Bootstrap Conditions in a Soluble Model*

Y. S. JIN AND KYUNGSIK KANG
Department of Physics, Brown University, Providence, Rhode Island
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A soluble model obtained by a slight extension of the Lee model is considered in a study of the bootstrap mechanism. By examining the general solution that is obtained by using properties of the Herglotz function, it is found that the bootstrap mechanism can be achieved if and only if two further restrictions in addition to the general requirements of analyticity, unitarity, and crossing symmetry are imposed on the solution. They are that (i) the scattering amplitude satisfy the asymptotic condition $\lim_{\omega\to\infty}\omega^{-2}t^{-1}(\omega)=0$ and (ii) the scattering amplitude have no Castillejo-Dalitz-Dyson (C.D.D.) zeros. It is also proved that the condition (i) is equivalent to $\lim_{\omega\to\infty}\omega^{-1}D(\omega)=0$ when $N(\omega)=O(\omega^{-1})$ as $\omega\to\infty$ or $Z_3=0$ or the Levinson theorem holds, while condition (ii) is equivalent to assuming the two familiar bootstrap equations based on the N/D method and implies in particular a nonpositive scattering length. Either of the conditions (i) and (ii) alone gives in general only an inequality between the mass and coupling constant, and it is therefore concluded that the possibility of the bootstrap mechanism depends in a very sensitive way on the low-energy behavior as well as the high-energy behavior of the scattering amplitude. It is further argued that destroying the crossing symmetry in the approximate solutions will not give any physically meaningful conditions for determining the parameters unless one introduces a subtraction or one C.D.D. zero in the Low equation.

I. INTRODUCTION

DURING the past couple of years there have appeared many papers in which various conditions for bootstrap mechanism have been discussed, either in soluble models or in approximation frameworks. In particular, in a recent paper by Huang and Low,[1] one finds thorough discussions of bootstrap solutions in many soluble models, where the bootstrap solution is defined to be the one which satisfies the Levinson theorem. Meanwhile, however, it has been claimed by various authors that the assumption of no C.D.D. pole[2] or a certain assumption on the high-energy behavior of the scattering amplitude[3] or the condition $Z_3=0$ [4] gives rise to a bootstrap mechanism. In view of these diverse forms of the bootstrap criteria, it is highly desirable to study the relations between different bootstrap conditions which have been used by various people.

In this paper we shall study various bootstrap conditions in a soluble model[5] which is a slight extension of the Lee model and contains an antiparticle of θ which is identical to the θ particle itself. It is well known that the general requirements such as analyticity, unitarity, and crossing symmetry alone do not give any relation whatsoever between the coupling constants (residue at the poles) and the masses of the particles (positions of the poles).[6] Hence, the bootstrap mechanism is possible only if some further specific restrictions on the solution, which we shall call *bootstrap conditions*, are imposed. By starting with the general solution given by C.D.D.,[7] we prove that the bootstrap is possible if and only if

(i) *at high energy, the scattering amplitude does not decrease as fast as* ω^{-2}, *i.e.*, $\lim_{\omega\to\infty}\omega^{-2}t^{-1}(\omega)=0$,

and

(ii) *there are no C.D.D. zeros of the scattering amplitude.*

Obviously, condition (ii) is equivalent to assuming that

(iia) *the scattering length is negative* $[t(\mu)<0]$ *and at any energy in the physical region there is interaction, i.e.* $t(\omega)\neq 0$ *for* $\omega>\mu$,

or

(iib) *the mass and coupling constant are determined by the first and second bootstrap equations*[8] *based on the* N/D *method.*

We shall see that the two bootstrap equations alone do not *a priori* exclude the high-energy behavior $t(\omega)=O(\omega^{-2})$ and without condition (i) in general they give rise to only an inequality between the bootstrap parameters.

Condition (i) will be shown to be equivalent to assuming that

(ia) $D(\omega)$ *does not increase as fast as* ω, *i.e.*, $\lim_{\omega\to\infty}D(\omega)/\omega=0$ *with* $t(\omega)=N(\omega)/D(\omega)$ *and* $N(\omega)=(g^2/4\pi)(\omega+\Delta)^{-1}$,

or

(ib) *The wave-function renormalization constant of the* V *particle vanishes, i.e.* $Z_V=0$,

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[1] K. Huang and F. E. Low, Phys. Rev. Letters **13**, 596 (1964); J. Math. Phys. **6**, 795 (1965).

[2] E. J. Squires, Nuovo Cimento **37**, 1749 (1965). The amplitude given in this reference, however, is in error. The correct one should be the same as our Eq. (43).

[3] S. Aramaki, University of Illinois report (unpublished).

[4] For instance, S. Saito and T. Akiba, Progr. Theoret. Phys. (Kyoto) **33**, 307 (1965).

[5] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).

[6] B. Diu and H. R. Rubinstein, Phys. Letters **8**, 203 (1964).

[7] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1955).

[8] For instance, F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).

or

(ic) *The Levinson theorem, i.e.,* $\delta(\mu)-\delta(\infty)=(n_b-n_c)\pi$, *holds.*

Here n_b and n_c are the numbers of bound states and C.D.D. zeros of the scattering amplitude, respectively.

It is clear from these results that a bootstrap, at least in this model, is impossible, for instance, *if the scattering length is positive or the scattering amplitude at high energy decreases like* $t(\omega)=O(\omega^{-2})$. For a realistic case, however, it is an open question whether the aforementioned situation might persist. Nevertheless, it is evident that the positiveness of the scattering length would necessarily imply the existence of at least one C.D.D. zero between the bound state and the two-particle threshold, and hence introduce at least two more parameters, i.e., the position of the zero and the slope of the scattering amplitude at the zero, irrespective of the complexity due to the left-hand cut. As to the high-energy behavior, it is also highly unlikely that in a realistic case the bootstrap mechanism is less sensitive to the high-energy behavior of the scattering amplitude than in a simple model. In short, our conclusion is that the possibility of the bootstrap mechanism depends in a very sensitive way on the *low-energy behavior as well as the high-energy behavior* of the scattering amplitude, even if one excludes the C.D.D. zeros in the physical region.

In the next section, we shall give the Low equation for the scattering amplitude in an extended Lee model[5] by making use of the one-meson approximation, and present the general solution for the scattering amplitude following the discussions given by C.D.D.[7] Here, we shall also show that the general solution can be obtained in the framework of the N/D method. It will be seen that the number of subtractions necessary in the denominator function for a given choice of the numerator function is determined by *the lower bound of the scattering amplitude* and that in general we will get only an inequality relation between the coupling constant and the mass of the bound state unless conditions (i) and (ii) are assumed. In Sec. III, we state further restrictions imposed on the general solution in addition to the general requirements so that a definite relation between the parameters of the bound state can be given. Implications of these bootstrap conditions are also discussed.

