From Bardakci *et al.*,[24] we have

$$(f_{\omega\rho\pi}^2/4\pi)\simeq 32/9, \quad (8)$$

and taking $\Gamma(\rho^0|\pi^0\gamma)\simeq 0.103$ from Table I, we obtain

$$(f_{\rho\pi\gamma}^2/4\pi)\simeq 1.27\times 10^{-4}.$$

Taking $\Gamma(\omega^0|\pi^0\gamma)\simeq 1$ MeV, we get

$$\Gamma(\omega^0|\pi^+\pi^-\gamma):\Gamma(\omega^0|\pi^0\gamma)\simeq 1:100. \quad (9)$$

Again from $\rho\to\omega\pi\to\pi\pi\gamma$, we have[25]

$$\Gamma(\rho^0|\pi^0\pi^0\gamma)\simeq(f_{\rho\omega\pi}^2/4\pi)(f_{\omega\pi\gamma}^2/4\pi)\times(m_\rho^5/48\pi m_\pi^4)(1.05\times 10^{-3}). \quad (10)$$

For $\Gamma(\omega^0|\pi^0\gamma)\simeq 1$ MeV, we have

$$(f_{\omega\pi\gamma}^2/4\pi)\simeq 1.09\times 10^{-3}.$$

Thus

$$\Gamma(\rho^0|\pi^0\pi^0\gamma):\Gamma(\rho^0|\pi^0\gamma)\simeq 1:5. \quad (11)$$

Thus, from the above we find that, compared to $\omega^0\to\pi^0\gamma$, the $\omega^0\to\pi^+\pi^-\gamma$ mode is highly suppressed. The discrepancy in such a conclusion is due most probably to the failure of the pole model for such processes, which has been clearly mentioned in Ref. 23. For processes $V\to P+P'+\gamma$, one has to evaluate the amplitudes $(\gamma_{\pm 1}V_{\pm 1}|P_0P_0')$ and can estimate their partial widths from the observed width[19] of $\omega^0\to\pi^+\pi^-\gamma$.

The author acknowledges his deep indebtedness to Professor S. N. Biswas for suggestions and useful discussions and to Professor R. C. Majumdar for his kind interest.

[24] K. Bardakci *et al.*, Phys. Rev. Letters **14**, 264 (1965). We would like to note here that Eq. (8) of these authors is derived on the basis of a $U(12)$-type theory and it is not a prediction of $SU(6)_W$ symmetry.

[25] P. Singer, Phys. Rev. **130**, 2441 (1963).

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Matrix N/D Method with Arbitrary Inelasticity*

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The N/D method for n two-body channels is extended to allow arbitrary inelastic effects. The functions regarded as given are the left-cut discontinuities and the "absorption matrix." (The absorption matrix F is the term of the unitarity condition accounting for transitions between the n channels and any other open channels: $\mathrm{Im}T_+=T_+\rho T_-+F$.) The procedure is based on a new D matrix, which is defined by means of generalized complex phase shifts. The basic equation is a singular but linear integral equation for ImD. The singular equation is related to an equivalent Fredholm equation. The method may provide a useful general framework in which to discuss absorptive corrections to the peripheral model of high-energy processes. It includes other ways of treating such corrections as special approximations. The technique may also be valuable as a tool in constructing a more rigorous bootstrap theory. A by-product of the investigation is an elegant parametrization of an n-channel submatrix of the entire S matrix.

1. INTRODUCTION

THE matrix N/D representation proposed by Bjorken[1] can be used to solve systems of coupled partial-wave dispersion relations. One employs the matrix analog of the Chew-Mandelstam method.[2] The Bjorken representation refers only to n two-body channels, so the procedure has the drawback of always violating unitarity at high energies. Mandelstam[3] proposed a way of handling three-body channels, but the possibility of an explicit treatment of all necessary channels at high energies seems completely out of reach. If only one channel is treated explicitly, the unitarity condition referring to that channel can be met at all energies by a modified N/D procedure.[4,5] In the latter approach the inelasticity factor η is put into the equations as a given function. In the applications of this method carried out so far,[6] empirical information on the low-energy behavior of η is combined with hypotheses about its high-energy behavior.

In the following I show how to extend the matrix N/D method to allow n two-body channels with arbitrary, given inelasticity. The given inelasticity function is the term in the unitarity condition which accounts for transitions between the n explicit channels and any other open channels. I shall call this term the "absorption matrix."

The equations derived may turn out to be useful as a

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[1] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960).

[2] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[3] S. Mandelstam, Phys. Rev. **140**, B375 (1965).

[4] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

[5] M. Froissart, Nuovo Cimento **22**, 191 (1961).

[6] See, for example, P. W. Coulter and G. L. Shaw, Phys. Rev. **138**, B1273 (1965).

general framework in which to investigate absorptive corrections to the peripheral model of high-energy processes. The plan would be to treat a production process such as $\pi+N\to\rho+N$ as a two-channel problem. All channels but $\pi-N$ and $\rho-N$ are accounted for by the given absorption matrix, for which some model is to be assumed. In accordance with the peripheral picture, the only left-hand singularities of the two-channel amplitude would be those due to exchange of single particles of small mass. In principle the number of explicit channels may be greater than two, but in that case the equations become rather complicated unless the absorption matrix has a simple form. Białas and van Hove[7] suggested a reasonable model of the absorption matrix based on minimal correlation of produced particles and minimal overlap of states produced in different two-particle collisions. Their assumptions imply that the absorption matrix is a real multiple of unity. With that simplification the N/D equation is tractable with a good many channels treated explicitly. If the absorption matrix is merely real, the equation is still relatively simple. Also, effects, of a small imaginary part of the absorption matrix may be assessed by a perturbation method without solving the complete equation. The advantage of the approach suggested is its generality. Other schemes (e.g., the distorted-wave Born approximation[8] or the Białas–van Hove[7] method) are comprehended as special approximations. The general N/D equation might suggest improvements to these approximations.

