couplings, we found that

(i) $(f/d)_{V\bar{B}B} \simeq (F/D)_{T\bar{B}B} \simeq -2$.
(ii) The $\langle T\rangle\langle\bar{B}B\rangle$ coupling strength must necessarily be large.

Finally, we might emphasize that more precise measurements of the total cross sections will permit considerable refinement of the present analysis (especially for the Σ_{AB}) and will check the validity of this approach to high-energy scattering.

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Application of an Approximate Solution of Partial-Wave Dispersion Relations to Yukawa Potential Scattering

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The time-reversal symmetrization of the multichannel scattering amplitude proposed by Fulton and Shaw is used to construct the amplitude for nonrelativistic single-channel Yukawa potential scattering. It provides a modified determinantal method for the solution of the N/D equations. This amplitude is compared to the exact solution of the Schrödinger equation. It is found that the scattering lengths predicted by this method are qualitatively the same as those predicted by the full N/D equations and significantly better than the results of the determinantal method. The computational simplicity of the determinantal method has been retained and combined with the accuracy of the full N/D solution, which, with first-order Born approximation, gives quite a reliable picture of the qualitative features of the scattering.

I. INTRODUCTION

The N/D method of finding the relativistic scattering amplitude for single-channel scattering entails the solution of an integral equation and the evaluation of an integral. One frequently uses the determinantal method, which reduces the problem to the evaluation of a single integral. Although simple to use, this method gives results which often differ significantly from the results predicted by the full N/D method. It was shown by Luming[1] that for nonrelativistic Yukawa potential scattering the exact Schrödinger solution lies close to the full N/D solution with both first and second Born approximation as input to the N/D equations. The determinantal solution, again using Born approximation for input, is quite unreliable in predicting the features of the scattering.

The ordinary determinantal method has no time-reversal symmetry when applied to the multichannel problem. A modification of the N/D equations was proposed by Fulton[2] and Shaw[3] to restore the time-reversal symmetry and the main purpose of this paper is to examine how this modification affects the single channel nonrelativistic scattering for which the exact solution can be found for comparison. Another such modification, proposed by Nath and Srivastava,[4] was examined by Smith.[5]

It is found that in the first-order Born approximation both of these methods give results in qualitative agreement with those of the N/D equations for all the angular-momentum states and coupling strengths examined, so that a considerable amount of computational labor can be saved by using a modified determinantal method. Calculations are in progress at present to include second-order Born terms in both methods and preliminary results show that one can obtain fairly good quantitative agreement between them.

The application of the N/D and determinantal method to Yukawa scattering was examined by Luming,[1] and the reader is referred to that paper for details. A short discussion of the Fulton-Shaw method is given in Sec. II. The application to potential theory is discussed in Sec. III, where the effective-range expressions are derived. Finally, in Sec. IV, we give the results of the calculations, which were performed on the AMTRAN self-programming computer system, and our conclusions.

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[3] G. Shaw, Phys. Rev. Letters **12**, 345 (1964).
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[5] J. Smith (to be published).

II. FULTON-SHAW METHOD

The problem of constructing a partial-wave scattering amplitude from a knowledge of the discontinuity across the nearby cuts is usually solved by the N/D method. However, as is well known, there are several features in the N/D method which are undesirable. The problem of imposing the correct threshold behavior of the amplitude for partial waves with $l>0$ is, by the introduction of the ν^l factor, transferred to a problem of how to cut off the divergent integrals. Subtractions can, of course, be easily made, but it is difficult to decide where to make them. Many authors have stated that the solution to the full N/D equations is independent of the subtraction point, and, if that is the case, it should be possible to find a modification of the N/D equations in which subtraction points do not occur. This was first done by Fulton and Shaw. Hassoun and Kang[6] examined their result, and, in order to have a better understanding of the approximations made, calculated correction terms. However, the method of elimination of subtraction points due to Hassoun and Kang is lengthy and not really necessary. One can just write down dispersion relations for a suitable combination of the input "potential" and the inverse of the scattering amplitude to obtain the same result.