Section IV contains the particular solutions obtained by the N/D method. When the two bootstrap equations of the N/D formalism are applied to the general solution, it turns out that there should be no C.D.D. zeros in the amplitude. This implies that the two bootstrap equations which are referred to as the bootstrap conditions in some literature correspond merely to our condition (ii). Moreover these bootstrap equations do not restrict in any way the high-energy behavior of the denominator function, and thus condition (ia) is needed to achieve bootstrap mechanism. We shall also see that the two bootstrap equations give only the same relation between the mass and coupling constant even under conditions (i) and (ii) if there is crossing symmetry, and hence it is impossible to determine the bootstrap parameters completely. We shall notice that the particular solutions become atuomatically crossing-symmetric as soon as the bootstrap conditions (i) and (ii) are assumed.

Other equivalent bootstrap conditions are discussed in Sec. V. They are $Z_V=0$ as a condition for the V particle being composite, and the condition imposed by the Levinson theorem. We shall show that these conditions (ib) and (ic) are completely equivalent to our condition (i). Finally in Sec. VI, we shall examine the consequences of destroying the crossing symmetry in the solution, in order to see if the two bootstrap equations give two independent conditions that determine the bootstrap parameters completely. It will be seen, however, that breaking crossing symmetry does not give any more useful conditions, and therefore there is no way of determining the parameters completely in our model.

II. GENERAL SOLUTION

Let us consider an extended Lee model[5] in which the θ particle has an identical antiparticle $\bar{\theta}$ and the interaction between the θ particle and the source N takes place through the virtual processes $V\leftrightarrow N+\theta$ and $N\leftrightarrow V+\bar{\theta}$. In this model, the N-θ scattering has an identical crossed process and the Low equation for the scattering amplitude in the one-meson approximation turns out to be[5,6]

$$t(\omega)=-\frac{g^2}{4\pi}\left(\frac{1}{\omega-\Delta}-\frac{1}{\omega+\Delta}\right)+\frac{1}{\pi}\int_\mu^\infty d\omega'|f(\omega')|^2|t(\omega')|^2, \quad (1)$$

where $\Delta=m_V-m_N$, $f(\omega)$ is a cutoff function, and g is the renormalized coupling constant. The crossing symmetry

$$t(\omega)=t(-\omega) \quad (2)$$

holds since $\bar{\theta}=\theta$.

By rewriting (1) in the form

$$t(\omega)=\frac{\Delta g^2}{2\pi}(\Delta^2-\omega^2)^{-1}+\frac{1}{\pi}\int_\mu^\infty d\omega'^2|f(\omega')|^2|t(\omega')|^2\frac{(\omega'^2-\mu^2)^{1/2}}{\omega'^2-\omega^2} \quad (3)$$

it is clear that $t(\omega)$ is a Herglotz function[9] of ω^2, i.e., $\mathrm{Im}t(\omega)/\mathrm{Im}\omega^2>0$. From this it follows that $-t^{-1}(\omega)$ is also a Herglotz function of ω^2 and admits the representa-

[9] J. A. Shohat and J. D. Tamarkin, *The Problems of Moments* (American Mathematical Society, New York, 1943).

tion[7]

$$-t^{-1}(\omega)=A+B\omega^2+\frac{\omega^2}{\pi}\int_{\mu^2}^{\infty}d\omega'^2\frac{|f(\omega')|^2(\omega'^2-\mu^2)^{1/2}}{\omega'^2(\omega'^2-\omega^2)}+\sum_i\frac{R_i}{\omega_i^2-\omega^2},\quad(4)$$

where $\pm\omega_i(i=0,1,\cdots,n-1)$ are zeros of $t(\omega)$, $R_i>0$, and

$$\lim_{\omega\to\infty}\frac{-1}{t(\omega)\omega^2}=B\geqslant 0,\quad(\epsilon<\arg\omega^2<\pi-\epsilon).\quad(5)$$

By imposing the condition that $t^{-1}(\omega)=0$ at $\omega^2=\Delta^2$, which is obvious from Eq. (3), we obtain[10]

$$-t^{-1}(\omega)=B(\omega^2-\Delta^2)+\frac{\omega^2-\Delta^2}{\pi}\int_{\mu^2}^{\infty}d\omega'^2\times\frac{|f(\omega')|^2(\omega'^2-\mu^2)^{1/2}}{(\omega'^2-\Delta^2)(\omega'^2-\omega^2)}+\sum_i\frac{R_i(\omega^2-\Delta^2)}{(\omega_i^2-\omega^2)(\omega_i^2-\Delta^2)},\quad(6)$$

and this gives us the most general solution for the scattering amplitude $t(\omega)$. Now by using the condition that

$$[dt^{-1}(\omega)/d\omega^2]_{\omega^2=\Delta^2}=-2\pi/\Delta g^2,$$

which follows also from Eq. (3), one gets

$$\frac{\Delta g^2}{2\pi}B+\frac{\Delta g^2}{2\pi^2}\int_{\mu^2}^{\infty}d\omega^2\frac{|f(\omega)|^2(\omega^2-\mu^2)^{1/2}}{(\omega^2-\Delta^2)^2}+\frac{\Delta g^2}{2\pi}\sum_i\frac{R_i}{(\omega_i^2-\Delta^2)^2}=1,\quad(7)$$

and consequently the general solution reads

$$t(\omega)=\frac{\Delta g^2}{2\pi}(\Delta^2-\omega^2)^{-1}\left[1+\frac{\Delta g^2}{2\pi}\frac{\omega^2-\Delta^2}{\pi}\times\int_{\mu^2}^{\infty}d\omega'^2\frac{|f(\omega')|^2(\omega'^2-\mu^2)^{1/2}}{(\omega'^2-\omega^2)(\omega'^2-\Delta^2)^2}+\frac{\Delta g^2}{2\pi}\sum_i\frac{R_i(\omega^2-\Delta^2)}{(\omega_i^2-\Delta^2)^2(\omega_i^2-\omega^2)}\right]^{-1}.\quad(8)$$

This general solution can also be obtained by the familiar N/D method,[11]

$$t(\omega)=n(\omega^2)/d(\omega^2),\quad(9)$$

where $n(\omega^2)$ is taken as

$$n(\omega^2)=(\Delta g^2/2\pi)(\Delta^2-\omega^2)^{-1}.\quad(10)$$

Since then $-1/d(\omega^2)$ can be shown to be a Herglotz function of ω^2, the function $d(\omega^2)$ itself is a Herglotz function of ω^2. The high-energy behavior implied by the Low equation (1) is

$$C_1|\omega|^{-2}\leqslant|t(\omega)|<C_2|\omega|^2,\quad(11)$$

and therefore it follows that

$$C_1'\leqslant|d^{-1}(\omega^2)|<C_2'|\omega|^4,\quad(12)$$

and hence

$$C_2''|\omega|^{-4}<|d(\omega^2)|\leqslant C_1''.\quad(13)$$

From the relation (13) and from the fact that $d(\omega^2)$ is a Herglotz function of ω^2, it follows that[7]

$$\int_{\mu^2}^{\infty}d\omega^2\frac{\mathrm{Im}d(\omega^2)}{\omega^2}<\infty,\quad(14)$$

and we have a representation for $d(\omega^2)$:

$$d(\omega^2)=a+\frac{1}{\pi}\int_{\mu^2}^{\infty}d\omega'^2\frac{\mathrm{Im}d(\omega'^2)}{\omega'^2-\omega^2}+\sum_i\frac{r_i}{\omega_i^2-\omega^2},\quad(15)$$

where $\pm\omega_i(i=0,1,\cdots,n-1)$ are zeros of $t(\omega)$, $r_i>0$, and

$$\lim_{\omega^2\to\infty}d(\omega^2)=a\geqslant 0.\quad(\epsilon<\arg\omega^2<\pi-\epsilon).\quad(16)$$

