation of space by a vector $\mathbf{a}$ acts upon a state as

$$e^{i\mathbf{p}\cdot\mathbf{a}}|p\rangle\,. \tag{4.11}$$

Therefore a state of two particles with mean relative momentum $\mathbf{k}$, translated relatively by a vector $\mathbf{a}$ normal to $\mathbf{k}$ will be given by

$$\psi(\mathbf{P}\cdot\mathbf{p})=e^{-P^2/2A^2}e^{-(\mathbf{p}\cdot\mathbf{k})^2b^2/2}e^{i\mathbf{p}\cdot\mathbf{a}}(A^2b^{-2})^{-3/2}\,. \tag{4.12}$$

This last expression gives a precise meaning to the width b^2 and the impact parameter a mentioned in property P. In practice, we shall let A be very large so that the expression (4.12) is in fact a delta function of the total momentum $\mathbf{P}$.

Let us now make a few remarks:

(1) Our definition of the impact parameter coincides with the usual meaning of that term only for large values. This cannot lead to any ambiguity since we are precisely interested only in large values of this parameter.

(2) We shall make property P more precise by assuming that the total probability for two particles giving

[3] R. Newton and E. P. Wigner, Rev. Mod. Phys. 21, 400 (1949).

rise to a channel α behaves for large values of a like

$$P_\alpha(a) < \text{constant} \times e^{-2\mu_\alpha a}. \quad (4.13)$$

(3) Generally μ_α could depend upon λ, $\mathbf{k}$, and α. We shall assume that it does not depend upon $\mathbf{k}$ because in fact all values of the relative momentum $\mathbf{p}$ are always present in Eq. (4.12), whatever the value of $\mathbf{k}$ be. Furthermore, we shall assume that there exists an absolute lower bound μ independent of the channel α.

(4) Property P can be expressed as a statement about measurements: If we compare the results of experiments made by accelerators which are increasingly far from a target and increasingly precise in energy as in Eq. (4.18), then the probability of any process induced by strong interactions vanishes exponentially.

(5) According to Eq. (4.9) and the fact that $H_{5/4}^{(1)}(imx)$ decreases like e^{-mx}, the preceding interpretation as a *gedanken* experiment will only be meaningful if

$$\mu \leq m, \quad (4.14)$$

if we specify position by means of the Newton-Wigner operator.

5. PROBABILITY AND ABSORPTIVE PART

In this section, we shall express the probability $P_\alpha(a)$ as an integral upon the contribution of a given channel α to the absorptive part of the scattering amplitude. For a two-body collision we shall use the conventional notation s, t, u for the invariants.

Let us consider a reaction initiated by two particles:

$$a_1 + a_2 \to a_1' + a_2' + \cdots + a_n', \quad (5.1)$$

where the set of final particles is in a channel α. The collision matrix element $T_{i\alpha}(p_1,p_2;p_1',\cdots,p_n')$ relates the initial wave packet $\psi_i(p_1,p_2)$ and the final wave packet in the channel α, $\psi_\alpha(p_1',\cdots,p_n')$ by

$$\psi_\alpha(\mathbf{p}_1',\cdots,\mathbf{p}_n') = \int T_{i\alpha}(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1',\cdots,\mathbf{p}_n') \times \delta^{(4)}(p_1+p_2-p_1'\cdots-p_n')\psi_i(\mathbf{p}_1,\mathbf{p}_2)\frac{d^3p_1d^3p_2}{p_1^0p_2^0}. \quad (5.2)$$

We can compute the norm of ψ_α, i.e., the total probability for reaction (5.8) as

$$P_\alpha = \langle\psi_\alpha|\psi_\alpha\rangle. \quad (5.3)$$

P_α depends only upon the absorptive part $A_\alpha(s,t)$

$$A_\alpha(s,t) = \int T_{i\alpha}(p_1,p_2;p_1',\cdots,p_n') \times T_{f\alpha}^*(-p_3,-p_4;p_1',\cdots,p_n') \times \delta^{(4)}(p_1'+p_2'+\cdots+p_n'-p_1-p_2)\frac{d^3p_1'}{p_1'^0}\cdots\frac{d^3p_n'}{p_n'^0} \quad (5.4)$$

