"Incoherent Droplet" Model of High-Energy Large-Angle Scattering*

KERSON HUANG†
Institute for Advanced Study, Princeton, New Jersey
(Received 17 January 1966)

The high-energy large-angle scattering between two hadrons is thought of as incoherent scattering between two objects of complicated internal structure. A model is proposed whereby the incoherence is simulated by a simple recipe. Consequences of the model can be deduced simply. The salient feature of the result is that transverse momentum transfer is strongly inhibited for kinematical reasons. Orear's empirical fit for large-angle p-p scattering is obtained.

1. INTRODUCTION

A STRIKING general feature of two-body collisions of hadrons at high energies (≥ 10 BeV) is that the angular distribution displays qualitatively different behavior in different angular ranges:

(a) Near the forward direction it is sharply peaked, and has the form $\exp(-\alpha t)$, where t is the invariant 4-momentum transfer.[1] There is indication that a similar peak occurs near the backward direction.[2]

(b) At large c.m. angles (i.e., near 90°), it decreases very rapidly with increasing transverse momentum transfer $P_\perp = P \sin\theta$, where P and θ are, respectively, c.m. momentum and scattering angle.[3] For p-p scattering at $t > 2.3$ $(\mathrm{BeV}/c)^2$, Orear[4] has pointed out that all existing data fit the formula

$$d\sigma/d\Omega = A \exp(-aP_\perp),\quad A = 3.0\times 10^{-26}\ \mathrm{cm}^2/\mathrm{sr},\quad a^{-1} = 152\ \mathrm{MeV}/c. \tag{1}$$

The marked difference between forward and large-angle scattering reminds one of a more familiar phenomenon, the scattering of waves by a liquid droplet, where coherent scattering produces a sharp forward peak, while incoherent scattering accounts for the rest. That is, for forward scattering the droplet acts as a whole and may be approximated by a dispersive medium; for large-angle scattering, subunits of the droplet act independently. Motivated by the analogy, we suppose that hadrons have complicated internal structure, and that the ideas of coherence and incoherence might be useful in understanding the behavior of the angular distribution.

Byers and Yang[5] have proposed a "coherent droplet" model for forward scattering, in which the scattering process was pictured as the traverse of a particle through an optical medium. We propose here a "droplet" model in which incoherent rather than coherent effects are simulated, in an attempt to reproduce the experimental features of large angle scattering. Wu and Yang[6] have suggested some of the ideas of the present model, which may be called an "incoherent droplet" model. Compared to the "coherent droplet" model, it emphasizes momentum space properties rather than configuration space properties. The two models should complement each other, for they apply under different physical conditions.

2. DEFINITION OF THE MODEL

For simplicity we discuss the elastic scattering of two spinless protons of mass M. In the limit of infinite energy, where spin and mass probably make little difference, the gross features of our result should be shared by all two-body reactions.

We suppose that the colliding protons in the initial state are made up of N subunits each, where $N \to \infty$. The subunits in a proton have a definite energy and momentum distribution. The momentum distribution is spherical symmetric in the rest frame of the proton. The individual 4-momentum of a subunit may be space-like or time-like, but the sum of the 4-momenta should be the physical 4-momentum of the proton. Intuitively we think of N as the potential number of pieces into which a proton can be "fractured" when it is hit. Since, in a naive view, this number depends on how hard the proton is hit, we expect N to increase with total c.m. energy. We shall see later that this is in fact necessary in order to agree with experiments. Thus the subunits are not properties of isolated protons, but

* Research sponsored by the U. S. Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR Nr 42-65.

† John Simon Guggenheim Fellow on leave of absence from Massachusetts Institute of Technology, Cambridge, Massachusetts.

[1] See, e.g., K. Foley, S. Lindenbaum, W. Love, S. Ozaki, J. Russel, and L. Yuan, Phys. Rev. Letters **11**, 425 (1963); **11**, 503 (1963); O. Czyzewski, B. Escoubes, Y. Goldschmidt-Clermont, M. Guinea-Moorhead, D. Morrison, and S. Unamuno-Escoubes, Phys. Letters **15**, 188 (1965), and the work of many other groups quoted in these papers.