An incidental motivation for this work has to do with the Castillejo-Dalitz-Dyson (CDD) ambiguity[9,4] in the single-channel N/D method with inelasticity, and also with the general formulation of bootstrap dynamics. Sufficient conditions for the lack of arbitrary CDD parameters in the solution of a single-channel dispersion relation with inelasticity were given in Ref. 4, Sec. V and also in Ref. 10. These conditions are likely to be met in the higher partial waves, but in the lower waves arbitrary CDD parameters are expected to appear in the general solution of the dispersion relation. Such parameters may appear in the physically relevant solution, even if the existence of "elementary" (noncomposite)[11] particles is ruled out. This was mentioned by Chew and Frautschi,[12] who noticed that in the Dalitz-Tuan[13] model of the Y_0^* (1405 MeV) resonance a CDD pole should appear in the denominator function of the $\pi-\Sigma$ channel. Here the Y_0^* is regarded as a virtual bound state of the $\bar{K}-N$ system; i.e., a state which would be a $\bar{K}-N$ bound state if the coupling between the $\bar{K}-N$ and $\pi-\Sigma$ channels were reduced to zero. The CDD pole in the $\pi-\Sigma$ channel is a reflection of the virtual bound state. In the two-channel treatment of the problem there is no such CDD pole. This situation has led several authors[14] to comment on inequivalence between the single-channel inelastic equation and the many-channel N/D equations. Actually, there is full equivalence between the single-channel equation with CDD terms [(Eq.) VI.7 of Ref. 4] and the many-channel equations, since the former equation provides the general solution to the single-channel dispersion relation.[15] The present paper suggests an escape from the CDD problem while still allowing arbitrary inelasticity. If there are no elementary particles, it is reasonable to assume that so many channels may be included explicitly that CDD poles become unnecessary. In this way one can imagine a relatively complete and general formulation of a bootstrap theory: A limited number of two-body or quasi-two-body channels participate in the basic bootstrap dynamics, while corrections from all other channels are provided by an appropriate absorption matrix. Of course, the above remarks are not meant to imply that the CDD ambiguity is diminished in a mathematical sense by increasing the number of channels. On the contrary, arbitrarily many CDD parameters remain. The supposition is merely that a limited number of such parameters have any importance for the bootstrap dynamics of states of fairly small mass, and that these few parameters may have their values determined by allowing sufficiently many explicit channels.

2. DEFINITION OF THE D MATRIX

The partial-wave scattering matrix referring to n two-body channels is denoted by $S=[S_{ij}(s)]$. The variable s is the squared energy in the zero-momentum frame. The transition matrix $T=[T_{ij}(z)]$ is related to S by the equation

$$S=1+2i\rho^{1/2}T_+\rho^{1/2}, \tag{2.1}$$

[7] A. Białas and L. van Hove, Nuovo Cimento **38**, 1385 (1965).

[8] N. J. Sopkovich, Nuovo Cimento **26**, 186 (1962); K. Gottfried and J. D. Jackson, *ibid.* **34**, 735 (1964); L. Durand, III, and Y. T. Chiu, Phys. Rev. **139**, B646 (1965).

[9] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[10] R. L. Warnock, Phys. Rev. **131**, 1320 (1963).

[11] See, for instance, S. C. Frautschi, *Regge Poles and S-matrix Theory* (W. A. Benjamin, Inc., New York, 1963).

[12] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).

[13] R. H. Dalitz and S. F. Tuan, Ann. Phys. (N. Y.) **3**, 307 (1960).

[14] M. Bander, P. Coulter, and G. Shaw, Phys. Rev. Letters **14**, 207 (1965); E. J. Squires, Nuovo Cimento **34**, 1751 (1964); E. J. Squires and P. J. S. Watson, *ibid.* **42**, 77 (1966); H. Munczek, Phys. Letters **13**, 92 (1964); H. Munczek and A. Pignotti, *ibid.* **16**, 198 (1965); D. Atkinson, K. Dietz, and D. Morgan, CERN Reports TH. 531 and TH. 564 (unpublished); J. B. Hartle and C. E. Jones, Phys. Rev. Letters **14**, 801 (1965); Phys. Rev. **140**, B90 (1965); Ann. Phys. (N. Y.) (to be published); P. Hertel, Z. Physik **186**, 288 (1965); T. Kanki, Nuovo Cimento **37**, 1769 (1965); Y. Fujii, Phys. Rev. **139**, B472 (1965); J. Finkelstein, *ibid.* **140**, B175 (1965).

[15] More precisely, it provides the general solution within the class of amplitudes having bounded Hölder continuous phase shifts and such that the N/D equation is of Fredholm type. This follows because such amplitudes have an N/D representation, and Eq. (VI.7) of Ref. 4 is a necessary condition on ImD. In some of the papers of Ref. 14, there seems to have been doubt as to whether inclusion of CDD poles would establish complete equivalence between the single-channel and many-channel formulations.

where $\rho=[\rho_i(s)\delta_{ij}]$ is a diagonal matrix of threshold factors of the form $\rho_i(s)=\theta(s-s_{0i})\sigma_i(s)$, and $T_+(s)=T(s+i0)$. Here s_{0i} is the threshold of channel i. The positive function $\sigma_i(s)$ is chosen in such a way that the analytic function $T_{ij}(z)$ does not have branch points at the thresholds s_{0i}, s_{0j}, and so that σ_i behaves as $s^{1/2}$ at infinity.[16] By time-reversal invariance, T may be chosen to be symmetric: $T=T^T$. (Superscript T means the transpose.) We also assume the reality condition $T(z)=T^*(z^*)$. The unitarity equation referring to the n channels that are treated explicitly is

$$(T_+-T_-)/2i=T_+\rho T_-+F. \tag{2.2}$$

For simplicity it is assumed that all unphysical singularities of T_{ij} lie to the left of s_{0i}, s_{0j}. The absorption matrix F accounts for virtual transitions from the n channels to any other physical states that are allowed at the energy $s^{1/2}$. In terms of the S matrix the unitarity condition is

$$SS^\dagger=1-4\rho^{1/2}F\rho^{1/2}=H. \tag{2.3}$$

Since S is symmetric, $[S,S^\dagger]=2i\,\mathrm{Im}H$. In general the Hermitian matrix H is not real, so S does not commute with its adjoint. Hence there is no guarantee that S may be diagonalized. This makes the inelastic N/D problem a good deal more complicated in general than in the special case of real H. Since H has the form $SS^\dagger$, it is clearly a non-negative matrix. In the following work the stronger requirement that H be positive-definite is imposed. This is very reasonable physically, since it only means that the eigenvalues of the absorption term $4\rho^{1/2}F\rho^{1/2}$ are forbidden to reach their upper limits.