by

$$P_\alpha = \int A_\alpha(s,\hat{p}\cdot\hat{p}')d\Omega d\Omega'\psi_i^*(s,\hat{p})\psi_i(s,\hat{p}')\frac{p^2}{W^2}dW, \quad (5.5)$$

where we have used the Jacobian

$$d^3p_1d^3p_2 = d^4PWpd\Omega, \quad (5.6)$$

calling $W=\sqrt{s}$ and $d\Omega$ a solid-angle element in the center-of-mass system. The integration upon P has been made by assuming that the wave packet depended upon P as in Eq. (4.12). In practice, we shall take for the initial wave packet the Gaussian form

$$\psi_i(s,\hat{p}) = e^{-(\mathbf{p}-\mathbf{k})^2/A}e^{-i\mathbf{p}\cdot\mathbf{a}}. \quad (5.7)$$

We are now going to choose a more convenient variable for the angular integrations in Eq. (5.5). To that end we define two-coordinate systems (Σ_0) and (Σ). The system (Σ_0) has its z_0 axis along the direction of the mean momentum $\mathbf{k}$. We take the impact parameter $\mathbf{a}$ to be along its x_0 axis. The system (Σ) is linked to the vectors $\mathbf{p}$ and $\mathbf{p}'$, the z axis being normal to the plane which contains $\mathbf{p}$ and $\mathbf{p}'$, the x and y axes being directed along the bisecting lines of the angle defined by $\mathbf{p}$ and $\mathbf{p}'$. We shall call (α,β,γ) and θ the Euler angles of the rotation which brings (Σ_0) upon (Σ) and the scattering angle between $\mathbf{p}$ and $\mathbf{p}'$.

By straightforward calculations, one gets

$$(\mathbf{p}+\mathbf{p}')\cdot\mathbf{k} = -2\sin\beta\cos\gamma\cos(\theta/2)pk = 2(\mathbf{e}_x\cdot\mathbf{e}_{z_0})pk\cos(\theta/2),$$

$$(\mathbf{p}-\mathbf{p}')\cdot\mathbf{a} = -2pa(\cos\alpha\cos\beta\sin\gamma+\sin\alpha\cos\gamma)\sin(\theta/2) = 2(\mathbf{e}_y\cdot\mathbf{e}_{x_0})pa\sin(\theta/2), \quad (5.8)$$

$$d\Omega d\Omega' = \sin\beta\sin\theta d\alpha d\beta d\gamma d\theta. \quad (5.9)$$

In Eqs. (5.8) we have called, for instance, $\mathbf{e}_z$ the unit vector along the z axis.

The expression (5.5) for the probability becomes finally

$$P_\alpha(a) = \int A_\alpha(s,\cos\theta)e^{-(p^2+k^2)b^2} \times \exp[-2(\mathbf{e}_x\cdot\mathbf{e}_{z_0})b^2pk\cos(\theta/2)] \times \exp[2ipa\sin(\theta/2)(\mathbf{e}_y\cdot\mathbf{e}_{x_0})] \times (p^2/W^2)dWd\alpha d\cos\beta d\gamma d\cos\theta. \quad (5.10)$$

6. LAPLACE TRANSFORM OF THE PROBABILITY

In this section, we shall replace property P by analytic properties of the Laplace transform of the probability $P_\alpha(a)$.

If we introduce the Laplace transform of $P_\alpha(a)$

$$L_\alpha(\nu) = \int_0^\infty e^{-\nu a}P_\alpha(a)\,da, \quad (6.1)$$

property P is equivalent to the statement that $L_\alpha(\nu)$ is an analytic function of ν inside the domain defined by

$$-2\mu<\mathrm{Re}\nu. \tag{6.2}$$

When Re>0, we can replace $P_\alpha(a)$ in Eq. (6.1) by its expression (5.10) and invert the order of the integrations to get

$$L_\alpha(\nu)=\int_0^\infty I_\alpha(\nu,p^2)p^2W^{-2}\,dW\,, \tag{6.3}$$

where

$$I_\alpha(\nu,p^2)=\int_{-1}^{+1} d\cos\theta\, A_\alpha(s,\cos\theta)F(\nu,p,\lambda,\theta)\,, \tag{6.4}$$

and

$$F(\nu,p,\lambda,\theta)=\int\frac{d\alpha\, d\cos\beta d\gamma}{\nu+(p^2+k^2)\lambda+2(\mathbf{e}_x\cdot\mathbf{e}_{z_0})\lambda pk\cos(\theta/2)-2ip(\mathbf{e}_y\cdot\mathbf{e}_{x_0})\sin(\theta/2)}. \tag{6.5}$$