[2] C. T. Coffin N. Dikmen, L. Ettlinger, D. Meyer, A. Saulys, K. Terwilliger, and D. Williams, Phys. Rev. Letters **15**, 838 (1965); H. Brody, R. Lanza, R. Marshall, J. Niederer, W. Selove, M. Shochet, and R. Van Berg, *ibid.* **16**, 828 (1966).

[3] D. S. Narayan and K. V. L. Sarma, Phys. Letters **5**, 365 (1963); A. D. Krisch, Phys. Rev. Letters **11**, 217 (1963); J. Orear, *ibid.* **12**, 112 (1954); G. Cocconi, V. Cocconi, A. Krisch, J. Orear, R. Rubinstein, D. Scarl, B. Ulrich, W. Baker, E. Jenkins, and A. Read, Phys. Rev. **138**, B165 (1965).

[4] See Ref. 3. Krisch suggested $\exp(-aP_\perp^2)$ which fits small-angle data better than large-angle data. Orear's form fits large-angle data better.

[5] N. Byers and C. N. Yang, Phys. Rev. **142**, 976 (1966).

[6] T. T. Wu and C. N. Yang, Phys. Rev. **137**, B708 (1965).

of the colliding two-body system. The number N is a Lorentz invariant associated with a given collision.

During the collision the $2N$ subunits in both protons are freely regrouped into two groups representing the outgoing protons. The individual 4-momentum of a subunit remains unchanged in the regrouping, and there is no restriction on the possible regrouping except that the final protons have physically permissible 4-momenta. It is not required, for example, that the final protons each have N subunits; nor do they have to have any specific 4-momentum distributions. This is consistent with the philosophy that the subunits are meaningful only for the colliding system.

Given the final 4-momenta of the protons, the relative probability for the final state is the number of ways in which the $2N$ initial subunits can be regrouped to yield the final state. The relative probability gives a relative angular distribution at a fixed energy.

To calculate an absolute probability, the model has to be supplemented by a description of small angle scattering (for which this model does not claim validity), and multiple production processes such as $p+p\to p+p+n\pi$. The former seems difficult, whereas the latter can be accommodated by a natural extension. All we have to do, for multiple production, is to regroup the $2N$ initial subunits into more than two groups, requiring only that each group have total 4-momentum corresponding to an appropriate final particle. Assuming that the elastic cross section can be neglected compared to the production cross section, we can obtain an absolute probability for p-p elastic scattering by dividing the relative probability obtained earlier by the sum of all relative probabilities for multiple production. In this paper, however, we calculate only the relative probability for elastic scattering.

Some consequences of the model are immediately obvious. We recall that the momentum distribution in one of the colliding protons is assumed to be spherically symmetric in the Lorentz frame in which that proton is at rest. It follows that in the c.m. frame the momentum distribution is constant over the surface of an ellipsoid of revolution, with major axis along the incident direction (the x axis). In the limit of infinite energy, the ratio of minor to major axis approaches zero. The momentum distributions of the two initial protons are mirror images of each other with respect to the y-z plane. Since in the limit of infinite energy the combined distributions of the two protons contain overwhelmingly more longitudinal than transverse momentum, the model implies a small probability of transverse momentum transfer.

The maximum transverse momentum transfer is achieved when one of the final protons is given all those subunits in both initial protons that have a positive component along the transverse direction. For any reasonable momentum distribution it is a finite number proportional to N. Since physically the maximum momentum transfer increases with energy, we see that unless N increases with total c.m. energy, the angular distribution will be strictly zero in a finite neighborhood of 90°, and this neighborhood will enlarge with increasing energy. Such a behavior does not agree with existing experiments. Therefore N should increase with total c.m. energy, in accordance with our intuitive expectation discussed earlier.

We now work out the model in more detail. In the c.m. frame, let the initial and final 4-momentum of one of the protons be respectively $P_\mu=(\mathbf{P},E)$, and $Q_\mu=(\mathbf{Q},E)$. We choose the x axis along the initial momentum, and let the final momentum lie in the x-y plane. Thus

$$\begin{aligned}\mathbf{P}&=(P,0,0)\,,\\ \mathbf{Q}&=(P\cos\theta,P\sin\theta,0)\,.\end{aligned}\tag{2}$$

Let v be the proton velocity in the c.m. frame, with

$$\begin{aligned}P&=Mv/(1-v^2)^{1/2}\,,\\ E&=M/(1-v^2)^{1/2}.\end{aligned}\tag{3}$$

Then a Lorentz transformation along the x axis with velocity $\pm v$ will bring either proton to rest.