Suppose we take $B_l(\nu)$, $\nu=k^2$ as our known input or "potential," and write down a dispersion relation for the function

$$g_l(\nu)=B_l(\nu)[1/f_l(\nu)]B_l(\nu)-B_l(\nu)\,, \tag{2.1}$$

where $f_l(\nu)$ is the partial-wave scattering amplitude. The unitarity condition for $f_l(\nu)$ is $\operatorname{Im} f_l^{-1}(\nu)=-\rho_l(\nu)$ so that, on the right-hand cut where $B_l(\nu)$ is real,

$$\operatorname{Im} g_l(\nu)=-B_l(\nu)\rho_l(\nu)B_l(\nu) \tag{2.2}$$

while, on the left-hand cut, $\operatorname{Im}B_l(\nu)=\operatorname{Im}f_l(\nu)$ so

$$\operatorname{Im} g_l(\nu)=\operatorname{Re}(B_l(\nu)-f_l(\nu))\operatorname{Im}[B_l(\nu)/f_l(\nu)].$$

Hence

$$B_l(\nu)[1/f_l(\nu)]B_l(\nu)=-\frac{1}{\pi}\int_{\mathrm{RHC}}\frac{B_l(\nu')\rho_l(\nu')B_l(\nu')}{\nu'-\nu}d\nu' +\frac{1}{\pi}\int_{\mathrm{LHC}}\frac{\operatorname{Re}(B_l(\nu')-f_l(\nu'))}{\nu'-\nu}\operatorname{Im}\left[\frac{B_l(\nu')}{f_l(\nu')}\right]d\nu'\,,$$

or

$$\frac{B_l(\nu)}{f_l(\nu)}=1-\frac{1}{\pi B_l(\nu)}\int_{\mathrm{RHC}}\frac{B_l(\nu')\rho_l(\nu')B_l(\nu')}{\nu'-\nu}d\nu' +\frac{1}{\pi B_l(\nu)}\int_{\mathrm{LHC}}\frac{\operatorname{Re}(B_l(\nu')-f_l(\nu'))}{\nu'-\nu}\operatorname{Im}\left[\frac{B_l(\nu')}{f_l(\nu')}\right]d\nu'. \tag{2.3}$$

In the approximation where we neglect the contribution over the left-hand cut, on the grounds that $\nu'-\nu$ is large for values of ν on the right-hand cut, then we arrive at the expression

$$\frac{B_l(\nu)}{f_l(\nu)}=1-\frac{1}{\pi B_l(\nu)}\int_{\mathrm{RHC}}\frac{B_l(\nu')\rho_l(\nu')B_l(\nu')}{\nu'-\nu}d\nu'. \tag{2.4}$$

Equation (2.4) can be derived from the full N/D equations by the following trick, designed to eliminate the dependence on the subtraction point. Suppose we consider the N/D equations with one subtraction,

$$N_l(\nu)=B_l(\nu)+\frac{1}{\pi}\int_{\mathrm{RHC}}\left[B_l(\nu')-\frac{\nu-\nu_0}{\nu'-\nu_0}B_l(\nu)\right]\times\frac{\rho_l(\nu')N_l(\nu')}{\nu'-\nu}d\nu'\,, \tag{2.5}$$

$$D_l(\nu)=1-\frac{(\nu-\nu_0)}{\pi}\int_{\mathrm{RHC}}\frac{\rho_l(\nu')N_l(\nu')d\nu'}{(\nu'-\nu)(\nu'-\nu_0)}\,, \tag{2.6}$$

replace $N_l(\nu)$ by $C_l(\nu)B_l(\nu)$, and approximate $C_l(\nu')$ under the integral sign by $C_l(\nu)$. Equations (2.5) and (2.6) become

$$1=\frac{1}{C_l(\nu)}+\frac{1}{\pi B_l(\nu)}\int\frac{B_l(\nu')\rho_l(\nu')B_l(\nu')}{\nu'-\nu}d\nu' -\frac{\nu-\nu_0}{\pi}\int\frac{\rho_l(\nu')B_l(\nu')d\nu'}{(\nu'-\nu)(\nu'-\nu_0)}$$

and

$$\frac{D_l(\nu)}{C_l(\nu)}=\frac{1}{C_l(\nu)}-\frac{(\nu-\nu_0)}{\pi}\int\frac{\rho_l(\nu')B_l(\nu')}{(\nu'-\nu)(\nu'-\nu_0)}d\nu'.$$

Hence,

$$\frac{B_l(\nu)}{f_l(\nu)}=\frac{D_l(\nu)}{C_l(\nu)}=1-\frac{1}{\pi B_l(\nu)}\int\frac{B_l(\nu')\rho_l(\nu')B_l(\nu')d\nu'}{\nu'-\nu}\,, \tag{2.7}$$

which is the result given by (2.4). The drawback of using dispersion relations for $f_l^{-1}(\nu)$ is the fact that $f_l(\nu)$ may have zeros which would produce additional poles in the dispersion relation for $f_l^{-1}(\nu)$. Suppose we take the first-order Born approximation, for example, and replace $f_l(\nu)$ by $B_l(\nu)$, then $f_l(\nu)$ would be a product of Legendre functions of the first and second kind. The Legendre function of the first kind has a number of zeros depending on its order, but they are usually situated on the left-hand cut below the physical threshold, and are probably so far away as to have no influence in the physical region. This problem is discussed in more detail in the paper of Franklin.[7] Our case of Yukawa-potential scattering is special because the input term does not contain any zeros. Therefore we do not go any further into this question.