Although F by itself is rather intractable, it needs only a little algebra to compute $\partial F/\partial\nu$ which, for our purposes, will be just as good. One has

$$\frac{\partial F}{\partial\nu}(\nu,p,\lambda,\theta)=\prod_{\epsilon=\pm1,\epsilon'=\pm1}[\nu+(p^2+k^2)\lambda+2\epsilon pk\lambda\cos(\theta/2)+2i\epsilon' p\sin(\theta/2)]^{-1/2}, \tag{6.6}$$

or

$$\frac{\partial F}{\partial\nu}(\nu,p,\lambda,\theta)=(A\cos^2\theta-2B\cos\theta+C)^{-1/2}, \tag{6.7}$$

where

$$\begin{aligned}A&=4p^4(k^2\lambda^2-1)^2,\\ B&=4p^2[\nu+(p^2+k^2)\lambda]^2(k^2\lambda^2+1)-8p^4(k^4\lambda^4-1),\\ C&=[\nu+(p^2+k^2)\lambda]^4-4p^2[\nu+(p^2+k^2)\lambda]^2(k^2\lambda^2-1)\\&\quad+4p^6(k^2\lambda^2+1)^2.\end{aligned} \tag{6.8}$$

It is clear from Eq. (6.1) that $L_\alpha(\nu)$ is well-defined and analytic for $\mathrm{Re}\nu>0$. In order to extend ν into the strip $-2\mu<\mathrm{Re}\nu<0$, we shall need to consider the possible singularities of the integral in Eq. (6.3).

The case where $k\lambda=1$ is particularly simple and gives

$$\frac{\partial F}{\partial\nu}(\nu,p,\lambda,\theta)=\beta^{-1/2}(y-\cos\theta)^{-1/2}, \tag{6.9}$$

where

$$\beta=16p^4+[\nu+(p^2+k^2)/k]^4, \tag{6.10}$$

$$y\beta=8p^2[\nu+(p^2+k^2)/k]^2. \tag{6.11}$$

In the present paper, we shall restrict our attention to this special case. It will make our considerations much simpler. On the other hand, if k is too small, the restriction to $k\lambda=1$ will give too large values of λ. Accordingly, we shall also restrict our considerations to the case where p and k are restricted by

$$p>\mu,\quad k>\mu. \tag{6.12}$$

7. GEOMETRIC CONSIDERATIONS

Before entering into the discussion of the relation between the analyticity properties of $I_\alpha(\nu)$ and of $A_\alpha(s,x)$ as a function of $x=\cos\theta$, we need to make some geometric discussion of the strip Δ_ν and its image under some changes of variables.

Under the conditions where Eq. (6.9) holds, the integral (6.4) for $I_\alpha(\nu,p^2)$ reads

$$\frac{\partial I_\alpha(\nu,p^2)}{\partial\nu}=\frac{1}{\beta^{1/2}}\int_{-1}^{+1}A_\alpha(s,x)\frac{dx}{(y-x)^{1/2}}. \tag{7.1}$$

The singularity y of the square root is related to ν more easily through the expressions

$$y=\tfrac{1}{2}(\xi+\xi^{-1}), \tag{7.2}$$

$$\xi=(\nu/2p+\rho)^2, \tag{7.3}$$

$$\rho=(p^2+k^2)/2pk. \tag{7.4}$$

When ν varies inside the strip Δ_ν, ξ varies inside a domain Δ_ξ and y inside a domain Δ_y. These domains depend upon p and ρ and we are now going to discuss them.