In order to carry the single-channel inelastic method over to the many-channel case, one must find an analog of the complex phase-shift description on which that method is based. With a single channel the denominator function $D(z)=D^*(z^*)$ is defined[4] as an analytic solution of the equation

$$D_-(s)=e^{2i\delta(s)}D_+(s), \tag{2.4}$$

where the functions $D_\pm(s)=D(s\pm i0)$ are the limits of $D(z)$ as z approaches the physical cut. In (2.4), δ is the real part of the phase shift. The many-channel analog of (2.4) should have the form

$$D_-(s)=M(s)D_+(s), \tag{2.5}$$

where M is a matrix such that

$$M^{-1}(s)=M^*(s). \tag{2.6}$$

Equation (2.6) follows from complex conjugation of (2.5), the requirement $D(z)=D^*(z^*)$, and the demand that $D_\pm(s)$ have an inverse at all points s on the physical cut. For the many-channel problem without inelasticity, one has $M(s)=1+2i\rho(s)T_+(s)$, and (2.6) is just the unitarity condition on T. When inelasticity is allowed it is appropriate to make use of the following general solution of the unitarity condition (2.3):

$$S=\textstyle\sum_i \Psi_i\alpha_i\Psi_i^T, \quad H=\sum_i \Psi_i|\alpha_i|^2\Psi_i^\dagger, \tag{2.7}$$

where the Ψ_i form a complete, orthonormal set of eigenvectors of H, and the α_i are complex numbers such that $0<|\alpha_i|\leqslant 1$. Equation (2.7) is proved in the Appendix; the only conditions for (2.7) are symmetry of S and the positive-definite property of H. A factorization of S as a product of an Hermitian and a unitary matrix follows from (2.7); viz., $S=H^{1/2}\Omega$, where

$$\Omega=\textstyle\sum_i \Psi_i e^{i\phi_i}\Psi_i^T, \quad \alpha_i=|\alpha_i|e^{i\phi_i}. \tag{2.8}$$

Since $\Omega^{-1}=\Omega^*$, Ω is a good candidate for identification with the matrix M of Eq. (2.5). Furthermore, the quantity ϕ_i is the closest many-channel analog of the phase 2δ appearing in Eq. (2.4). The generalized complex phase shift is Δ_i, where $\alpha_i=e^{2i\Delta_i}$. Actually, $M=\Omega$ is not precisely what is needed. To find further conditions on M one must look at the equations giving the jumps of the N and D matrices over their cuts. If $D(z)$ is an analytic function satisfying (2.5), and $N(z)$ is defined as $N(z)=T(z)D(z)$, the following relations hold:

$$\begin{aligned}(1+2i\rho T_+)D_+&=[\rho^{1/2}S\rho^{-1/2}M^{-1}]D_-=D_++2i\rho N_+,\\ (1-2i\rho T_-)D_-&=[\rho^{1/2}S^*\rho^{-1/2}M]D_+=D_--2i\rho N_-.\end{aligned} \tag{2.9}$$

In the single-channel case the square bracket expressions of (2.9) are equal to the real inelasticity factor η: $\rho^{1/2}S\rho^{-1/2}M^{-1}=\rho^{1/2}S^*\rho^{-1/2}M=\eta$. In that case Eqs. (2.9) lead to the integral equation of Ref. 4, via dispersion relations for N and D. In the many-channel case it does not seem possible to make the two square bracket expressions of (2.9) real, but at least they can be made to depend only on the absorption matrix. In fact, the identification

$$M=\rho^{1/2}\Omega\rho^{-1/2} \tag{2.10}$$

ensures $M^{-1}=M^*$ and also yields

$$\begin{aligned}\eta D_-&=D_++2i\rho N_+,\\ \eta^*D_+&=D_--2i\rho N_-.\end{aligned} \tag{2.11}$$

The generalized inelasticity factor η is the matrix

$$\eta=\rho^{1/2}H^{1/2}\rho^{-1/2}. \tag{2.12}$$

The divisions by $\rho^{1/2}$ in (2.10) and (2.12) are justified later on [it turns out that there is no trouble from the vanishing of elements of ρ at closing of channels—Eqs. (2.10) and (2.12) are still well-defined]. By addition and subtraction of Eqs. (2.11) the discontinuity equations follow:

$$\begin{aligned}(\mathrm{Re}\eta-1)\mathrm{Re}D+\mathrm{Im}\eta\,\mathrm{Im}D&=-2\rho\,\mathrm{Im}N,\\ (\mathrm{Re}\eta+1)\mathrm{Im}D-\mathrm{Im}\eta\,\mathrm{Re}D&=-2\rho\,\mathrm{Re}N.\end{aligned} \tag{2.13}$$

[16] The behavior of σ_i at infinity may depend on the relation between invariant and partial-wave amplitudes. The behavior $\sigma_i\sim s^{1/2}$ is commonly assumed for spin 0–spin $\frac{1}{2}$ scattering, while $\sigma_i\sim 1$ is standard for spin 0–spin 0 scattering. Modifications of the discussion for behaviors other than $s^{1/2}$ are straightforward; subtractions may be introduced in the dispersion relation (3.2). It should also be mentioned that formulation of the spin 0–spin $\frac{1}{2}$ problem in the complex plane of $w=s^{1/2}$ is more suitable than the s-plane formulation given here. The only reason for avoiding the w plane is the notational complication that it involves.

In the following section an integral equation for ImD is deduced from Eqs. (2.13).

3. THE SINGULAR INTEGRAL EQUATION

To set up the integral equation, some assumptions about the behavior of T and D at infinity are needed. The behavior of T on the physical cut is restricted by unitarity. This is seen from the formula (2.7) for the S matrix. Since Ψ_i is normalized to unit length, and $|\alpha_i| \leqslant 1$, S_{ij} is bounded. From (2.1) and the fact that ρ_i is asymptotic to $cs^{1/2}$, a bound on $T_+(s)$ is fixed:

$$T_+(s)=O(s^{-1/2}), \quad s\to\infty . \tag{3.1}$$

(As was noted in footnote 16, the behavior of ρ at infinity may depend on the spins of the particles, as a result of differing relations between invariant and partial-wave amplitudes. The choice $\rho_i \sim cs^{1/2}$ is appropriate for scattering of spin-0 from spin-$\frac{1}{2}$ particles.) From (3.1) it follows that the Cauchy integral of Im$T_+(s)$ along the physical cut converges. If the behavior of $T(Re^{i\theta})$ as $R\to\infty$ is sufficiently uniform in θ, an unsubtracted dispersion relation for $T(z)$ will hold. For simplicity the condition $T(z)=O(|z|^{-\epsilon})$, $\epsilon>0$, will be assumed. Then the unsubtracted dispersion relation is valid:

$$T(z)=T^U(z)+T^P(z)=\frac{1}{\pi}\int_U \frac{\Delta T(z')dz'}{z'-z} + \frac{1}{\pi}\int_P \frac{T_+(s)\rho(s)T_-(s)\,ds}{s-z} + \frac{1}{\pi}\int_P \frac{F(s)\,ds}{s-z} . \tag{3.2}$$

The letters U and P denote the unphysical and physical cuts, respectively.