[6] G. Q. Hassoun and K. Kang, Phys. Rev. **137**, B955 (1965).

[7] J. Franklin, Phys. Rev. **139**, B912 (1965).

III. APPLICATION TO POTENTIAL THEORY

We consider the case of an attractive Yukawa potential

$$V(r)=-g^2e^{-\mu r}/r\,, \tag{3.1}$$

with the partial wave amplitude defined by

$$\begin{aligned}h_l(\nu)&=\nu^{-1/2}\exp i\delta_l(\nu)\sin\delta_l(\nu)\\&=\nu^{-1/2}[\cot\delta_l(\nu)-i]^{-1}.\end{aligned} \tag{3.2}$$

The first Born approximation for the amplitude is

$$h^{(1)}(\nu,\cos\theta)=\frac{g^2}{\mu^2+2\nu(1-\cos\theta)} \tag{3.3}$$

with partial-wave projection

$$b_l^{(1)}(\nu)=(g^2/2\nu)Q_l(1+\mu^2/2\nu)\,. \tag{3.4}$$

Because of the analytic properties of the Legendre function of the second kind, $b_l^{(1)}(\nu)$ is an analytic function of ν with a cut from $\nu=-\mu^2/4$ to $-\infty$. We now approximate the whole of the left-hand cut by assuming it to be the same as this first Born cut, so we let

$$B_l(\nu)=b_l^{(1)}(\nu)\,. \tag{3.5}$$

Unfortunately, (3.4) does not satisfy the correct threshold behavior for higher partial waves, so we disperse in the new amplitude

$$f_l(\nu)=h_l(\nu)/\nu^l \tag{3.6}$$

with a new first-order Born approximation given by

$$B_l(\nu)=\frac{g^2}{2\nu^{l+1}}Q_l(1+\mu^2/2\nu)\,. \tag{3.7}$$

The N/D equations for $f_l(\nu)$ now diverge, so that a subtraction should be made. Luming preferred to use a simple cutoff and examined the dependence of the results on the variation of the cutoff. We would like to use (2.4), with (3.7) as input, i.e.,

$$\frac{1}{f_l(\nu)}\frac{g^2}{2\nu^{l+1}}Q_l\left(1+\frac{\mu^2}{2\nu}\right)=1-\frac{g^2}{2\pi}\frac{\nu^{l+1}}{Q_l(1+\mu^2/2\nu)}\int_0^\infty\left(\frac{Q_l(1+\mu^2/2\nu')}{\nu'^{l+1}}\right)^2\frac{\nu'^{l+1/2}}{\nu'-\nu}d\nu'.$$

Let us now put $\mu^2=1$, and use the abbreviation

$$K_l(\nu)=\frac{1}{\pi}P\int_0^\infty\left(\frac{Q_l(1+1/2\nu')}{\nu'^{l+1}}\right)^2\frac{\nu'^{l+1/2}}{\nu'-\nu}d\nu', \tag{3.8}$$

where P denotes the Cauchy principal value integral. The expression for the effective range is now

$$\nu^{l+1/2}\cot\delta_l(\nu)=\left(1-\frac{g^2}{2}\frac{\nu^{l+1}}{Q_l(1+1/2\nu)}K_l(\nu)\right)\times\left[\frac{g^2}{2\nu^{l+1}}Q_l\left(1+\frac{1}{2\nu}\right)\right]^{-1}. \tag{3.9}$$

Unfortunately the integral $K_l(\nu)$ cannot be evaluated analytically in terms of elementary functions. It is possible, however, to express $K_l(\nu)$ in terms of Clausen's integral as shown in Appendix A.