To a line parallel to the imaginary axis

$$\nu=\nu_0+i\eta, \tag{7.5}$$

where ν_0 is fixed and η varies from $-\infty$ to $+\infty$, there corresponds in the ξ plane a parabola $\pi(\nu_0)$:

$$\begin{aligned}\xi_1&=\mathrm{Re}\xi(\nu_0/2p+\rho)^2-\eta^2,\\ \xi_2&=\mathrm{Im}\xi=2(\nu_0/2p+\rho)\eta.\end{aligned} \tag{7.6}$$

This parabola goes to infinity in the negative ξ_1 direction. Its axis is along the real axis and its apex at $\xi_1=[(\nu_0/2p)+\rho]^2$. All the parabolas corresponding to different values of ν_0 are equal and translate.

When ν varies inside Δ_ν, ξ varies inside Δ_ξ which is bounded by two equal parabolas π_1 and π_2 with their apexes respectively at $\xi=\rho^2$ and $\xi=(\rho-\mu/p)^2$. Let us note that, since the correspondance between ξ and ν is not one-to-one, π_1 and π_2 would be in two different Riemann sheets if $\rho-\mu/p$ were negative. However, since ρ is larger than 1 and p restricted by (6.12), this possibility will not arise.

The correspondence (7.2) between ξ and y applies a circle $|\xi|=r$ onto an ellipse with its foci at $y=\pm1$ and semi-axes $\frac{1}{2}(r+r^{-1})$, $\frac{1}{2}(r-r^{-1})$, if $r>1$. To a given value of y correspond two values of ξ which are the inverse

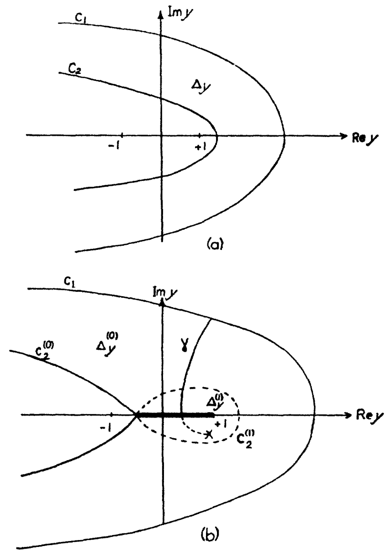

FIG. 1. The domain Δ_α image of the strip Δ_γ. (a) Case (i), $\rho-\mu/p\geq 1$. (b) Case (ii), $\rho-\mu/p<1$.

of each other. The unit circle in the ξ plane which separates these two values of ξ is applied upon the segment $y=1$ to $+1$ and the y Riemann surface is two-sheeted.

The topological structure of Δ_y can be different if π_1 and π_2 cross or do not cross the unit circle. It is easily checked that the parabola $\pi(\nu_0)$ crosses the unit circle if and only if $(\rho+\nu_0/2p)^2$ is smaller than 1. Accordingly π_1 never crosses the unit circle and only two cases have to be distinguished

$$\text{Case (i)}: \quad \rho-\mu/p\geq 1\,, \tag{7.7}$$

$$\text{Case (ii)}: \quad \rho-\mu/p<1\,. \tag{7.8}$$

Let us call C_1 and C_2 the images of π_1 and π_2. In case (i) C_1 and C_2 are in the same Riemann sheet and C_2 encloses the segment $y=-1$ to $+1$. In case (ii), C_2 crosses the segment $y=-1$ to $+1$ and it consists of two parts $C_2^{(0)}$ and $C_2^{(1)}$ (see Fig. 1), where $C_2^{(0)}$ is in the same Riemann sheet as C_1 whereas $C_2^{(1)}$ is in another sheet. Δ_y consists then of two parts: $\Delta_y^{(0)}$ bounded by C_1 and $C_2^{(0)}$ and $\Delta_y^{(1)}$ bounded by $C_2^{(1)}$ in another sheet.

Let us note that π_2 does not cross the circle, with its center at the origin, which touches π_2 at its apex. Accordingly, $\Delta_y^{(1)}$ is completely contained inside an ellipse, with its foci at $y+\pm 1$, which is tangent to $C_2^{(1)}$ at its apex on the real axis. This remark will prove to be important in the future.