To investigate the asymptotic behavior of D, it is necessary to inquire about the possible solutions of Eq. (2.5). To my knowledge this problem is not discussed in the literature, even in the case of the matrix N/D method without inelasticity. I intend to treat the question in a later paper by transforming Eq. (2.5) to the "Hilbert problem in several unknown functions."[17] There certainly are solutions of (2.5) which are bounded by polynomials at infinity, under weak conditions on $M(s)$.[17] On the other hand, one does not yet have a complete classification of all such solutions by which the "minimal" asymptotic behavior of D is related to the asymptotic behavior of M.[18] In the single-channel problem such a classification was given in Refs. 4 and 10. In those papers it was proved that there are solutions $D(z)$ of (2.4) such that $D(z)=O(|z|^{\epsilon})$, with $\epsilon>0$ arbitrarily small.[19] Such solutions may or may not have poles, depending on the asymptotic behavior of $\delta(s)$. The "normal" case is that in which there are no poles of D, while in the "CDD" case poles are necessary. If $\delta(\infty)$ exists, then the larger the value of $\delta(\infty)$, the more poles are required to maintain the behavior $D(z)=O(|z|^{\epsilon})$. In the following discussion of the many-channel case it will be assumed that for the class of scattering amplitudes of interest (equivalently, for the class of M matrices of interest) there is a normal solution of (2.5); i.e., a solution without poles and such that $D(z)=O(|z|^{\epsilon})$, $\epsilon>0$ arbitrarily small. Extension of the work to allow CDD poles depends on an analysis of Eq. (2.5), which is deferred for the time being. In the normal case, $D(z)$ satisfies a once-subtracted dispersion relation

$$D(z)=1+\frac{z}{\pi}\int_P \frac{\mathrm{Im}D(s)\,ds}{s(s-z)} . \tag{3.3}$$

The N matrix satisfies an unsubtracted dispersion relation, since $T(z)=O(|z|^{-\epsilon})$, $\epsilon>0$ and $D(z)=O(|z|^{\delta})$, $\delta<\epsilon$. Hence

$$N(z)=\frac{1}{\pi}\int_U \frac{\Delta T(z')D(z')}{z'-z}+\frac{1}{\pi}\int_P \frac{\mathrm{Im}N(s)\,ds}{s-z} . \tag{3.4}$$

Of course, Im$N=0$ in that part of the physical region P in which there is no absorption from the n explicit channels (i.e., where $H=1$).

Define the function $n(s)$ by $\mathrm{Im}D(s)=-\rho(s)n(s)$. In terms of n the Eqs. (2.13) read

$$\begin{aligned} \mathrm{Im}N&=\tfrac{1}{2}\rho^{-1}(1-\mathrm{Re}\eta)\mathrm{Re}D+\tfrac{1}{2}\rho^{-1}\mathrm{Im}n , \\ \mathrm{Re}N&=\tfrac{1}{2}\rho^{-1}(1+\mathrm{Re}\eta)\rho n+\tfrac{1}{2}\rho^{-1}\mathrm{Im}\eta\mathrm{Re}D . \end{aligned} \tag{3.5}$$

Substitute (3.3) for $D(z')$ in Eq. (3.4), and (3.5) for ImN in the same equation. Then take the real part of (3.4) after letting z go to $s+i0$, and substitute (3.5) for ReN on the left side. The second integral of (3.4) now involves repeated principal value integrations, since ImN is proportional to ReD. The order of these principal value integrals is reversed[4] by the Poincaré-Bertrand formula, and the outer integral becomes proper. These manipulations parallel the work of Ref. 4, to which the reader is referred. The resulting equation is

$$\alpha(s)n(s)+\frac{P}{\pi i}\int_P \frac{K(s,s')n(s')\,ds'}{s'-s}=f(s) , \tag{3.6}$$

where

$$\begin{aligned} \alpha(s)&=\rho^{-1}(s)\mathrm{Re}\eta(s)\rho(s) , \\ K(s,s')/i&=[sB(s)-s'B(s')]\rho(s')/s' \\ &\quad -\tfrac{1}{2}[\rho^{-1}(s)\mathrm{Im}\eta(s)s/s'+\rho^{-1}(s')\mathrm{Im}\eta(s')]\rho(s') , \\ f(s)&=B(s)-\tfrac{1}{2}\rho^{-1}(s)\mathrm{Im}\eta(s) , \\ B(s)&=\frac{1}{\pi}\int_U \frac{dz\,\Delta T(z)}{z-s} \\ &\quad +\frac{P}{\pi}\int_U \frac{ds'\,\frac{1}{2}\rho^{-1}(s')[1-\mathrm{Re}\eta(s')]}{s'-s} . \end{aligned} \tag{3.7}$$

[17] R. L. Warnock, Bull Am. Phys. Soc. **9**, 116 (1964).

[18] The D with minimal asymptotic behavior is that for which $D(z)=O(|z|^{a})$, with a as small as possible. Note that if $D(z)$ is a solution of (2.5), then so is $\Phi(z)D(z)$, where $\Phi(z)$ is a polynomial with real coefficients.

[19] This is true if the phase is bounded and Hölder continuous.

If $\mathrm{Im}\eta=0$, Eq. (3.6) is a straightforward matrix analog of the single-channel equation of Ref. 4. Equation (3.6) then has no singularity at $s'=s$. In fact, it will be a Fredholm equation if $B(s)$ meets appropriate boundedness conditions at $s=\infty$. Sufficient conditions on B (discussed in Ref. 4) are met in those single-particle exchange models which are usually considered nonsingular; viz., those based on exchange of spin-0 or spin-$\frac{1}{2}$ particles.

To account for $\mathrm{Im}\eta\neq 0$, two procedures come to mind. The first is a simple-minded perturbation procedure in which (3.6) is first solved with $\mathrm{Im}\eta=0$. This "unperturbed" solution is substituted under the integral sign in (3.6). Then an evaluation of (3.6) with the complete expressions for K and f gives the first-order perturbed solution. If $\rho^{-1}\,\mathrm{Im}\eta$ is small compared to B, in some sense, this method might work. Convergence of the perturbation series has not been investigated. A second method is to relate (3.6) to an equivalent Fredholm equation. This approach has the advantage of mathematical rigor, but the Fredholm equation (derived in the following section) is somewhat complicated. However, it too can be treated by a perturbation method when $\rho^{-1}\,\mathrm{Im}\eta$ is small. The perturbation method can probably be justified in this case by a standard trick.[20]

4. THE NONSINGULAR INTEGRAL EQUATION

Equations of the type (3.6) are dealt with by Muskhelishvili.[21] I shall review the relevant part of the theory very briefly. Write Eq. (3.6) as

$$\mathbf{K}n=f. \tag{4.1}$$

By addition and subtraction of $K(s,s)$ in the numerator of the integral, the equation becomes

$$\mathbf{K}n=\mathbf{K}_0n+\mathbf{k}n=\alpha(s)n(s)+\frac{\beta(s)}{\pi i}\int_P\frac{n(s')\,ds'}{s'-s}+\frac{1}{\pi i}\int_P k(s,s')n(s')\,ds'=f(s), \tag{4.2}$$

where

$$\beta(s)=K(s,s);\quad k(s,s')=\frac{K(s,s')-K(s,s)}{s-s'}. \tag{4.3}$$

If K is Hölder continuous (as it will be in the physical applications of the theory) the kernel $k(s,s')$ is "quasi-regular" at $s=s'$ [20,21]; i.e., it is subject to the Fredholm theory provided its behavior at infinity is satisfactory. In the following it will be assumed that the behavior of k at infinity is all right, so that k is a Fredholm kernel.