The analyticity properties of $K_l(\nu)$ now allow one to extend the result (A7) to all values of l. We need to construct a function whose discontinuity across the left-hand cut is

$$i\nu^{l+1/2}[Q_l(1+1/2\nu)/\nu^{l+1}]^2 \tag{3.10}$$

and which has no other singularities. Remember that

$$Q_l(z)=P_l(z)Q_0(z)-W_{l-1}(z)\,,$$

where $W_{l-1}(z)$ is a polynomial of order $(l-1)$. The functions $P_l(1+1/2\nu)$ and $W_{l-1}(1+1/2\nu)$ are analytic everywhere in the ν plane except at the origin, where they have lth- and $(l-1)$th-order poles, respectively. Suppose we take

$$\begin{aligned}A_l(\nu)=&\left(\frac{P_l(1+1/2\nu)}{\nu^{l/2}}\right)^2\left\{\frac{1}{\nu^{3/2}}\ln|1+4\nu|\arctan(2\nu^{1/2})-\frac{4}{\nu}\ln 2+\frac{1}{\nu^{3/2}}\mathrm{Cl}_2(\pi-2\arctan(2\nu^{1/2}))+i\nu^{1/2}\left(\frac{Q_0(1+1/2\nu)}{\nu}\right)^2\right\}\\&-\frac{2W_{l-1}(1+1/2\nu)}{\nu^{l/2+1}}\frac{P_l(1+1/2\nu)}{\nu^{l/2}}\left\{\frac{\arctan(2\nu^{1/2})}{\nu^{1/2}}+\frac{i\nu^{1/2}Q_0(1+1/2\nu)}{\nu}\right\}+i\nu^{1/2}\left(\frac{W_{l-1}(1+1/2\nu)}{\nu^{l/2+1}}\right)^2,\end{aligned} \tag{3.11}$$

where $\mathrm{Cl}_2(\theta)$ is the Clausen integral

$$\mathrm{Cl}_2(\theta)\equiv-\int_0^\theta\ln(2\sin\tfrac{1}{2}t)dt\,.$$

The imaginary part of this function reduces to

$$\mathrm{Im}A_l(\nu)=\nu^{l+1/2}[Q_l(1+1/2\nu)/\nu^{l+1}]^2$$

as required. However, the real part of $A_l(\nu)$ given by

$$\mathrm{Re}A_l(\nu)=\left(\frac{P_l(1+1/2\nu)}{\nu^{l/2}}\right)^2\left\{\frac{1}{\nu^{3/2}}\ln|1+4\nu|\arctan(2\nu^{1/2})-\frac{4}{\nu}\ln2+\frac{1}{\nu^{3/2}}\mathrm{Cl}_2(\pi-2\arctan(2\nu^{1/2}))\right\}$$
$$-\frac{2W_{l-1}(1+1/2\nu)P_l(1+1/2\nu)}{\nu^{l+1}}\frac{\arctan(2\nu^{1/2})}{\nu^{1/2}} \qquad (3.12)$$

has poles at the origin, which have to be subtracted out before we can identify $\mathrm{Re}A_l(\nu)$ with $\mathrm{Re}K_l(\nu)$. The expansion of all the terms for small ν involves the multiplication of several terms and a lot of tedious algebra. Depending on l, one has to go to several orders of ν. Using the series expansion and a recurrence relation of Clausen's integral,[8] expanding the logarithm and arctan, and collecting all terms of the curly bracket of (3.12), we obtain for the expansion in terms of ν:

$$\{\text{bracket of (3.12)}\}=\tfrac{2}{3}\alpha^3(1-\ln2)$$
$$+\alpha^5\nu[\tfrac{2}{5}\ln2-(7/15)]+\alpha^7\nu^2[-(2/7)\ln2+(2/15)+(11/135)+(106/810)]$$
$$+\alpha^9\nu^3[(2/9)\ln2-(2/21)-(11/225)-(106/2835)-(1399/28\,350)]. \qquad (3.13)$$

This is enough for s and p waves. We now subtract the divergent terms (3.12) to find,

$$K_1(\nu)=\frac{(1+1/2\nu)^2}{\nu}\left\{\frac{1}{\nu^{3/2}}\ln|1+4\nu|\arctan(2\nu^{1/2})-\frac{4}{\nu}\ln2+\frac{1}{\nu^{3/2}}\mathrm{Cl}_2(\pi-2\arctan(2\nu^{1/2}))-\frac{16}{3}(1-\ln2)\right\}$$
$$-32\left(\frac{2}{5}\ln2-\frac{7}{15}\right)\left(\frac{1}{\nu}+\frac{1}{4\nu^2}\right)-\frac{32}{\nu}\left(\frac{103}{315}-\frac{2}{7}\ln2\right)-\frac{2}{\nu}\left(1+\frac{1}{2\nu}\right)\left(\frac{\tan^{-1}(2\nu^{1/2})}{\nu^{1/2}}-2\right)-\frac{8}{3\nu}. \qquad (3.14)$$

Actually, to find $K(\nu)$ for d waves, would require terms up to the sixth order in the expansion of the large bracket of (3.12). This is probably not worth the labor involved.