8. ANALYTIC PROPERTIES OF THE ABSORPTIVE PART

Since the absorptive part A_α $(s, \cos\theta)$ is not an analytic function of s, the analyticity domain of $L_\alpha(\nu)$ is the intersection of the analyticity domains of $I_\alpha(\nu,p^2)$ as a function of ν. Therefore $I_\alpha(\nu,p^2)$ must be analytic inside the strip Δ_ν defined by

$$-2\mu<\text{Re}\nu\leq 0\,.$$

In fact it will be shown below that the integration over p in the expression of $L_\alpha(\nu)$ can be restricted to a finite range of values of p around k.

We shall call $\bar{\Delta}_y(p,k)$ the domain $\Delta_y^{(1)}$ without the cut along the real axis. We have indicated explicitely its dependence upon p for a given value of k. For the values of p which satisfy Eq. (7.7), the domain is empty. We shall also call it $\bar{\Delta}_x(p)$ when the notation x replaces the notation y. We are now ready to prove the following theorem:

Theorem: A necessary and sufficient condition for $I_\alpha(\nu,p^2,k)$ to be analytic inside the strip Δ_ν is that the absorptive part $A_\alpha(s,x)$ be an analytic function of x inside the domain $\bar{\Delta}_y(p,k)$.

When one continues the expression (7.1) for $I_\alpha(\nu,p^2)$ along a path Γ which starts from a point with $\text{Re}\nu=0$ inside the strip Δ_ν, the singularity at $x=y$ of $F(\nu,p^2,x)$ will vary along a path γ inside Δ_y which starts from a point of C_1. As long as γ does not cross the integration segment from $x=-1$ to $+1$, the integral (7.1) will remain an analytic function of ν. This is always so when condition (7.7) is satisfied, i.e., when the domain $\bar{\Delta}_y(p,k)$ is empty.

Under the conditions (7.8) and (6.12) we shall first note that ξ is an analytic function of ν inside Δ_ν so that it is equivalent to discuss the analyticity properties of $I_\alpha(\nu,p^2)$ as a function of ν inside Δ_ν or as a function of y inside Δ_y.

Dropping all unnecessary parameters and calling $f(y)$ the function $(\partial I_\alpha(\nu,p^2)/\partial\nu)\beta^{1/2}$, Eq. (7.1) takes the form

$$f(y)=\int_{-1}^{+1} A(x)(y-x)^{-1/2}\,dx\,. \tag{8.1}$$

The determination of the square root is fixed from values of ν with $\text{Re}\nu>0$, i.e., from the right of C_1 where it is taken to be positive-definite.

The difference $2F(y)$ between the two determinations of $f(y)$ in the two Riemann sheets is given by

$$F(y)=\int_x^1 \frac{A(x)\,dx}{(y-x)^{1/2}} \tag{8.2}$$

which shows immediately that, if $A(x)$ is analytic in $\bar{\Delta}_x$, $F(y)$ is analytic in $\Delta_y^{(1)}$.

To prove the necessity part of the theorem, we solve Eq. (8.2) which is an Abel equation:

$$A(x)=\frac{1}{\pi}\int_y^1 \frac{F'(y)\,dy}{(x-y)^{1/2}}\,. \tag{8.3}$$

If $F(y)$ is analytic, $F'(y)$ exists and the integral is well defined. It shows that $A(x)$ is an analytic function of x in $\Delta_x^{(1)}$. In order to show that it is analytic in $\bar{\Delta}_x$, i.e., that it has no singularity at $x=1$, we note that $f(y)$ is an analytic function of ξ. It can therefore be written as a uniformly converging series in a neighborhood of

$y=1$ as

$$f(y)=\sum_{n=0}^{\infty} b_n(y-1)^n+\sum_{n=0}^{\infty} a_n(y-1)^{n+\frac{1}{2}}, \quad (8.4)$$

from which

$$F(y)=\sum_{n=0}^{\infty} a_n(y-1)^{n+\frac{1}{2}}. \quad (8.5)$$

From Eq. (8.3) we then get

$$A(x)=\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} a_n y^n, \quad (8.6)$$

which has the same circle convergence as the series (8.5). This then proves the theorem.