We denote by $p_\mu=(\mathbf{p},p_0)$ the 4-momentum of a subunit in either proton, in the c.m. frame. There are then altogether $2N$ values of p_μ. From this set of $2N$ values choose a subset G, arbitrary except for the condition

$$\sum_{p\in G} p_\mu=Q_\mu\,.\tag{4}$$

Then G is a possible grouping that constitutes one of the outgoing protons. The complement of G automatically satisfies all the kinematic requirements for constituting the other final outgoing proton. The relative probability $W(Q_\mu)$ for the final state specified by Q_μ is the number of subsets G that satisfies (4).

To illustrate the counting method in calculating $W(Q_\mu)$ we consider a simpler example in which p_μ is replaced by a one-component quantity p. We assume that by choosing the unit of momentum sufficiently small, all the $2N$ values of p will become integers $\{p_1,p_2,\cdots,p_{2N}\}$. Consider the polynomial $\prod_p(1+z^p)$, where the product extends over all the $2N$ values of p. The coefficient of z^Q in the polynomial is the number of partitions of Q into integers belonging to the set $\{p_1,p_2,\cdots,p_{2N}\}$, with each p_i appearing at most once in a partition. This coefficient is given by

$$W(Q)=\frac{1}{2\pi i}\oint\frac{dz}{z^{Q+1}}\prod_p(1+z^p)\,.\tag{5}$$

In the limit $N\to\infty$, the integral above can be evaluated by the method of saddle-point integration. Putting $\lambda=\ln z$, we find the leading contribution to the integral to be

$$\ln W(Q)=\sum_p \ln(1+e^{\lambda p})-\lambda Q\,,\tag{6}$$

where λ is such as to maximize $\ln W$:

$$Q=\sum_p p/(1+e^{-\lambda p}). \tag{7}$$

The generalization to our actual problem is straightforward. We merely give the result:

$$\ln W(Q_\mu)=\sum_p \ln(1+e^{\lambda\cdot p})-\lambda\cdot Q, \tag{8}$$

$$Q_\mu=\sum_p p_\mu/(1+e^{-\lambda\cdot p}), \tag{9}$$

where $\lambda_\mu=(\lambda_x,\lambda_y,\lambda_z,\lambda_0)$ is a 4-vector, $\lambda\cdot p$ denotes 4-vector scalar product, and the sums extend over all $2N$ subunits of the initial protons. To calculate $W(Q_\mu)$, we have to solve for λ_μ from (9) and substitute the result into (8).

The method of counting can be immediately generalized to obtain the relative probability for multiple production processes. Suppose in the final state there are in addition to a proton n particles of 4-momenta $Q_{1\mu},\cdots,Q_{n\mu}$. Again consider first a simpler example in which $Q_{i\mu}$ is replaced by a one-component quantity Q_i, and consider the polynomial

$$\prod_p(1+z_1{}^p+z_2{}^p+\cdots+z_n{}^p).$$

The number of ways to divide the $2N$ values of p into $n+1$ groups, n of which having sums of p values equal, respectively, to Q_1, $Q_2,\cdots Q_n$, is the coefficient of $z_1{}^{Q_1}z_2{}^{Q_2}\cdots z_n{}^{Q_n}$ in the polynomial. The relative probability for the final state in our problem can now be immediately written down:

$$W(Q_{1\mu},\cdots,Q_{n\mu})=\sum_p \ln(1+e^{\lambda_1\cdot p}+\cdots+e^{\lambda_n\cdot p}) -(\lambda_1\cdot Q_1+\cdots+\lambda_n\cdot Q_n), \tag{10}$$

where the n 4-vectors $\lambda_{1\mu},\cdots,\lambda_{n\mu}$ are determined by

$$Q_{i\mu}=\sum_p (p_\mu e^{\lambda_i\cdot p})/(1+e^{\lambda_1\cdot p}+\cdots e^{\lambda_n\cdot p}). \qquad (i=1,\cdots,n). \tag{11}$$