In fact, it will be assumed that all kernels encountered have good behavior at infinity, so that the only singularities to be concerned about are the Cauchy singularities at $s=s'$. The Cauchy singularity of (4.2) occurs in the term proportional to β. For this reason $\mathbf{K}_0$ is called the dominant part of the operator $\mathbf{K}$, where

$$\mathbf{K}_0n=\alpha(s)n(s)+\frac{\beta(s)}{\pi i}\int_P\frac{n(s')\,ds'}{s'-s}. \tag{4.4}$$

Multiplication of two operators of the type of $\mathbf{K}$ yields[21]

$$\mathbf{K}_1(\mathbf{K}_2\phi)=[\alpha_1(s)\alpha_2(s)+\beta_1(s)\beta_2(s)]\phi(s)+\frac{P}{\pi i}\int\frac{[\alpha_1(s)K_2(s,t)+K_1(s,t)\alpha_2(t)]\phi(t)\,dt}{t-s}+\frac{1}{(\pi i)^2}\int\left[P\int\frac{K_1(s,u)K_2(u,t)\,du}{(u-s)(t-u)}\right]\phi(t)\,dt. \tag{4.5}$$

The second integral over t in (4.5) is proper, so the operator $\mathbf{K}_1\mathbf{K}_2$ can be made into a Fredholm operator if the dominant part of the operator represented by the first t integral in (4.5) can be made to vanish. If $\mathbf{K}^*=\mathbf{K}_1\mathbf{K}_2$, then

$$\alpha^*=\alpha_1\alpha_2+\beta_1\beta_2,\qquad \beta^*=\alpha_1\beta_2+\beta_1\alpha_2. \tag{4.6}$$

If $\beta^*=0$, $\mathbf{K}^*$ is Fredholm. In terms of the "sum" and "difference" functions $S=\alpha+\beta$, $D=\alpha-\beta$, Eqs. (4.6) read

$$S^*=\alpha^*+\beta^*=(\alpha_1+\beta_1)(\alpha_2+\beta_2)=S_1S_2,\qquad D^*=\alpha^*-\beta^*=(\alpha_1-\beta_1)(\alpha_2-\beta_2)=D_1D_2. \tag{4.7}$$

The condition $\beta^*=0$ is the same as

$$S^*=D^*=S_1S_2=D_1D_2. \tag{4.8}$$

Suppose $\mathbf{K}_2$ is given, and $\mathbf{K}_1$ is to be determined so that (4.8) holds. Then $\mathbf{K}_1$ may be chosen in infinitely many ways, provided S_2 and D_2 are nonsingular everywhere. This is seen by noting that for arbitrary S^*, the following expressions solve (4.8):

$$S_1=S^*S_2^{-1},\quad D_1=S^*D_2^{-1}. \tag{4.9}$$

The dominant part of $\mathbf{K}_1$ is determined from S_1 and D_1, while the nondominant part is arbitrary; it may be set equal to zero. The choice $S^*=1$ is convenient for present purposes, and it also has the nice property of making $\mathbf{K}_1\mathbf{K}_2$ and $\mathbf{K}_2\mathbf{K}_1$ both Fredholm.

If a singular integral equation is written

$$\mathbf{K}_2\phi=f \tag{4.10}$$

and $\mathbf{K}_1\mathbf{K}_2$ is a Fredholm operator, then all of the solutions of (3.17) are included among the solutions of the Fredholm equation

$$\mathbf{K}_1\mathbf{K}_2\phi=K_1f. \tag{4.11}$$

[20] The kernel of a Fredholm equation may be divided into a large part and a small part, and the small part treated by an iterative method. See S. G. Mikhlin, *Linear Integral Equations* (Hindustan Publications Corporation, Delhi, 1960).

[21] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953).

[Assume that $f\in L^2$, and that $\mathbf{K}_1$ maps L^2 into itself, so that (4.11) is indeed a Fredholm equation.] In the physical application, Eq. (4.11) presumably has a unique L^2 solution, since in the usual models vanishing of the Fredholm determinant would be an incredible accident. Thus, if (4.10) has an L^2 solution at all, it is unique, and is the same as the unique solution of (4.11). But how can one be sure that (4.10) has an L^2 solution? Note that (4.11) may be expressed as

$$\mathbf{K}_1(\mathbf{K}_2\phi-f)=0, \tag{4.12}$$

so the L^2 solution of (4.11) also solves (4.10) provided $\mathbf{K}_1\Psi=0$ has no nontrivial solution in L^2. If the latter equation has a nontrivial solution Ψ, then so does the equation

$$\mathbf{K}_2\mathbf{K}_1\Psi=0. \tag{4.13}$$

This is not possible if $\mathbf{K}_2\mathbf{K}_1$ is a Fredholm operator with nonvanishing determinant. In that case if follows that the solution of (4.11) is also a solution of (4.10). With the choice $S^*=1$ mentioned above, $\mathbf{K}_2\mathbf{K}_1$ is a Fredholm operator, and it undoubtedly has nonvanishing determinant in the physical examples of interest. Therefore, for the purposes of the N/D method the regularized equation (4.11) is equivalent to the singular equation (4.10), provided $S^*=1$ and S_2 and D_2 are nonsingular.

If $\mathbf{K}_2=\mathbf{K}$ in (4.2), then

$$\begin{aligned} S_2&=\rho^{-1}\eta^*\rho=\rho^{-1/2}H^{1/2*}\rho^{1/2},\\ D_2&=\rho^{-1}\eta\rho=\rho^{-1/2}H^{1/2}\rho^{1/2}, \end{aligned} \tag{4.14}$$

by (3.7) and (4.3). Since H is positive-definite, $\det S_2=\det H^{1/2*}$ and $\det D_2=\det H^{1/2}$ are nonvanishing. From (4.9) and $S^*=1$ the regularizing operator $\mathbf{K}_1$ has the coefficients

$$\alpha_1=\rho^{-1}\,\mathrm{Re}(\eta^{-1})\rho,\quad \beta_1=-i\rho^{-1}\,\mathrm{Im}(\eta^{-1})\rho. \tag{4.15}$$

The nondominant part of the regularizing operator is set equal to zero.