Expressions for the scattering length for s and p waves can be found from the above expansions. At $\nu=0$

$$Q_l(1+1/2\nu)/\nu^{l+1}=\pi^{1/2}\Gamma(l+1)/\Gamma(l+\tfrac{3}{2})$$

so, for s waves

$$a_0^{-1}=[1-\tfrac{1}{4}g^2K_0(0)]/g^2, \qquad (3.15)$$

where

$$\tfrac{1}{4}K_0(0)=\tfrac{4}{3}(1-\ln2)=0.409;$$

while for p waves

$$a_1^{-1}=[1-0.375g^2K_1(0)]/\tfrac{2}{3}g^2 \qquad (3.16)$$

and

$$\frac{K_1(0)}{8}=32[\tfrac{2}{5}\ln2-(7/15)]$$
$$+128[-(2/7)\ln2+(103/315)]$$
$$+128[(2/9)\ln2-(1297/5670)]\simeq-0.02. \qquad (3.17)$$

The effective range values were then calculated by two different methods. Equation (3.8) was integrated in Huntsville using the AMTRAN computer system and the result substituted into (3.9) to give a plot of the effective range values for s, p and d waves. At the same time the scattering lengths and two effective-range values were calculated in Copenhagen using formulas

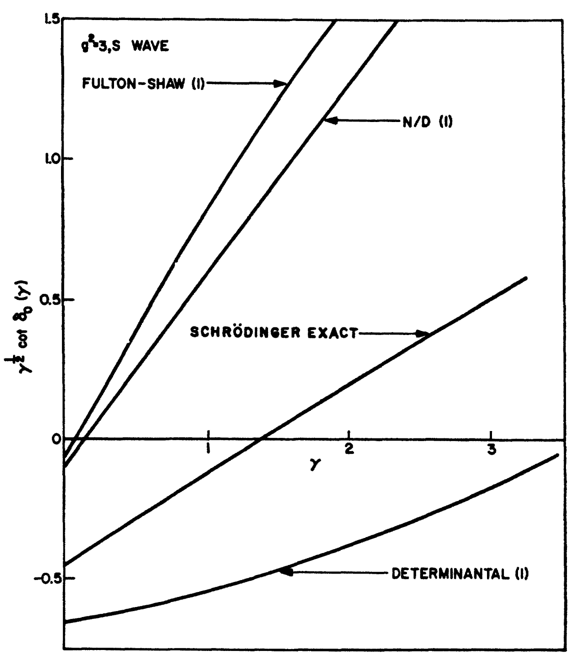

FIG. 1. Effective-range plot for S wave, $g^2=3$, $\mu=1$. The Schrödinger equation is solved exactly while the N/D Fulton-Shaw and determinantal methods are evaluated using first Born approximation only.

[8] L. Lewin, *Dilogarithms and Associated Functions* (MacDonald and Company Ltd., London, 1958).

(A7), (3.14), (3.15), (3.16), and a table of Clausen's integral. The resulting numbers checked with each other which gives us confidence that d-wave values from the computer are also reliable. The resulting curves for $g^2=3$ and $g^2=1$ are plotted in Figs. 1 to 6. We did not calculate the D function for negative energies to look for bound-state poles because we were mainly interested in the application of this method to scattering problems.

The determinantal method of Baker[9] assumes that the $N_l(\nu)$ function is the same as its first Born approximation, so we get the following expression for the partial-wave scattering amplitude.

$$\frac{B_l(\nu)}{f_l(\nu)}=1-\frac{1}{\pi}\int\frac{B_l(\nu')\nu'^{l+1/2}}{\nu'-\nu}d\nu'. \qquad (3.18)$$

After substituting for $B_l(\nu')$ from (3.7) the effective-range expansion becomes

$$\nu^{l+1/2}\cot\delta_l(\nu)=(1-\tfrac{1}{2}g^2M_l(\nu))\times\left[\frac{g^2}{2\nu^{l+1}}Q_l\left(1+\frac{1}{2\nu}\right)\right]^{-1}, \qquad (3.19)$$

where

$$M_l(\nu)=\frac{1}{\pi}P\int_0^\infty\frac{Q_l(1+\mu^2/2\nu')\nu'^{l+1/2}}{\nu'^l(\nu'-\nu)}d\nu'. \qquad (3.20)$$