Upon normalizing $d(\omega^2)=1$ at $\omega^2=\Delta^2$, one obtains

$$a=1-\frac{1}{\pi}\int_{\mu^2}^{\infty}d\omega^2\frac{\mathrm{Im}d(\omega^2)}{\omega^2-\Delta^2}-\sum_i\frac{r_i}{\omega_i^2-\Delta^2}.\quad(17)$$

Inserting Eq. (17) into Eq. (15), the denominator function becomes

$$d(\omega^2)=1+\frac{\Delta g^2}{2\pi}\frac{\omega^2-\Delta^2}{\pi}\int_{\mu^2}^{\infty}d\omega'^2\frac{|f(\omega')|^2(\omega'^2-\mu^2)^{1/2}}{(\omega'^2-\omega^2)(\omega'^2-\Delta^2)^2}+\sum_i\frac{r_i(\omega^2-\Delta^2)}{(\omega_i^2-\omega^2)(\omega_i^2-\Delta^2)}.\quad(18)$$

In writing (18), we have used the unitarity relation

$$\mathrm{Im}d(\omega^2)=\frac{\Delta g^2}{2\pi}|f(\omega)|^2(\omega^2-\mu^2)^{1/2}\times(\omega^2-\Delta^2)^{-1}\theta(\omega^2-\mu^2).\quad(19)$$

By inserting Eqs. (10) and (18) into Eq. (9), the general solution (8) is rederived if one sets

$$r_i=(\Delta g^2/2\pi)R_i(\omega_i^2-\Delta^2).\quad(20)$$

We remark in passing that only when no C.D.D. zeros are present in $t(\omega)$ can it be shown that the N/D solution is independent of the subtraction point; we

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[10] Along the real axis Eq. (5) does not necessarily hold. However, if one uses the Phragmen-Lindehöf theorem, it follows that

$$\liminf_{\omega^2\to\infty}|t^{-1}(\omega)\omega^{-2}|\leqslant B\quad\text{and}\quad\limsup_{\omega^2\to\infty}|t^{-1}(\omega)\omega^{-2}|\geqslant B.$$

Therefore, if $t(\omega)$ does not oscillate as $\omega^2\to\infty$, (5) holds on the real axis as well. This remark applies also to the bounds like (11).

[11] G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

also remark that *the lower bound of* $t(\omega)$ determines the number of subtractions in the denominator function for a given left-hand-cut contribution or the numerator function.

Although the general solution (8) meets the general requirements of analyticity, unitarity, and crossing symmetry, it does not give any relationship between Δ and g^2 except for an inequaltiy

$$B=\frac{2\pi}{\Delta g^2}-\frac{1}{\pi}\int_{\mu^2}^{\infty} d\omega^2 \frac{|f(\omega)|^2(\omega^2-\mu^2)^{1/2}}{(\omega^2-\Delta^2)^2} -\sum_i \frac{R_i}{(\omega_i^2-\Delta^2)^2}\geqslant 0, \quad (21)$$

which is clear from the relations (5) and (7) or from (16) and (17). Since R_i are positive and unknown, one gets from (21)

$$\frac{\Delta g^2}{2\pi}\leqslant\left(\frac{1}{\pi}\int_{\mu^2}^{\infty} d\omega^2 \frac{|f(\omega)|^2(\omega^2-\mu^2)^{1/2}}{(\omega^2-\Delta^2)^2}\right)^{-1}. \quad (22)$$

This is the well-known L.S.Z. inequality[12] which holds in the general framework of the quantum field theory.

III. BOOTSTRAP CONDITIONS

Though in this model the general requirements such as analyticity, unitarity, and crossing symmetry do not lead to an equation between g^2 and Δ which does not contain any other unknown parameters, it is possible to obtain such an equation by imposing more specific conditions. These we shall call bootstrap conditions.

In our model, such an equation is furnished by Eq. (7), namely

$$\frac{2\pi}{\Delta g^2}=B+\frac{1}{\pi}\int_{\mu^2}^{\infty} d\omega^2 \frac{|f(\omega)|^2(\omega^2-\mu^2)^{1/2}}{(\omega^2-\Delta^2)^2} +\sum_i \frac{R_i}{(\omega_i^2-\Delta^2)^2}, \quad (23)$$

in which if we impose further conditions

$$B=0, \quad (24a)$$

and

$$R_i=0 \quad \text{for all } i, \quad (24b)$$

the resulting equation then reads

$$\frac{2\pi}{\Delta g^2}=\frac{1}{\pi}\int_{\mu^2}^{\infty} d\omega^2 \frac{|f(\omega)|^2(\omega^2-\mu^2)^{1/2}}{(\omega^2-\Delta^2)^2}, \quad (25)$$

and Eqs. (24) are *necessary and sufficient* conditions for the relation (25). From the relation (5), however, $B=0$ is equivalent to the requirement that

$$\lim_{\omega\to\infty} 1/t(\omega)\omega^2=0, \quad (26)$$

if one excludes too violent oscillatory behavior of $t(\omega)$ along the real axis in the ω^2 plane. Hence, the bootstrap conditions in our model are that

(i) $\lim_{\omega\to\infty} 1/t(\omega)\omega^2=0$,

and

(ii) there are no C.D.D. zeros in $t(\omega)$.

It is clear that condition (i) does not exclude the C.D.D. zeros and that without condition (i), condition (ii) alone will give in general only the inequality (22). Thus either one of the conditions (i) and (ii) alone does not give the relation (25). It is claimed by Aramaki[3] that the assumption on the high-energy behavior of $t(\omega)$ alone ensures the bootstrap mechanism. We observe, however, that this was so only because the author tacitly assumed no C.D.D. zeros of $t(\omega)$ in his discussion.

Still another author[2] claimed that the bootstrap condition is equivalent to excluding C.D.D. zeros and used in his argument an approximate solution. Here again one can easily notice that condition (i) was assumed without mentioning it. To see this, we remark that $(\omega+\Delta)t(\omega)$ is a Herglotz function and is bounded by

$$C_1|\omega|^{-1}\leqslant|(\omega+\Delta)t(\omega)|\leqslant C_2|\omega|. \quad (27)$$

It follows then that $-D(\omega)$ is again a Herglotz function in ω when $N(\omega)$ is taken as

$$N(\omega)=(g^2/4\pi)(\omega+\Delta)^{-1}, \quad (28)$$

and that

$$C_2'|\omega|^{-1}\leqslant|D(\omega)|\leqslant C_1'|\omega|, \quad (29)$$

which implies that $-D(\omega)$ should have a representation

$$-D(\omega)=A_1+B_1\omega-\frac{\omega}{\pi}\int_{\mu}^{\infty} d\omega'\frac{\mathrm{Im}D(\omega')}{\omega'(\omega'-\omega)} -\frac{\omega}{\pi}\int_{-\infty}^{-\mu} d\omega'\frac{\mathrm{Im}D(\omega')}{\omega'(\omega'-\omega)}, \quad (30)$$

if no C.D.D. zeros are assumed. Here

$$\lim_{\omega\to\infty} -D(\omega)/\omega=B_1\geqslant 0, \quad (31)$$

and $B_1=0$ implies the condition (i). Indeed B_1 is set equal to zero in the discussion of Ref. 2, and thus condition (i) was assumed in addition to the general requirements of analyticity, unitarity, and crossing symmetry. This solution will be discussed in detail later on.