We can increase the analyticity domain by noticing that in the Legendre expansion (8.3) of $A_\alpha(s,x)$, all the coefficients $a_n(s)$ are positive. Accordingly, if $A_\alpha(s,x)$ is analytic for x real between 1 and $x_1>1$, it has to be analytic inside the ellipse with its foci at $x=\pm 1$ and semi-axis x_1. Taking into account the remark at the beginning of Sec. 8 about the analyticity domains of $I_\alpha(\nu,p^2)$ and $L_\alpha(\nu)$. we get the following new theorem:

Theorem: A necessary and sufficient condition for property P to be satisfied is that $A_\alpha(s,x)$ be an analytic function of x inside the smallest ellipse with its foci at $x=\pm 1$ which contains $\Delta_x^{(1)}$.

This smallest ellipse has for major semi-axis

$$a=\tfrac{1}{2}(\xi_0+\xi_0^{-1}),$$
$$\xi_0=\left(\frac{p^2+k^2}{2pk}-\frac{\mu}{p}\right)^2, \quad \xi_0<1. \quad (8.7)$$

For a given value of p, the largest value of a will be obtained for the smallest value of ξ_0, i.e., for $k=p$, or

$$a_{\max}=\tfrac{1}{2}[(1-\mu/p)^2+(1-\mu/p)^{-2}]. \quad (8.8)$$

The corresponding value of the momentum transfer is

$$t_0=2p^2(a_{\max}-1)=\mu^2(2p-\mu)^2/(p-\mu)^2; \quad (8.9)$$

it is a decreasing function of p and tends to $4\mu^2$ when p tends to infinity.

As a final remark, let us note that the case (i) where $\rho-\mu/p$ is larger than 1 does not lead to any condition on the absorptive part so that the discussion is not modified if we cut off the wave packet to values of p which satisfy

$$(p-k)^2<2\mu k. \quad (8.10)$$

9. CONCLUSIONS

We have obtained that the contribution of any channel α to the absorptive part $A_\alpha(s,t)$ is an analytic function of t inside an ellipse. This ellipse contains positive values of t up to $t=4\mu^2$ when the energy tends to infinity. Since the total absorptive part $A(s,t)$ is a sum of $A_\alpha(s,t)$ over the finite number of channels open at energy s, it is also analytic in the same region.

When α is a two-particle channel, unitarity tells us that the amplitude for the two initial particles going to channel α is also an analytic function of t.

It has to be emphasized that these results depend only upon the finite-range hypothesis as expressed in an experimental way by property P. They do not involve any reference to quantum field theory.

The same results have been obtained in a recent work by Martin as a consequence of quantum field theory.[4] This is a beautiful achievement; however, we feel that it involves going a very long way from the axioms of field theory as compared to the very simple arguments given here. Since our results are in the form of a necessary and sufficient condition, the result of Martin together with ours gives a proof that property P is satisfied in quantum field theory. This is marked progress with respect to the cluster properties of this type which have been obtained up to now.[5]) It also shows that Martin's result in fact does not depend upon causality but only upon the spectral properties.

Not all the consequences of our technique have been drawn. In particular, we shall have to examine the analytic properties of $A_\alpha(s,t)$ in the low-energy region.

In our considerations, the mass μ appears as a parameter. In the case of pion-pion scattering, using a dispersion relation in s and crossing, it is easy to show that $\mu=m$.

It is of foremost importance to investigate the derivation of dispersion relations along the same lines of measurement theory as we have done here. It is well known that it has been impossible up to now to derive analyticity properties in s directly from causality (i.e., the observed signal does not precede the initial signal in time) because the spectrum of energy has a gap for systems of particles with a finite mass. It is our opinion that this difficulty is spurious. Indeed, in order to produce a signal which is zero for negative times, one must take into account explicitly the generation of particles, i.e., for instance the accelerator. This breaks up the invariance of the subsystem made up by the particles with respect to translation of time and therefore suppresses the gap in energy. We intend to investigate whether a careful analysis of the production of particles, together with the down-to-earth notion of causality, does not in fact imply dispersion relations. A preliminary analysis of the Schrödinger equation supports this view.

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[5] K. Hepp, Helv. Phys. Acta 37, 659 (1964).