The final integral equation $\mathbf{K}_1\mathbf{K}_2 n=\mathbf{K}_1 f$ is as follows:

$$\begin{aligned} n(s)&=g(s)+\frac{1}{\pi}\int_P\frac{H(s,s')n(s')\,ds'}{s-s'},\\ g(s)&=\rho^{-1}(s)\mathrm{Re}[\eta^{-1}(s)][\rho(s)B(s)-\tfrac{1}{2}\,\mathrm{Im}\eta(s)]\\ &\quad-\rho^{-1}(s)\mathrm{Im}[\eta^{-1}(s)]\rho(s)\frac{P}{\pi}\int_P\\ &\quad\times[B(s')-\tfrac{1}{2}\rho^{-1}(s')\mathrm{Im}\eta(s')]ds'/(s'-s),\\ \rho(s)H(s,s')&=-i\,\mathrm{Re}[\eta^{-1}(s)]\rho(s)K(s,s')\\ &\quad-\mathrm{Im}[\eta^{-1}(s)]\rho(s)\rho^{-1}(s')\mathrm{Re}\eta(s')\rho(s')+\frac{i}{\pi}\,\mathrm{Im}[\eta^{-1}(s)]\\ &\quad\times\rho(s)\frac{P}{\pi}\int_P\frac{K(s'',s')\,ds''}{(s''-s)(s'-s'')}(s-s'). \end{aligned} \tag{4.16}$$

Here $K(s,s')$ and $B(s)$ are the same as in Eq. (3.7). As is expected, $H(s,s)=0$. Also, the equation reduces to Eq. (3.6) if $\mathrm{Im}\eta=0$ in both equations.

Perhaps the most direct way to obtain the scattering matrix from the solution n of (4.16) is by first constructing D from Eq. (3.3) and the definition $\mathrm{Im}D=-\rho n$. Then from $S=H^{1/2}\Omega$ and (2.5), (2.10),

$$S=H^{1/2}\rho^{-1/2}D_-D_+^{-1}\rho^{1/2}. \tag{4.17}$$

An equivalent expression is

$$1+2i\rho T_+=\eta D_-D_+^{-1}. \tag{4.18}$$

High-energy conditions on H and g sufficient to make (4.16) a Fredholm equation will not be studied here, since the general problem is too academic. It seems certain that "regular" Born terms appearing in $B(s)$ (i.e., Born terms for exchange of spin-0 or spin-$\frac{1}{2}$ particles) will have suitable high-energy behavior. As usual, higher spin exchanges must be subjected to a cut-off procedure. As in the single-channel problem, the eigenvalues of η must not approach zero too rapidly at infinity, since they occur as divisors in the kernel. In the Białas–van Hove[7] model of the η matrix, the eigenvalues tend rapidly to a constant.

5. SYMMETRY OF THE SCATTERING MATRIX

Since the absorption matrix F is not symmetric in general, it is by no means obvious that the solution of the dispersion relation (3.2) will be symmetric, even if the jump ΔT over the unphysical cuts is a symmetric matrix. It is, therefore, gratifying to be able to prove that the solution T obtained through Eq. (4.16) is indeed symmetric for arbitrary F, so long as ΔT is symmetric. The argument follows Bjorken and Nauenberg,[22] who prove (neglecting inelasticity) that

$$D^T(T-T^T)D\equiv 0 \tag{5.1}$$

provided $\Delta T=\Delta T^T$. Since D is nonsingular except possibly at isolated points, it follows that $T=T^T$. The jump of the matrix (5.1) over the unphysical cuts U is

$$\Delta[D^T(T-T^T)D]=D^T(\Delta T-\Delta T^T)D=0 \tag{5.2}$$

since D is analytic on those cuts. On the physical cut P, substitute $T=ND^{-1}$ and compute

$$\begin{aligned} \Delta[D^T(T-T^T)D]&=\mathrm{Im}D^T\,\mathrm{Re}N+\mathrm{Re}D^T\,\mathrm{Im}N\\ &\quad-\mathrm{Im}N^T\,\mathrm{Re}D-\mathrm{Re}N^T\,\mathrm{Im}D. \end{aligned} \tag{5.3}$$

Eliminate N by means of (2.13). The Hermitian property of H and Eq. (2.12) show that

$$\mathrm{Im}\eta^T=-\rho^{-1}\,\mathrm{Im}\eta\rho;\quad \mathrm{Re}\eta^T=\rho^{-1}\,\mathrm{Re}\eta\rho. \tag{5.4}$$

By using (5.4), it is easy to show that the expression (5.3) is identically zero. That is true even if D has

[22] J. D. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961).

CDD poles, which can be accounted for by delta functions in ImD. Thus, the matrix $D^T(T-T^T)D$ has zero discontinuity over the cuts of D and T, and it vanishes at infinity by the assumed asymptotic behavior of T and D (cf. Sec. 3). Consequently, Eq. (5.1) is true.

6. MULTIPLICATIONS BY $\rho^{-1/2}$ AND ρ^{-1}

One small problem that remains is to show that the multiplications by $\rho^{-1/2}$ and ρ^{-1} in Eqs. (2.10), (2.12), (3.5), (3.7), and (4.16) cause no difficulty when elements of ρ vanish due to closing of channels. Suppose that the nth channel has the threshold s_{0n}, and that the nth diagonal element of ρ is proportional to Q just above threshold. Here Q is a suitable power of the momentum of the nth channel (e.g., $Q=q_n{}^{2l+1}$). To make the discussion simple, suppose that only the nth channel has its threshold at s_{0n}. All matrices involved in multiplications with $\rho^{-1/2}$ or ρ^{-1} are constructed from the eigenvectors Ψ of H. Therefore, it is appropriate to study Ψ at small Q by a perturbation method. First decompose H into terms of zeroth, first, and second orders in $Q^{1/2}$:

$$H=1-4\rho^{1/2}F\rho^{1/2}=H_0+H_1+H_2,$$

$$H_0=\begin{bmatrix}1-4\rho_1{}^{1/2}F_1\rho_1{}^{1/2} & 0\\ 0 & 1\end{bmatrix},\quad H_1=\begin{bmatrix}0 & -4\rho_1{}^{1/2}G\rho_2{}^{1/2}\\ -4\rho_2{}^{1/2}G^\dagger\rho_1{}^{1/2} & 0\end{bmatrix},\quad H_2=\begin{bmatrix}0 & 0\\ 0 & -4\rho_2{}^{1/2}F_2\rho_2{}^{1/2}\end{bmatrix},$$

$$F=\begin{bmatrix}F_1 & G\\ G^\dagger & F_2\end{bmatrix},\quad \rho=\begin{bmatrix}\rho_1 & 0\\ 0 & \rho_2\end{bmatrix}. \tag{6.1}$$