As for condition (ii), it is equivalent to assuming that the scattering length $t(\mu)<0$ and $|t(\omega)|>0$ for

[12] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 2, 425 (1955).

$\omega^2>\mu^2$, since $t(\mu)>0$ entails necessarily the existence of a C.D.D. zero of $t(\omega)$ in the region $\omega^2<\mu^2$. We note that this situation occurs whenever one starts with an unsubtracted dispersion relation. Thus, one concludes that the bootstrap mechanism depends very sensitively on *the high-energy behavior as well as the low-energy behavior of the scattering amplitude* $t(\omega)$.

IV. PARTICULAR SOLUTIONS OBTAINED BY THE N/D METHOD

In this section, we shall discuss the bootstrap mechanism for particular solutions of the scattering amplitude $t(\omega)$. For this purpose, it is convenient to use the N/D method.[11] Let us first start with the numerator function given by (28). From the discussions given in Sec. III, $-D(\omega)$ will have in general a representation

$$-D(\omega)=A_1+B_1\omega-\frac{\omega}{\pi}\int_\mu^\infty d\omega'\frac{\operatorname{Im}D(\omega')}{\omega'(\omega'-\omega)}-\frac{\omega}{\pi}\int_{-\infty}^{-\mu} d\omega'\frac{\operatorname{Im}D(\omega')}{\omega'(\omega'-\omega)}-\sum_i\frac{R_i}{\omega-\omega_i}, \quad (32)$$

where ω_i are C.D.D. zeros of $t(\omega)$, $R_i>0$, and B_1 is given by (31).

By normalizing $D(\omega)=1$ at $\omega=-\Delta$, we get

$$D(\omega)=1-B_1(\omega+\Delta)+\frac{\omega+\Delta}{\pi}\int_\mu^\infty d\omega'\frac{\operatorname{Im}D(\omega')}{(\omega'-\omega)(\omega'+\Delta)}+\frac{\omega+\Delta}{\pi}\int_{-\infty}^{-\mu} d\omega'\frac{\operatorname{Im}D(\omega')}{(\omega'-\omega)(\omega'+\Delta)}+\sum_i\frac{R_i(\omega+\Delta)}{(\omega-\omega_i)(\Delta+\omega_i)}. \quad (33)$$

From unitarity and crossing symmetry, it follows that

$$\operatorname{Im}D(\omega)=-(g^2/4\pi)|f(\omega)|^2(\omega^2-\mu^2)^{1/2}(\omega+\Delta)^{-1},\quad \omega>\mu \quad (34)$$

and

$$\operatorname{Im}D(-\omega)=-(g^2/4\pi)|f(\omega)|^2(\omega^2-\mu^2)^{1/2}(\omega-\Delta)^{-1},\quad -\omega<-\mu. \quad (35)$$

Thus Eq. (33) becomes[13]

$$D(\omega)=1-B_1(\omega+\Delta)-\frac{g^2}{4\pi}\frac{\omega+\Delta}{\pi}\int_\mu^\infty d\omega'|f(\omega')|^2\times(\omega'^2-\mu^2)^{1/2}\left(\frac{1}{(\omega'-\omega)(\omega'+\Delta)^2}+\frac{1}{(\omega'+\omega)(\omega'-\Delta)^2}\right)+\sum_i\frac{R_i(\omega+\Delta)}{(\omega-\omega_i)(\omega_i+\Delta)}. \quad (36)$$

The first and second bootstrap equations[8] based on the N/D formalism are given by

$$D(\Delta)=0, \quad (37)$$

and

$$\left[N(\omega)\Big/\frac{dD(\omega)}{d\omega}\right]_{\omega=\Delta}=-\frac{g^2}{4\pi}. \quad (38)$$

By imposing Eq. (37), one gets from Eq. (36) that

$$1=2\Delta\left(B_1-\sum_i\frac{R_i}{\Delta^2-\omega_i^2}\right)+\frac{\Delta g^2}{\pi^2}\int_\mu^\infty d\omega\frac{|f(\omega)|^2\omega(\omega^2-\mu^2)^{1/2}}{(\omega^2-\Delta^2)^2}. \quad (39)$$

From Eqs. (28) and (36), it follows that

$$\left[N(\omega)\Big/\frac{dD(\omega)}{d\omega}\right]_{\omega=\Delta}=\frac{g^2}{4\pi}\left(-2\Delta B_1-\frac{\Delta g^2}{\pi^2}\int_\mu^\infty d\omega\times\frac{|f(\omega)|^2\omega(\omega^2-\mu^2)^{1/2}}{(\omega^2-\Delta^2)^2}-\sum_i\frac{2\Delta R_i}{(\Delta-\omega_i)^2}\right)^{-1}. \quad (40)$$

By inserting Eq. (39) into Eq. (40), we obtain

$$\left[N(\omega)\Big/\frac{dD(\omega)}{d\omega}\right]_{\omega=\Delta}=\frac{g^2}{4\pi}\left(-1-\sum_i\frac{4\Delta^2R_i}{(\Delta-\omega_i)(\Delta^2-\omega_i^2)}\right)^{-1}. \quad (41)$$

It is clear from Eqs. (38) and (41) that in order to satisfy the second bootstrap equation (38), there should be no C.D.D. zeros present in the scattering amplitude, i.e., all R_i should be zero unless $\Delta=0$. Even if no C.D.D. zeros appear in $t(\omega)$, one would still end up from Eq. (39) with the inequality relation (22), since B_1 is positive definite and unknown. Therefore, it is obvious that the bootstrap relation (25) is achieved only when $B_1=0$ or equivalently

$$\lim_{\omega\to\infty}-D(\omega)/\omega=0 \quad (42a)$$

and

$$\text{all}\quad R_i=0. \quad (42b)$$

[13] Notice that the denominator function (36) contains a linear term $B_1(\omega+\Delta)$. The N/D method of solution does not exclude this term unless the asymptotic condition (42a) is assumed. See, for instance, A. P. Balachandran, Ann. Phys. (N. Y.) **35**, 209 (1965). This term was also noticed by M. L. Whippman and I. S. Gerstein of Ref. 14. When $B_1\neq0$ and no C.D.D. zeros are present in Eq. (36), then $D(\mu)<0$ and the s-wave phase shift is always negative and approaches $-\pi$ as $\omega\to\infty$. When $B_1\neq0$ but with $D(\mu)>0$, then we will have at least one C.D.D. zero in the interval $\Delta<\omega_1<\mu$ and a resonance behavior could emerge at some energy in the physical region, i.e., the phase shift goes through $\pi/2$ at some energy and approaches π as $\omega\to\infty$ in the ω plane. This resonant solution, however, contains at least two more parameters due to the presence of a C.D.D. zero.