Obviously $M_l(\nu)$ and $K_l(\nu)$ have different properties with regard to convergence. Both integrals are convergent in potential theory while their equivalents in relativistic scattering problems do not have this behavior. The equivalent of (3.8) is convergent, but the equivalent of (3.20) is divergent, and requires a subtraction. Luming has integrated (3.20) for s, p, and d waves and we quote his result:

$$M_l(\nu)=P_l(1+1/2\nu)[\tan^{-1}(2\nu^{1/2})/\nu^{1/2}]-A_l(\nu),$$

where

$$A_0(\nu)=0;\quad A_1(\nu)=1/\nu;\quad A_2(\nu)=2/\nu+3/4\nu^2.$$

The effective-range values calculated from (3.19) are also shown in Figs. 1 to 6.

Finally, we have checked some of the results given by Luming for the effective-range values calculated from an exact solution of the Schrödinger equation and for the full N/D equations with the first-order Born term. These two curves are plotted for comparison in Figs. 1 to 6.

IV. RESULTS AND CONCLUSIONS

The original purpose of this investigation was to compare the effective-range values calculated from the full N/D equations and the Fulton-Shaw approximation. It is obvious from Figs. 1 to 6 that the Fulton-Shaw approximation predicts values fairly close to the N/D results, although, in the first-order Born approximation, both of them are poor representations of the Schrödinger-equation values. As an approximation

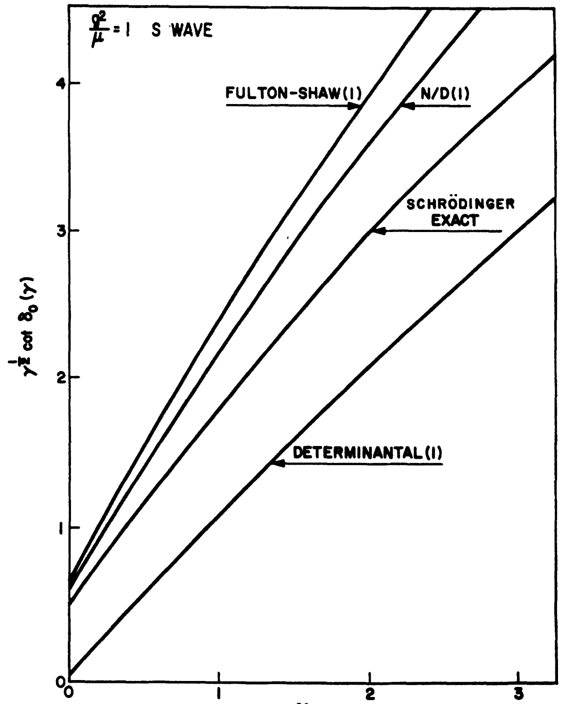

FIG. 2. Effective-range plot for S-wave scattering with $g^2=1$.

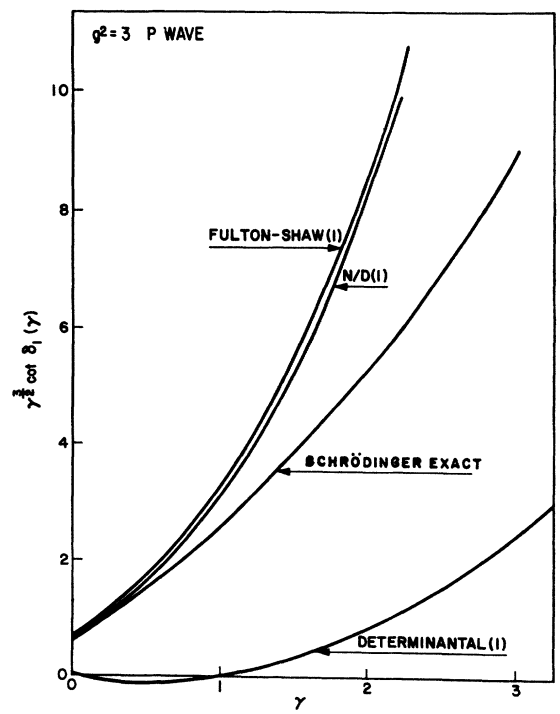

FIG. 3. Effective-range plot for P-wave scattering with $g^2=3$.

[9] M. Baker, Ann. Phys. (N. Y.) 4, 271 (1958).