3. ANGULAR DISTRIBUTION

Before introducing a specific assumption for the momentum distribution of the subunits, it is possible to deduce some consequences without it. First consider the equation for Q_z in (9). By noting that $Q_z=0$, and the fact that the momentum distribution is cylindrically symmetric about the z axis we find

$$\lambda_z=0. \tag{12}$$

Next consider the equation for Q_0 in (9). Noting that $Q_0=E$, and that by definition $\sum_p p_0=2E$, we subtract $\frac{1}{2}\sum_p p_0$ from both sides of the equation to obtain

$$0=\sum_p p_0 \tanh(\boldsymbol{\lambda}\cdot\mathbf{p}-\lambda_0 p_0), \tag{13}$$

which, by reason of symmetry, is satisfied by

$$\lambda_0=0. \tag{14}$$

To proceed further we need to specify the momentum distribution of the subunits. For N sufficiently large, we can introduce a 4-momentum distribution function $f(p)$, such that $f(p)d^4p$ is the number of subunits whose 4-momentum lies in the volume element d^4p about p_μ, with the requirements

$$\int d^4p\, f(p)=N, \qquad \int d^4p\, p_\mu f(p)=P_\mu, \tag{15}$$

where N is Lorentz-invariant. Under a Lorentz transformation $p_\mu\to p_\mu'$, the function f is transformed into f', with

$$f(p)=f'(p'). \tag{16}$$

Let the distribution functions of the two initial protons be denoted respectively by $f_1(p)$ and $f_2(p)$ in the c.m. frame. A Lorentz transformation of velocity v along the x axis transforms f_1 into a spherically symmetric function f', and one of velocity $-v$ transforms f_2 into f'. For convenience we take

$$f'(p)=Ce^{-\alpha^2\mathbf{p}^2-\beta^2(p_0-\mu)^2}, \tag{17}$$

where p_0 ranges from $-\infty$ to $+\infty$, and α and β are numerical constants. To satisfy (15) we put

$$C=(\alpha^3\beta/\pi^2)N, \qquad \mu=M/N. \tag{18}$$

Using (16) we find that

$$f_1(p)=C\exp\left\{-\alpha^2\left[\frac{(p_x-vp_0)^2}{1-v^2}+p_y{}^2+p_z{}^2\right] -\beta^2\left[\frac{p_0-vp_x}{(1-v^2)^{1/2}}-\mu\right]^2\right\}, \tag{19}$$

and $f_2(p)$ is the same except that v is replaced by $-v$, with C unchanged. The momentum distribution corresponding to f_1 is

$$F_1(\mathbf{p})=\int_{-\infty}^{+\infty} dp_0 f_1(p)=C(\pi^{1/2}\gamma/\alpha\beta) \times\exp[-\gamma^2(p_x-\sigma)^2-\alpha^2(p_y{}^2+p_z{}^2)], \tag{20}$$

where

$$\gamma^2=\alpha^2\beta^2(1-v^2)/(\beta^2+v^2\alpha^2), \qquad \sigma=\mu v/(1-v^2)^{1/2}. \tag{21}$$

To obtain $F_2(\mathbf{p})$ from $F_1(\mathbf{p})$, replace σ by $-\sigma$.

Using (12) and (14), and replacing sums by integrals, we reduce (8) and (9) to the following set of equations:

$$\ln W(Q_\mu)=\int d^3p(F_1+F_2)\ln[1+\exp(\lambda_xp_x+\lambda_yp_y)]-\lambda_xP\cos\theta-\lambda_yP\sin\theta, \quad (22)$$

$$P\cos\theta=\int d^3p(F_1+F_2)\times p_x[1+\exp(-\lambda_xp_x-\lambda_yp_y)]^{-1}, \quad (23)$$

$$P\sin\theta=\int d^3p(F_1+F_2)\times P_y[1+\exp(-\lambda_xp_x-\lambda_yp_y)]^{-1}. \quad (24)$$