We remark that Eq. (42a) is equivalent to the condition (i), while Eq. (42b) is the condition (ii) of Sec. III. In this particular solution, therefore, the necessary and sufficient condition in order to achieve bootstrap mechanism is that $\lim_{\omega\to\infty}[-\omega^{-1}D(\omega)]=0$ with $N(\omega)=O(\omega^{-0})$ and that there exist neither C.D.D. poles of $D(\omega)$ nor zeros of $N(\omega)$.

When the conditions of (42) are met, the resulting particular solution is

$$t(\omega)=\frac{g^2/4\pi}{\omega+\Delta}\left[1-\frac{g^2}{4\pi}\frac{\omega+\Delta}{\pi}\int_\mu^\infty d\omega'|f(\omega')|^2(\omega'^2-\mu^2)^{1/2}\times\left(\frac{1}{(\omega'-\omega)(\omega'+\Delta)^2}+\frac{1}{(\omega'+\omega)(\omega'-\Delta)^2}\right)\right]^{-1}. \quad (43)$$

This solution is not crossing-symmetric, as we have solved the N/D equations in the usual approximate way, i.e. we have tried to explain the bound-state pole in the direct channel from the force due to the one-particle exchange in the crossed channel.

In order to force crossing symmetry, let us rewrite the particular solution (43) as

$$t(\omega)=\frac{g^2/4\pi}{\omega+\Delta}\left[C-\frac{g^2}{4\pi}\frac{1}{\pi}\int_\mu^\infty d\omega'|f(\omega')|^2(\omega'^2-\mu^2)^{1/2}\times\left(\frac{1}{(\omega'-\omega)(\omega'+\Delta)}-\frac{1}{(\omega'+\omega)(\omega'-\Delta)}\right)\right]^{-1}, \quad (44)$$

with

$$C=1-\frac{\Delta g^2}{\pi^2}\int_\mu^\infty d\omega\frac{|f(\omega)|^2\omega(\omega^2-\mu^2)^{1/2}}{(\omega^2-\Delta^2)^2}. \quad (45)$$

Since the particular solution (43) is obtained under the conditions of (42), it is clear from (39) that $C=0$ and thus (43) becomes

$$t^{-1}(\omega)=-2\frac{\omega^2-\Delta^2}{\pi}\int_\mu^\infty d\omega'\frac{|f(\omega')|^2\omega'(\omega'^2-\mu^2)^{1/2}}{(\omega'^2-\omega^2)(\omega'^2-\Delta^2)}. \quad (46)$$

We shall show that the conditions of (42) will result again from Eq. (46) for the other particular solution, where

$$N(\omega)=-\frac{g^2}{4\pi}(\omega-\Delta)^{-1}. \quad (47)$$

After a manipulation similar to that used before, one obtains under the conditions of (42) a particular solution

$$t(\omega)=-\frac{g^2/4\pi}{\omega-\Delta}\left[1+\frac{g^2}{4\pi}\frac{\omega-\Delta}{\pi}\int_\mu^\infty d\omega'|f(\omega')|^2(\omega'^2-\mu^2)^{1/2}\times\left(\frac{1}{(\omega'-\omega)(\omega'-\Delta)^2}+\frac{1}{(\omega'+\omega)(\omega'+\Delta)^2}\right)\right]^{-1}, \quad (48)$$

which can be rewritten as

$$t(\omega)=-\frac{g^2/4\pi}{\omega-\Delta}\left[C+\frac{g^2}{4\pi^2}\int_\mu^\infty d\omega'|f(\omega')|^2(\omega'^2-\mu^2)^{1/2}\times\left(\frac{1}{(\omega'-\omega)(\omega'-\Delta)}-\frac{1}{(\omega'+\omega)(\omega'+\Delta)}\right)\right]^{-1}, \quad (49)$$

where C is given by Eq. (45) and is zero when the bootstrap conditions of (42) are met. Thus the particular solution (49) becomes Eq. (46), which is clearly crossing-symmetric. In other words, the particular solutions (43) and (48) become automatically crossing-symmetric, as soon as the bootstrap mechanism is achieved.

V. THE CONDITION $Z_3=0$ AND THE LEVINSON THEOREM

Recently, it has been suggested by many authors that in the framework of field theory the compositeness of a particle can be characterized by vanishing of its wave-function renormalization constant,[14] i.e. $Z_3=0$. From the standpoint of bootstrap philosophy, however, there is no elementary particle and every particle should be regarded as a composite of the others. Therefore it is very plausible that the condition $Z_3=0$ may turn out to be equivalent to the bootstrap condition. In this section let us study in our model what the condition $Z_3=0$ implies.

If we denote the V-particle propagator by $\Delta_V'(s)$, the Lehmann representation[15] is

$$\Delta_V'(s)=(\Delta^2-s)^{-1}+\frac{1}{\pi}\int_{\mu^2}^\infty ds'\sigma(s')(s'-s)^{-1}, \quad (50)$$

where

$$\sigma(s)=G^2\rho(s)|\Lambda(s)|^2, \quad (51)$$

with

$$\Lambda(s)=(2\pi)^{3/2}\pi(2P_{N0}P_{\theta 0})^{1/2}\langle V|N\theta\rangle, \quad (52)$$

$$\rho(s)=|f(s)|^2(s-\mu^2)^{1/2}, \quad (53)$$

and $s=\omega^2$. The quantity $\Lambda(s)$ is related to the form factor $F(s)$ by $F(s)=(\Delta^2-s)\Lambda(s)$ and to the vertex function by

$$\Gamma(s)=\Lambda(s)/\Delta_V'(s). \quad (54)$$

Since the propagator $\Delta_V'(s)$ is a Herglotz function, $-\Delta_V'^{-1}(s)$ is also a Herglotz function and admits the representation

$$\Delta_V'^{-1}(s)=(\Delta^2-s)\left[1-G^2\frac{\Delta^2-s}{\pi}\int_{\mu^2}^\infty ds'\frac{\rho(s')|\Gamma(s')|^2}{(s'-\Delta^2)^2(s'-s)}-\sum_n\frac{C_n(\Delta^2-s)}{(s_n-s)(s_n-\Delta^2)^2}\right], \quad (55)$$

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[14] A. Salam, Nuovo Cimento **25**, 224 (1962); Phys. Rev. **130**, 1287 (1963); S. Weinberg, *ibid.* **130**, 776 (1963); M. L. Whippman and I. S. Gerstein, *ibid.* **134**, B1123 (1964).