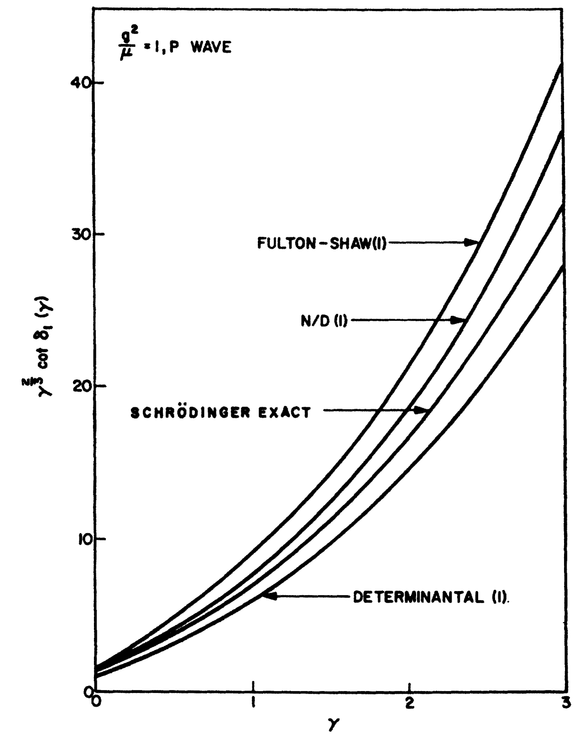

FIG. 4. Effective-range plot for P-wave scattering with $g^2=1$.

then, the Fulton-Shaw approach will give values which are qualitatively equivalent to the ones given by the full N/D equations. However, the amount of time saved in computation when using the Fulton-Shaw method is considerable. One does not have to solve the Fredholm equation for $N_l(\nu)$ and one does not have any subtraction points in the formalism.

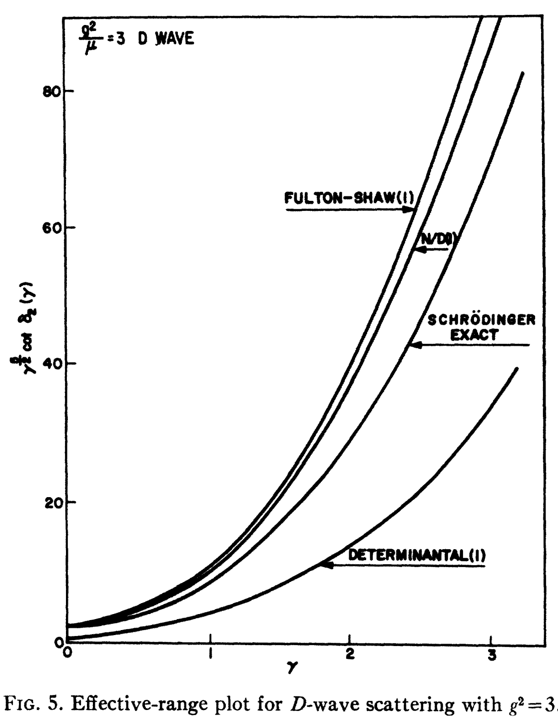

FIG. 5. Effective-range plot for D-wave scattering with $g^2=3$.

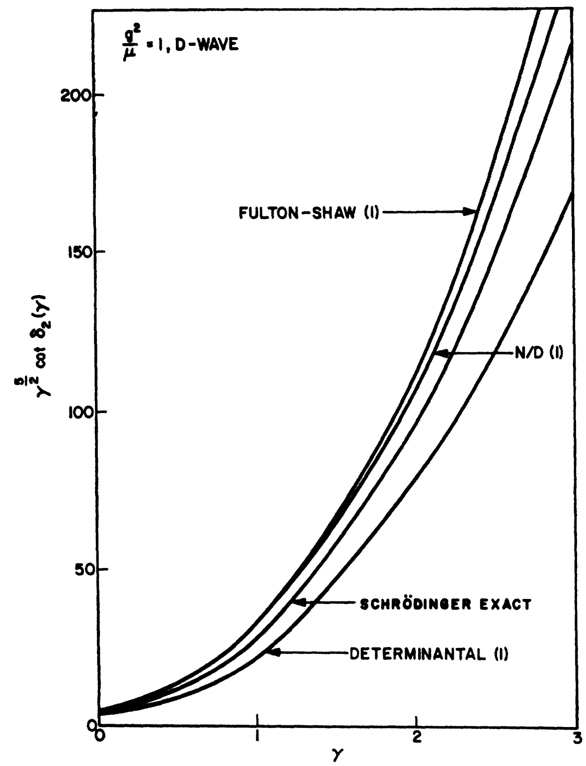

FIG. 6. Effective-range plot for D-wave scattering with $g^2=1$.