For $\theta=90°$, (23) requires $\lambda_x=0$. Thus for large angle scattering we expect λ_x to be small. Expanding (22)-(24) in powers of λ_x, and keeping only lowest order terms in λ_x, we find

$$\ln W(Q_\mu)=2\pi^{-1/2}N\int_z^\infty du\, u^{-2}g(u)-\lambda_yP\sin\theta-\lambda_xP\cos\theta, \quad (25)$$

$$P\cos\theta=\frac{N[\beta^2+v^2\alpha^2(1+2\beta^2\mu^2)]}{\alpha\beta^2(1-v^2)}\frac{\lambda_x}{\lambda_y}g(z), \quad (26)$$

$$P\sin\theta=\pi^{-1/2}\alpha^{-1}Ng(z), \quad (27)$$

where the terms neglected are at least of order λ_x^2, and

$$z=2\alpha/\lambda_y, \quad (28)$$

$$g(z)=\int_0^\infty dt(\mathrm{sech}\,t)^2\exp(-z^2t^2). \quad (29)$$

Table I contains numerical tables for $g(z)$ and $-dg(z)/dz$. The following expansions can be obtained:

$$g(z)=1+\sum_{n=1}^{\infty}\frac{\pi^{2n}(2^{2n-1}-1)B_{2n-1}}{n!2^{2n-1}}(-z^2)^n=1-\frac{\pi^2}{12}z^2+\frac{7\pi^4}{480}z^4+\cdots, \quad (30)$$

$$g(z)\approx(\pi^{1/2}/2z)(1-\tfrac{1}{2}z^{-2}+\tfrac{1}{2}z^{-4}+\cdots), \quad (31)$$

where B_{2n-1} is a Bernoulli number. Dividing (26) by (27), we get

$$\frac{\lambda_x}{\lambda_y}=\frac{\beta^2}{\beta^2+v^2\alpha^2(1+2\beta^2\mu^2)}\frac{1-v^2}{\tan\theta}, \quad (32)$$

which is small at high energies ($v\approx1$) and/or large angles ($\theta\approx90°$). Using (28) and (32), we can rewrite (25) in the form

$$\ln W(Q_\mu)=\frac{2N}{\pi^{1/2}}\int_z^\infty\frac{du}{u^2}g(u)-\frac{2\alpha}{z}P_\perp\left\{1-\frac{\beta^2(1-v^2)}{[\beta^2+v^2\alpha^2(1+2\beta^2\mu^2)]\tan^2\theta}\right\}, \quad (33)$$

TABLE I. The functions $g(z)$ and $h(z)$, defined, respectively, by Eqs. (29) and (39).

z	$g(z)$	$-dg(z)/dz$	$h(z)$
0	1.000	0	1.23
0.1	0.992	0.159	1.06
0.2	0.969	0.290	0.913
0.3	0.935	0.378	0.778
0.4	0.895	0.428	0.661
0.5	0.851	0.447	0.563
0.6	0.806	0.445	0.482
0.7	0.762	0.431	0.414
0.8	0.720	0.408	0.358
0.9	0.680	0.383	0.311
1.0	0.644	0.356	0.272
1.5	0.497	0.238	0.151
2.0	0.398	0.162	0.094
2.5	0.330	0.115	0.064
3.0	0.280	0.084	0.046
3.5	0.244	0.064	0.034
4.0	0.215	0.051	0.026
4.5	0.192	0.041	0.021
5.0	0.174	0.033	0.017
6.0	0.146	0.024	0.012
7.0	0.125	0.018	0.009
8.0	0.110	0.013	0.007
9.0	0.098	0.011	0.005
10.0	0.088	0.009	0.004
20.0	0.044	0.002	0.001

where $P_\perp=P\sin\theta$, and where z is to be found by solving (27). For high energies and large angles the last term in the bracket may be dropped. Thus the parameters β and μ become irrelevant.

We note that $g(z)$ is a monotonically decreasing function with $g(0)=1$. For (27) to have a solution, it is therefore necessary that

$$N\geqslant\pi^{1/2}\alpha Mv/(1-v^2)^{1/2}. \quad (34)$$

A simple reason for this requirement has been given earlier. There are now two possibilities. Either N increases faster than $(1-v^2)^{-1/2}$ as $v\to1$, or N is proportional to $(1-v^2)^{-1/2}$.

If the former were the case, the first term in (33) would dominate. This would lead to the prediction that $W(Q_\mu)$ is a decreasing function of z, or that it is an increasing function of θ at fixed v. Since this prediction disagrees with experiments, we conclude that

$$N=N_0/(1-v^2)^{1/2}, \quad (35)$$

where N_0 is a numerical constant.