[15] H. Lehmann, Nuovo Cimento **11**, 342 (1954).

where $C_n>0$, $s_n>\Delta^2$, and s_n gives the position of C.D.D. zeros of $\Delta_V'(s)$. In the interval $\Delta^2<s<\mu^2$, there can be at most one zero, depending on the sign of $\Delta_V'(\mu^2)$. Since the wave-function renormalization constant Z_V is given by $\lim_{s\to\infty}(\Delta^2-s)^{-1}\Delta_V'^{-1}(s)$, by making use of either (50) or (55), we obtain

$$Z_V^{-1}=1+\frac{1}{\pi}\int_{\mu^2}^{\infty}ds'\sigma(s')=1+\frac{G^2}{\pi}\int_{\mu^2}^{\infty}ds'\rho(s')|\Lambda(s')|^2 \quad (56a)$$

or

$$Z_V=1-\frac{G^2}{\pi}\int_{\mu^2}^{\infty}ds'\frac{\rho(s')|\Gamma(s')|^2}{(s'-\Delta^2)^2}-\sum_n\frac{C_n}{(s_n-\Delta^2)^2}. \quad (56b)$$

Let us make the following decomposition of the s-wave scattering amplitude $T(s)$,[16]

$$\begin{aligned}T(s)&=G^2\Gamma(s)\Delta_V'(s)\Gamma(s)+H(s)\\&=G^2\Lambda(s)\Delta_V'^{-1}(s)\Lambda(s)+H(s).\end{aligned} \quad (57)$$

Then by making use of the unitarity relations

$$T(s+i\epsilon)-T(s-i\epsilon)=2i\rho(s)T(s)T^*(s), \quad (58a)$$

$$\Lambda(s+i\epsilon)-\Lambda(s-i\epsilon)=2i\rho(s)\Lambda(s)T^*(s), \quad (58b)$$

$$\Delta_V'(s+i\epsilon)-\Delta_V'(s-i\epsilon)=2iG^2\rho(s)\Lambda(s)\Lambda^*(s), \quad (58c)$$

it can be easily seen that on the physical (right-hand) cut, i.e., $s>\mu^2$,

$$H(s+i\epsilon)-H(s-i\epsilon)=2i\rho(s)H(s)H^*(s), \quad (59a)$$

$$\Gamma(s+i\epsilon)-\Gamma(s-i\epsilon)=2i\rho(s)\Gamma(s)H^*(s). \quad (59b)$$

Hence, $H(s)$ by itself satisfies unitarity and has the same phase as $\Gamma(s)$, while $\Lambda(s)$ has the phase of $T(s)$.

We shall now proceed to discuss the condition $Z_V=0$ in our model. As we start from the Low equation (1) for $t(\omega)$ [$\equiv T(s)$], we do not *a priori* have quantities like $\Delta_V'(s)$, $\Lambda(s)$, and $\Gamma(s)$. In the following we shall rediscover these quantities in terms of the scattering amplitude $T(s)$ which is given by Eq. (8),

$$T(s)\equiv t(\omega)=\frac{\Delta g^2}{2\pi}(\Delta^2-s)^{-1}\left[1+\frac{\Delta g^2}{2\pi}\frac{s-\Delta^2}{\pi}\int_{\mu^2}^{\infty}ds'\times\frac{\rho(s')}{(s'-\Delta^2)^2(s'-s)}+\frac{\Delta g^2}{2\pi}\sum\frac{R_i(s-\Delta^2)}{(s_i-\Delta^2)^2(s_i-s)}\right]^{-1} \quad (8')$$

which is the most general solution which satisfies unitarity, analyticity, and crossing symmetry (in the ω plane). In the s plane the scattering amplitude does not have the left-hand cut, and in what follows we shall make much use of this situation. (Note that in the s plane our scattering amplitude is exactly the same as that in the Zachariasen model[17] in which G^2 is replaced by $\Delta g^2/2\pi$ and the cutoff factor $|f(s)|^2$ is introduced.)

By introducing the phase $\delta(s)$, which is defined by

$$\begin{aligned}\delta(s)&=\mathrm{Im}\log T(s), &&\text{for } s\geq\mu^2;\\&=0, &&\text{for } s<\mu^2;\end{aligned} \quad (60)$$

one has

$$T(s)=\frac{\Delta g^2}{2\pi}(\Delta^2-s)^{-1}\prod_{i=1}^{n_c}\frac{(s-s_i)}{(\Delta^2-s_i)}\mathfrak{D}^{-1}(s), \quad (61)$$

with

$$\mathfrak{D}(s)=\exp\left[\frac{\Delta^2-s}{\pi}\int_{\mu^2}^{\infty}ds'\frac{\delta(s')}{(s'-\Delta^2)(s'-s)}\right].$$

Since $\Lambda(s)$ has the phase of $T(s)$ and has a pole at $s=\Delta^2$ with no left-hand cut, it is appropriate to define

$$\Lambda(s)\equiv\frac{\Delta g^2}{2\pi}(\Delta^2-s)^{-1}\prod_{i=1}^{n_c}\frac{(s-s_i)}{(\Delta^2-s_i)}\mathfrak{D}^{-1}(s)=T(s). \quad (62)$$

Thus $\Lambda(s)$ so defined has zeros at the C.D.D. zeros of $T(s)$. In much the same manner, since $H(s)$ has the phase of $\Gamma(s)$ and in our model neither of them has a left-hand cut, it is evident that on the right-hand cut

$$H(s)=P_n(s)\Gamma(s), \quad (63)$$

with an arbitrary polynomial $P_n(s)$, and hence it follows that

$$T(s)=(\Delta g^2/2\pi)\Gamma(s)\Delta_V'(s)\Gamma(s)+P_n(s)\Gamma(s). \quad (64)$$

The ambiguity in the determination of $P_n(s)$ is closely related to the existence of C.D.D. zeros of $T(s)$ [or of $\Lambda(s)$]. If one attributes those zeros to the zeros of $\Delta_V'(s)$, then we must take

$$P_n(s)=\prod_i\frac{s_i-s}{s_i-\Delta^2},$$

and $\Gamma(s)$ will then have no zeros, unless $T(s)$ has multiple zeros. However, if one attributes all the C.D.D. zeros to those of $\Gamma(s)$, then $P_n(s)=\alpha$ (=const) and $\Delta_V'(s)$ will have no zeros. In the following we shall take the latter alternative in defining $\Delta_V'(s)$ and $\Gamma(s)$, so that all $C_n=0$ in Eqs. (55) and (56b) and

$$T(s)=(\Delta g^2/2\pi)\Gamma(s)\Delta_V'(s)\Gamma(s)+\alpha\Gamma(s), \quad (65)$$

$\Gamma(s)$ having zeros at the C.D.D. zeros $s=s_i(i=1,2,\cdots,n)$ of $T(s)$. From (54) and (65), it follows then immediately that

$$\Gamma(s)=T(s)[(\Delta g^2/2\pi)\Lambda(s)+\alpha]^{-1}, \quad (66a)$$

and hence

$$\Delta_V'(s)=(\Delta g^2/2\pi)\Lambda(s)+\alpha, \quad (66b)$$

and α is given by

$$\alpha=\frac{d}{ds}\left[(s-\Delta^2)\frac{T(s)}{\Gamma(s)}-\frac{\Delta g^2}{2\pi}(s-\Delta^2)\Lambda(s)\right]_{s=\Delta^2}. \quad (66c)$$

[16] S. D. Drell and F. Zachariasen, Phys. Rev. **105**, 1407 (1957); M. Ida, Phys. Rev. **136**, B1767 (1964); Y. S. Jin and S. W. MacDowell, *ibid.* **137**, B688 (1965); I. S. Gerstein and N. G. Deshpande, *ibid.* **140**, B1643 (1965).