When we compare the Fulton-Shaw approximation with the determinantal one, there is a large difference between their respective results. The fact is that both methods involve a mutilation of the N/D equations, designed to produce approximations involving only principal-value integrals. Nevertheless, the convergence properties of these integrals are very different and this, together with the multiplicative factor of $[B_l(\nu)]^{-1}$ with the Fulton-Shaw integral, produces significantly different results.

In nonrelativistic scattering there cannot be a resonance so the generalization to relativistic scattering is not obvious. However, we can safely say that the Fulton-Shaw approximation in nonrelativistic scattering is as reliable as the full N/D solution in predicting the qualitative features of the scattering.

One may also observe that the deviation between the various curves decreases as the angular momentum increases, but increases as the coupling strength increases. This is just as expected. The question is whether one can be satisfied with the results of a first-order calculation. All the methods of constructing the amplitude are bad in comparison with the Schrödinger-equation results. We expect that this deviation will be reduced considerably by finding a better approximation to the far away left-hand cuts in $N_l(\nu)$. Higher order terms in the potential will have to be taken into account and calculations are presently in progress to include the second-order Born term and thereby improve the agreement between the exact solution and the modified determinantal approximation.

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APPENDIX A

The integral $K_l(\nu)$ may be evaluated in the following way: Suppose we consider S waves for simplicity with

$$K_0(\nu)=\frac{1}{4\pi}\int_0^\infty \frac{\ln^2(1+4\nu')}{\nu'^{3/2}(\nu'-\nu)}d\nu'. \tag{A1}$$

The principal-value integral may be performed by using parametric differentiation. One introduces the parameter α and writes

$$K_0(\nu,\alpha)=\frac{1}{4\pi}\int_0^\infty \frac{\ln^2(1+\alpha^2\nu')}{\nu'^{3/2}(\nu'-\nu)}d\nu'. \tag{A2}$$

Introduce the basic integral

$$J(\nu,\alpha)=\int_0^\infty \frac{\ln(1+\alpha^2\nu')}{\nu'^{1/2}(\nu'-\nu)}d\nu'=\frac{2\pi i}{\nu^{1/2}}\ln(1-\alpha\nu^{1/2}), \tag{A3}$$

hence for $\nu>0$

$$J(\nu,\alpha)=\frac{2\pi\arctan(\alpha\nu^{1/2})}{\nu^{1/2}}+\frac{i\pi\ln(1+\alpha^2\nu)}{\nu^{1/2}}, \tag{A4}$$

but for $\nu<0$ we must go back to the integral of (A3) and obtain

$$J(-1/\alpha^2,\alpha)=2\pi\alpha\ln 2. \tag{A5}$$

If we now express $\partial K_0/\partial\alpha$ in terms of the basic integral using (A4) and (A5) and integrate with respect to α we get

$$K_0(\nu,\alpha)=\frac{2}{\nu^{1/2}}\int_0^\alpha \frac{\alpha'\arctan(\alpha'\nu^{1/2})}{1+\alpha'^2\nu}d\alpha'-2\ln 2\int_0^\alpha\frac{\alpha'^2d\alpha'}{1+\alpha'^2\nu}+\frac{i}{\nu^{1/2}}\int_0^\alpha\frac{\alpha'}{1+\alpha'^2\nu}\ln|1+\alpha'^2\nu|\,d\alpha'. \tag{A6}$$

Evaluating the integrals we obtain

$$K_0(\nu,\alpha)=\frac{1}{\nu^{3/2}}\ln|1+\alpha^2\nu|\arctan(\alpha\nu^{1/2})+\frac{1}{\nu^{3/2}}\mathrm{Cl}_2(\pi-2\arctan(2\nu^{1/2}))-\frac{2\alpha}{\nu}\ln 2+\frac{i\ln^2|1+\alpha^2\nu|}{4\nu^{3/2}}, \tag{A7}$$

where

$$\mathrm{Cl}_2(\theta)\equiv-\int_0^\theta \ln(2\sin\tfrac{1}{2}t)dt$$

is Clausen's integral, tables of which are readily available.[8] When $\alpha=0$, $K_0(\alpha,\nu)=0$, so there are no extra terms. The integration for higher partial waves is performed in a similar way.