In these matrices the block in the upper left corner is $(n-1)\times(n-1)$; the 1×1 matrix ρ_2 is proportional to Q for $s\gtrsim s_{0n}$, and zero for $s<s_{0n}$. One of the eigenvectors of H_0 is $\Psi_0{}^b=(00\cdots01)$. The other $n-1$ eigenvectors of H_0 are denoted $\Psi_{0i}{}^a$; they have vanishing nth components. The first-order perturbation H_1 causes transitions $a\leftrightarrow b$ but does not allow $a\leftrightarrow a$, $b\leftrightarrow b$. Standard perturbation theory yields the eigenvectors

$$\begin{aligned}\Psi^b&=\Psi_0{}^b-\sum_i \Psi_{0i}{}^a\frac{(\Psi_{0i}{}^a,H_1\Psi_0{}^b)}{\lambda_{0i}-1}+O(Q),\\ \Psi_i{}^a&=\Psi_{0i}{}^a-\Psi_0{}^b\frac{(\Psi_0{}^b,H_1\Psi_{0i}{}^a)}{1-\lambda_{0i}}+O(Q),\end{aligned} \tag{6.2}$$

where $H_0\Psi_{0i}{}^a=\lambda_{0i}\Psi_{0i}{}^a$. Now it is clear that the dyadics constructed from the Ψ's may be represented as follows:

$$\begin{aligned}\Psi^b(\Psi^b)^T&=\begin{bmatrix}O(Q) & O(Q^{1/2})\\ O(Q^{1/2}) & 1+O(Q)\end{bmatrix},\\ \Psi_i{}^a(\Psi_i{}^a)^T&=\begin{bmatrix}\Psi_{0i}{}^a(\Psi_{0i}{}^a)^T+O(Q) & O(Q^{1/2})\\ O(Q^{1/2}) & O(Q)\end{bmatrix}.\end{aligned} \tag{6.3}$$

From (6.3) it is apparent that the limit of $\rho^{1/2}\Psi\Psi^T\rho^{-1/2}$ as $Q\to0$ exists. A similar statement holds if Ψ^T is replaced by $\Psi^\dagger$. Hence, the definitions (2.10) and (2.12) of M and η are generally valid. The other possibily troublesome matrices that enter the discussion are $\rho^{-1}(1-\mathrm{Re}\eta)$, $\rho^{-1}\,\mathrm{Im}\eta$, $\rho^{-1}\,\mathrm{Re}\eta\rho$, $\rho^{-1}\,\mathrm{Re}[\eta^{-1}]\rho$, and $\rho^{-1}\,\mathrm{Im}[\eta^{-1}]\rho$. Using (6.3) and the definition of η one may check that these matrices are also well behaved. If more than one channel has the same threshold s_{0n}, a more elaborate discussion based on degenerate perturbation theory leads to similar conclusions.

7. REMARKS ON APPLICATIONS OF THE EQUATIONS

It is clear that numerical solution of the nonsingular integral Eq. (4.16) would be a complicated matter. We think, however, that Eq. (4.16) will be useful in the general mathematical analysis of coupled partial-wave dispersion relations—i.e., in generalization of the single-channel analysis of Refs. 4 and 10. In applications of the method to particular physical problems it will usually be necessary to make a suitably simple hypothesis about the behavior of the absorption matrix. In principle this matrix may be determined by experiment. In practice the determination will be far from complete, but we can expect at least some qualitative indications from experiment. At sufficiently high energies a theoretical argument of Białas and van Hove[7] may apply. These authors point out that the matrix element F_{ij} of Eq. (2.2) represents the overlap of the final states produced from initial states i and j. If $i\neq j$, these final states are likely to be nearly orthogonal at high energies where many channels are open. It is reasonable, therefore, to suppose that F is diagonal at high energies. Since F is Hermitian, it is then also real, and the relatively simple equation (3.6) becomes nonsingular. Better yet, F is actually a multiple of the unit matrix if it is diagonal (this was noted by Białas and van Hove[7], and also by Shaw,[23] who has independently studied matrix N/D equations with absorption). By additional physical arguments based on minimal correlation of the particles produced, Białas and van Hove obtain the following model of F:

$$\rho^{1/2}F\rho^{1/2}=f_0\exp(-b^2/r^2)I. \tag{7.1}$$

[23] G. L. Shaw, Bull. Am. Phys. Soc. **11**, 23 (1966).

The parameters f_0 and r are real constants, $b=l/q$ denotes the impact parameter, and I is the unit matrix. We intend to apply Eq. (7.1) with Eq. (3.6) to study absorptive corrections to the peripheral model. Of course, (7.1) may not hold at lower energies, and it remains to be seen whether its introduction in the low-energy region will be a serious error.

Another difficulty in the N/D approach to the pheripheral model is the difficulty of meeting the threshold conditions when left singularities are given by single-particle exchanges. To guarantee the correct q^{2l+1} momentum dependence at threshold,[4,24] one must choose the ρ_i of Eq. (2.1) to behave as q_i^{2l+1} at threshold, and to have the correct behavior at $s=\infty$ ($\rho_i \sim s^{1/2}$ at infinity for the spin 0–spin $\frac{1}{2}$ case). Except for s waves, such a choice of ρ necessarily introduces new poles in the T matrix; i.e., poles not present in the Born term for single-particle exchange. In the case of low partial waves, at least, the T matrix is known to depend strongly on the location (which is arbitrary) of the new poles.[24,25] In the peripheral model at high energies there are many partial waves participating, so it is not clear without calculation whether the dependence of T on the pole positions will be a severe difficulty in that model. In any case, one may attempt to adjust the pole positions to fit experiment. If the poles turn out to be far from the physical region, their existence may be ascribed to the unknown short-range forces necessary to produce correct threshold behavior. Since low partial waves will be largely absorbed, the dependence of results on pole positions may be less pronounced than it is in the case of bootstrap models.

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APPENDIX

Here I establish the parametrization of the S matrix stated in Eq. (2.7). Recall the definition (2.3) of H:

$$SS^\dagger = 1 - 4\rho^{1/2}F\rho^{1/2} = H, \tag{A1}$$

where F is the absorption matrix appearing in the unitarity relation (2.2). One wishes to find the most general symmetric matrix S satisfying (A1) for given positive-definite and Hermitian H. This problem was posed by Białas and van Hove.[7] Since $\rho^{1/2}F\rho^{1/2}$ is a unitarity sum, it is non-negative and Hermitian. Therefore,

$$0<h\leq 1, \quad 0\leq f<1, \tag{A2}$$

where h and f denote eigenvalues of H and $4\rho^{1/2}F\rho^{1/2}$, respectively. Equation (A2) permits the definition of the Hermitian matrix

$$H^{-1/2} = \sum_i \psi_i h_i^{-1/2} \psi_i^\dagger, \tag{A3}$$

where the ψ_i form a complete, orthonormal set of eigenvectors of H. From (A1) it follows that the matrix $H^{-1/2}S$ is unitary:

$$\Omega = H^{-1/2}S; \quad \Omega\Omega^\dagger = 1. \tag{A4}$$

Existence of Ω suggests the definition of a generalized reaction matrix $\mathcal{K}=\mathcal{K}^\dagger$:

$$\Omega = \frac{1+i\mathcal{K}}{1-i\mathcal{K}}; \quad S = H^{1/2}\frac{1+i\mathcal{K}}{1-i\mathcal{K}}. \tag{A5}$$

The representation (A5) of S has the disadvantage that $\mathcal{K}$ is subject to restrictions other than Hermiticity; viz., those due to symmetry of S. Białas and van Hove have applied Eq. (A5) in the special case in which H is a multiple of unity. In that event there is no difficulty, since the additional requirement on $\mathcal{K}$ is merely that it be symmetric. Note that the equation $S=H^{1/2}\Omega$ is the matrix analog of the exponential representation of a complex number: An arbitrary nonsingular matrix M may be factored as an Hermitian matrix times a unitary matrix.