[17] F. Zachariasen, Phys. Rev. **121**, 1851 (1961).

From the unitarity relations (58), some trivial manipulation gives

$$\mathrm{Im}\frac{\Lambda(s)}{\Gamma(s)}=[\Gamma(s)\mathrm{Im}\Lambda(s)-\Lambda(s)\mathrm{Im}\Gamma(s)]|\Gamma(s)|^{-2}. \quad (67)$$

Comparing (51) and (67) and using (58b) we get

$$\mathrm{Im}\Gamma^{-1}(s)=\frac{\Delta g^2}{2\pi}\rho(s)\Lambda^*(s)-\frac{\rho(s)T^*(s)}{\Gamma^*(s)}. \quad (68)$$

By substituting (65) in (68) and using (64), we obtain

$$\mathrm{Im}\Gamma^{-1}(s)=-\alpha\rho(s), \quad (69)$$

and hence also

$$\mathrm{Im}\Gamma(s)=\alpha\rho(s)|\Gamma(s)|^2, \quad (70)$$

and the following once-subtracted dispersion relations:

$$\Gamma(s)=1+\alpha\frac{s-\Delta^2}{\pi}\int_{\mu^2}^{\infty}ds'\frac{\rho(s')|\Gamma(s')|^2}{(s'-s)(s'-\Delta^2)}, \quad (71)$$

$$\Gamma^{-1}(s)=1-\alpha\frac{s-\Delta^2}{\pi}\int_{\mu^2}^{\infty}ds'\frac{\rho(s')}{(s'-\Delta^2)(s'-s)}-\sum_i\frac{a_i(s-\Delta^2)}{(s_i-s)(s_i-\Delta^2)}, \quad (72)$$

since from (62) and (66a), $\Gamma(s)=O(1)$ for $s\to\infty$. As we defined $\Delta_V'(s)$ so that it does not have zeros in Eq. (55), we do not have any C.D.D. contribution of $\Delta_V'(s)$ to Z_V given by Eq. (56b). From (71), however, we know that

$$\frac{1}{\alpha}\left[\frac{d\Gamma}{ds}\right]_{s=\Delta^2}=\frac{1}{\pi}\int_{\mu^2}^{\infty}ds'\frac{\rho(s')|\Gamma(s')|^2}{(s'-\Delta^2)^2}, \quad (73)$$

while from Eq. (72),

$$\frac{1}{\alpha}\left[\frac{d\Gamma}{ds}\right]_{s=\Delta^2}=\frac{1}{\pi}\int_{\mu^2}^{\infty}ds'\frac{\rho(s')}{(s'-\Delta^2)^2}+\sum_i\frac{a_i/\alpha}{(s_i-\Delta^2)^2}; \quad (74)$$

and finally it follows from Eq. (56b) that

$$Z_V=1-\frac{\Delta g^2}{2\pi^2}\int_{\mu^2}^{\infty}ds'\frac{\rho(s')}{(s'-\Delta^2)^2}-\frac{\Delta g^2}{2\pi}\sum_i\frac{a_i/\alpha}{(s_i-\Delta^2)^2}. \quad (75)$$

By using Eqs. (62) and (66a), it turns out that

$$a_i=\lim_{s\to s_i}(s-s_i)\Gamma^{-1}(s)=\lim_{s\to s_i}(s-s_i)\alpha T^{-1}(s)=\alpha R_i, \quad (76)$$

and consequently we obtain

$$Z_V=1-\frac{\Delta g^2}{2\pi^2}\int_{\mu^2}^{\infty}ds'\frac{|f(s')|^2(s'-\mu^2)^{1/2}}{(s'-\Delta^2)^2}-\frac{\Delta g^2}{2\pi^2}\sum_i\frac{R_i}{(s_i-\Delta^2)^2}. \quad (77)$$

By comparing (77) with (7), it is clear that

$$Z_V=(\Delta g^2/2\pi)B. \quad (78)$$

Hence, we conclude that $Z_3=0$ is equivalent to the condition

$$\lim_{\omega\to\infty}[\omega^2 t(\omega)]^{-1}=0$$

or

$$\lim_{\omega\to\infty}D(\omega)/\omega=0$$

which is our condition (i). But since this condition does not exclude the C.D.D. zeros, for a bootstrap it is not sufficient to assume $Z_3=0$ alone.

Before concluding this section, the following remark on the Levinson theorem is in order. Using the s variable, the phase representation of $T(s)$ given by Eq. (61) reads

$$T(s)=\frac{\Delta g^2}{2\pi}(\Delta^2-s)^{-1}\prod_{i=1}^{n_c}\frac{(s-s_i)}{(\Delta^2-s_i)}\times\exp\left[\frac{s-\Delta^2}{\pi}\int_{\mu^2}^{\infty}ds'\frac{\delta(s')}{(s'-\Delta^2)(s'-s)}\right]. \quad (79)$$

Then the high-energy behavior of $T(s)$ is given by

$$T(s)\sim|s|^{n_c-1-\delta(\infty)/\pi}, \quad (80)$$

up to a possible logarithmic factor in s. Hence, if the phase $\delta(x)$ satisfies the Levinson theorem,[18] i.e. if

$$\delta(\mu^2)-\delta(\infty)=(1-n_c)\pi \quad (81)$$

[note that in the s-plane we have one bound state and normalize so that $\delta(\mu^2)=0$], then from (80), we get

$$T(s)=O(1), \quad (82)$$

and consequently we get the condition (i),

$$\lim_{\omega\to\infty}[\omega^2 t(\omega)]^{-1}=0.$$

Conservely, if one assumes that $\lim_{\omega\to\infty}[\omega^2 t(\omega)]^{-1}=0$, then we have no linear term in ω^2 in (4), and for this solution one can prove the Levinson theorem. However, if the linear term is present, then the solution does not satisfy the Levinson theorem, since in this case the contribution from the large circle to the Cauchy integral

$$\int_c\frac{D'(z)}{D(z)}dz$$

does not vanish.

VI. APPROXIMATE SOLUTIONS

In the discussions of Sec. IV, we noticed that there should not be any C.D.D. zeros in the amplitude, in order for the particular solutions to satisfy the two

[18] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. 25, No. 9 (1949).

bootstrap equations (37) and (38). Yet these equations did not limit in any way the values of B_1, and thus gave only an inequality relation (22) for the parameters of Δ and g. In other words, the two bootstrap equations (37) and (38) are merely equivalent to one of our bootstrap conditions (ii) of Sec. II or (42b) of Sec. III when there is crossing symmetry, and we need the condition (i) or equivalently (42a) in order to have a definite equality relation between Δ and g which does not contain any other unknown parameters such as the relation (25). Even under the two conditions (i) and (ii), however, one can never determine both Δ and g completely for the given cutoff function $f(\omega)$ unless one develops an extra independent condition for them. We can only get a relation between them which gives one of the parameters in terms of the other.