We shall make the approximation $P=M/(1-v^2)^{1/2}$ in (27), so that z becomes purely a function of $\sin\theta$:

$$g(z)/g(z_0)=\sin\theta, \quad (36)$$

where z_0 is the root of

$$g(z_0)=\pi^{1/2}\alpha M/N_0. \quad (37)$$

The differential cross section for p-p elastic scattering is given by (33) up to a factor depending on the total c.m. energy:

$$d\sigma/d\Omega=A(v)\exp[-2\pi^{-1/2}N_0h(z)/(1-v^2)^{1/2}], \quad (38)$$

where

$$h(z)=\pi^{1/2}\int_0^\infty dt\, t(\operatorname{sech}t)^2\operatorname{erf}(zt)\,, \tag{39}$$

and z is a function of $\sin\theta$ determined by (36). A table of $h(z)$ is given in Table I. The only free parameters in (38) are N_0 and z_0, which are related through (37).

We discuss the two limiting cases $z_0 \ll 1$ and $z_0 \gg 1$, for which we can solve (36) by using respectively the expansions (30) and (31). The final results are

$$\frac{d\sigma}{d\Omega}=A(v)\exp\left[-\left(\frac{\pi}{3}\right)^{1/2}\times\frac{N_0(N_0\sin\theta-1)}{(1-v^2)^{1/2}(1-\sin\theta)^{1/2}}\right],\quad (z_0\ll 1)\,, \tag{40}$$

$$\frac{d\sigma}{d\Omega}=A(v)\exp\left[-\frac{\alpha M\sin^2\theta}{z_0(1-v^2)^{1/2}}\right],\quad (z_0\gg 1)\,. \tag{41}$$

Neither of these agree with experiments.

For z_0 of the order of unity, (36) has to be solved graphically. The results are illustrated in Fig. 1, where

$$\bar{h}(\sin\theta)\equiv h(z)\,. \tag{42}$$

We see that in a wide neighborhood of $\theta=90°$, $h(z)$ is a linear function of $\sin\theta$. Accordingly we expand $h(z)$ about $\theta=90°(z=z_0)$, neglecting terms of order $(1-\sin\theta)^2$, and obtain

$$d\sigma/d\Omega=A(v)\exp[-(2\alpha/z_0)P_\perp],\quad (z_0\approx 1)\,. \tag{43}$$

This conforms to Orear's empirical formula (1) for p-p elastic scattering, if $A(v)$ is independent of v, and $z_0/2\alpha=152$ MeV/c. There is still some freedom in the

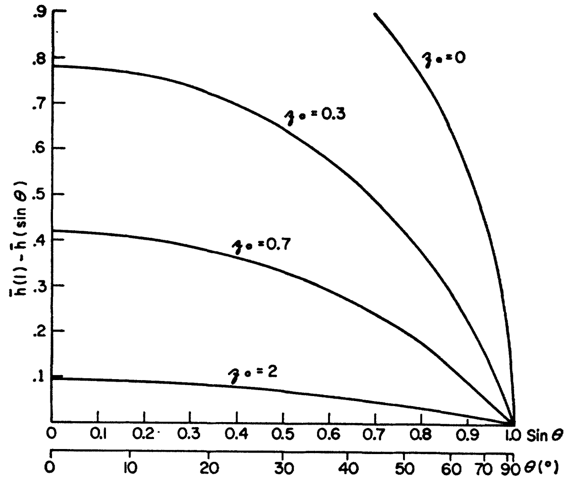

FIG. 1. The function plotted is proportional to $\ln(d\sigma/d\Omega)-\ln(d\sigma/\Omega)_{90°}$. The model is not designed to be valid for small θ.

choice of z_0. For illustration we choose $z_0=0.7$. Then

$$\begin{aligned}\alpha&=2.3(\text{BeV}/c)^{-1},\\ N_0&=5.4\,.\end{aligned} \tag{44}$$

At a laboratory momentum of 30 BeV/c, we have $(1-v^2)^{1/2}\approx\frac{1}{4}$, which gives $N\approx 22$.

ACKNOWLEDGMENTS

The author wishes to thank Professor Robert Oppenheimer for the hospitality extended him at the Institute for Advanced Study. He is indebted to Mr. Robert L. Thews for preparing Table I. Above all he must thank Professor C. N. Yang for many interesting discussions and helpful suggestions.