Since the reaction matrix method turns out to be inconvenient, it may be better to look for a generalized complex phase shift Δ. One naturally expects $e^{2i\Delta}$ to be something resembling an eigenvalue of S. Since there is no guarantee that S may be diagonalized, the resemblance cannot be perfect. An effective substitute for diagonalization is to exploit the symmetry of S by relating S to a real, symmetric matrix of twice the dimension. The first step is to show that there is a complete, orthonormal set of vectors ψ such that

$$S\psi^* = \alpha\psi, \tag{A6}$$

where the orthogonality is with respect to the usual Hermitian scalar product:

$$(\psi_i,\psi_j) = \psi_i^\dagger\psi_j = \delta_{ij}. \tag{A7}$$

The pseudo-eigenvalue α is a complex number. From (A6) it follows that ψ is an eigenvector of H:

$$SS^*\psi = H\psi = |\alpha|^2\psi. \tag{A8}$$

From the ψ's and α's a manifestly symmetric representation of S may be constructed:

$$S = \sum_i \psi_i\alpha_i\psi_i^T. \tag{A9}$$

An arbitrary vector may be represented as $\psi=\sum_i b_i\psi_i^*$, and Eq. (A6) implies that

$$S\psi = \sum_i b_i\alpha_i\psi_i. \tag{A10}$$

By (A7) this is the same as the result of applying (A9) to ψ, so (A9) is indeed correct. To establish the exist-

[24] A. W. Martin and J. L. Uretsky, Phys. Rev. 135, B803 (1964).
[25] L. M. Simmons, Jr., Phys. Rev. 145, 1157 (1966).

ence of the ψ_i, note that the complex equation (A6) is equivalent to the real equation

$$\begin{bmatrix} \mathrm{Re}S & \mathrm{Im}S \\ \mathrm{Im}S & -\mathrm{Re}S \end{bmatrix} \begin{bmatrix} \mathrm{Re}\psi \\ \mathrm{Im}\psi \end{bmatrix} = |\alpha| \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \mathrm{Re}\psi \\ \mathrm{Im}\psi \end{bmatrix}, \quad \text{(A11)}$$

where $\alpha = |\alpha| e^{i\phi}$. Multiplication by the inverse of the matrix in ϕ yields

$$\begin{bmatrix} \cos\phi \mathrm{Re}S + \sin\phi \mathrm{Im}S & \cos\phi \mathrm{Im}S - \sin\phi \mathrm{Re}S \\ \cos\phi \mathrm{Im}S - \sin\phi \mathrm{Re}S & -\cos\phi \mathrm{Re}S - \sin\phi \mathrm{Im}S \end{bmatrix} = |\alpha| \begin{bmatrix} \mathrm{Re}\psi \\ \mathrm{Im}\psi \end{bmatrix}. \quad \text{(A12)}$$

With fixed ϕ the matrix of (A12) is real and symmetric, and therefore has a complete, orthonormal set of $2n$ eigenvectors. Precisely n of the eigenvectors are solutions of (A12), since the matrix has n positive and n negative eigenvalues. To verify the latter statement, just notice that if $[\mathrm{Re}\psi, \mathrm{Im}\psi]$ is an eigenvector with eigenvalue λ, then $[\mathrm{Im}\psi, -\mathrm{Re}\psi]$ is a eigenvector with eigenvalue $-\lambda$. There are no zero eigenvalues, because if there were there would be a ψ such that $S\psi^* = 0$, or $SS^*\psi = 0$, contrary to Eq. (A2). A set of n real solutions $[\mathrm{Re}\psi_i, \mathrm{Im}\psi_i]$ of (A12) may be chosen in such a way that

$$\mathrm{Re}\psi_i{}^T \mathrm{Re}\psi_j + \mathrm{Im}\psi_i{}^T \mathrm{Im}\psi_j = \delta_{ij}, \quad i, j = 1, \cdots, n. \quad \text{(A13)}$$

Each of these solutions is orthogonal to all eigenvectors with negative eigenvalue:

$$\mathrm{Re}\psi_i{}^T \mathrm{Im}\psi_j - \mathrm{Im}\psi_i{}^T \mathrm{Re}\psi_j = 0, \quad i, j = 1, \cdots, n. \quad \text{(A14)}$$

By combination of (A13) and (A14) the complex orthonormality equation (A7) follows, and the proof is complete. Note that by (A9) the α's are not unique; a change $\phi \to \phi + \theta$ in the phase of α may be compensated by the change $\psi \to e^{-i\theta/2}\psi$. In fact, the ψ's may always be defined so that the α's are real.

The expressions for S and H solve the problem stated:

$$S = \sum_i \psi_i \alpha_i \psi_i{}^T, \quad H = \sum_i \psi_i |\alpha_i|^2 \psi_i{}^\dagger, \quad (\psi_i, \psi_j) = \delta_{ij}, \quad |\alpha_i| \leq 1. \quad \text{(A15)}$$

When H (equivalently, F) is given, and a definite set of eigenvectors of H chosen, there remain just n real, free, parameters—the phases of α_i. The analog of the usual complex phase shift is Δ_i, defined by $\alpha_i = e^{2i\Delta_i}$. The unitary matrix Ω takes the form

$$\Omega = \sum_i \psi_i e^{i\phi_i} \psi_i{}^T, \quad \text{(A16)}$$

where $\alpha_i = |\alpha_i| e^{i\phi_i}$. It is a pleasant surprise that Ω is symmetric. In the special case in which H is real, its eigenvectors ψ may be chosen real. These vectors are then simultaneous eigenvectors of H, S, Ω, and $\mathcal{K}$. The fact that S may be diagonalized in this case is an example of the well-known theorem that a matrix may be diagonalized if it commutes with its adjoint

$$([S, S^\dagger] = 2i \,\mathrm{Im}H = 0).$$

When H is nondegenerate, there is remarkably little freedom in determining S after H is fixed. In that case the eigenvectors of H are unique up to phase factors, so that only the n real parameters α_i are free. When H is degenerate the ψ_i are not unique in direction, and S depends on a greater number of free parameters. When $H = 1$ the number is $n(n+1)/2$, as is seen from (A5) and the symmetry of S.