One might think that if it were not for crossing symmetry, the two bootstrap equations (37) and (38) would become independent of each other, thus giving one more condition for the parameters. In order to examine this possibility, let us destroy the crossing-symmetric property of our model by introducing two different cutoff functions[4] $f_+(\omega)$ and $f_-(\omega)$ which are not identically zero along the cuts, so that the denominator function (36) can be modified as

$$D(\omega)=1-B_1(\omega+\Delta)-\frac{g^2}{4\pi}\frac{\omega+\Delta}{\pi}\int_\mu^\infty d\omega'(\omega'^2-\mu^2)^{1/2}\times\left[\frac{|f_+(\omega')|^2}{(\omega'-\omega)(\omega'+\Delta)^2}+\frac{|f_-(\omega')|^2}{(\omega'+\omega)(\omega'-\Delta)^2}\right]+\sum_i\frac{R_i(\omega+\Delta)}{(\omega-\omega_i)(\omega_i+\Delta)}. \quad (83)$$

Then the first bootstrap equation (37) results in

$$1=2\Delta\left(B_1-\sum_i\frac{R_i}{\Delta^2-\omega_i^2}\right)+\frac{\Delta g^2}{2\pi^2}\int_\mu^\infty d\omega\frac{(\omega^2-\mu^2)^{1/2}}{\omega^2-\Delta^2}\times\left[\frac{|f_+(\omega)|^2}{\omega+\Delta}+\frac{|f_-(\omega)|^2}{\omega-\Delta}\right], \quad (84)$$

while one obtains

$$\left[N(\omega)\Big/\frac{dD(\omega)}{d\omega}\right]_{\omega=\Delta}=\frac{g^2}{4\pi}\left\{-1-4\Delta^2\left[\sum_i\frac{R_i}{(\Delta-\omega_i)(\Delta^2-\omega_i^2)}+\frac{g^2}{4\pi^2}\times\int_\mu^\infty d\omega\frac{(\omega^2-\mu^2)^{1/2}}{(\omega^2-\Delta^2)^2}(|f_+(\omega)|^2-|f_-(\omega)|^2)\right]\right\}^{-1}, \quad (85)$$

where an identity relation from (84) is inserted. We notice that Eq. (85) becomes Eq. (41) when $|f_+(\omega)|^2=|f_-(\omega)|^2$, as it should. When Eq. (85) is compared with Eq. (38), one finds in general that $\Delta=0$ and/or

$$\frac{g^2}{4\pi^2}\int_\mu^\infty d\omega\frac{(\omega^2-\mu^2)^{1/2}}{(\omega^2-\Delta^2)^2}(|f_+(\omega)|^2-|f_-(\omega)|^2)=-\sum_i\frac{R_i}{(\Delta-\omega_i)(\Delta^2-\omega_i^2)}, \quad (86)$$

whether the parameter B_1 is zero or not. It should be noticed that the right-hand side of Eq. (86) is always negative if there exist C.D.D. zeros, since $R_i>0$.

If no C.D.D. zeros are present in the scattering amplitude, then the integral on the left-hand side of Eq. (86) should vanish in a delicate manner. While one may choose the two cutoff functions so as to make the integral vanish, it is plausible (if nature is simple) that the cutoff functions will behave more or less monotonically. The bootstrap equations therefore may lead to crossing symmetry when there exist no C.D.D. zeros. Moreover, it is clear that Eq. (86) in the case of no C.D.D. zeros does not provide any more independent conditions as far as determination of the parameters Δ and g is concerned.

Let us consider a particular case in which one neglects the left-hand-cut contribution[3,6] in the above general discussion. This means putting $f_-(\omega)=0$ identically. Then the bootstrap equations (37) and (38) can be satisfied by $\Delta=0$ and/or by

$$\frac{g^2}{4\pi^2}\int_\mu^\infty d\omega\frac{(\omega^2-\mu^2)^{1/2}}{(\omega^2-\Delta^2)^2}|f_+(\omega)|^2=-\sum_i\frac{R_i}{(\Delta-\omega_i)(\Delta^2-\omega_i^2)}, \quad (87)$$

for all possible values of B_1. Since the right-hand side of (87) can never become positive and the integrand stays always positive in the domain of integration, Eq. (87) can be satisfied only when all R_i are zero and $f_+(\omega)=0$. Thus we notice that the bootstrap equations lead to the retention of crossing symmetry in our approximate solution with no C.D.D. zeros of the amplitude present. This approximate solution, however, is clearly uninteresting to us, because the bootstrap equations restrict the parameters to $\Delta=0$ and/or to the case of no scattering at all. Again we see that breaking crossing symmetry in this way does not result in any nontrivial conditions for determining the parameters. In order to get meaningful results in the approximate solutions, one must either introduce a subtraction in the Low equation (1) from which we started, or introduce one C.D.D. zero. In the former case, it is anticipated that we can determine the parameters Δ and g^2 completely, while in the latter case we would get an equation between them, since one C.D.D. zero introduces two more parameters.

It seems, therefore, that there is no way of determining the parameters completely in our model. Even if one breaks the crossing-symmetry property, the two bootstrap equations never give any more useful conditions than a relation which merely relates the symmetry-breaking factors built in. Furthermore, using crossing symmetry rather than destroying it seems to be the reasonable approach, in the sense that the bootstrap equations then can result in a nontrivial relation between the bootstrap parameters. The bootstrap equations, when crossing symmetry is used, do not permit any C.D.D. zeros and hence exclude the possibility of having a positive scattering length in our model.

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Sum Rule Connecting Mesonic and Photonic Matrix Elements and the Rate for $\omega^0 \rightarrow \pi^0+\gamma$ †

S. GASIOROWICZ
School of Physics, University of Minnesota, Minneapolis, Minnesota
(Received 9 February 1966)

The Gell-Mann current-commutation relations are used to obtain a relation between matrix elements involving vector currents and matrix elements involving axial currents, in the infinite energy limit. This relation leads to a set of sum rules. One of these is used to calculate the rate for $\omega^0 \rightarrow \pi^0+\gamma$ in terms of the ρ-meson width and the axial-current renormalization constant, under the assumption that only states with spin and parity $0^{\pm}$ and 1^- need be taken into account. The result, $\Gamma(\omega^0 \rightarrow \pi^0+\gamma)=1.2$ MeV, is in reasonable agreement with experiment.

INTRODUCTION

THE remarkable calculation of the axial-current renormalization constant by Weisberger[1] and Adler[2] may be ranked with the determination of the Yukawa coupling constant from the forward scattering dispersion relations as one of the major successes of field theory in the domain of the strong interactions. This calculation, based on the equal-time current commutation relations of Gell-Mann,[3] has stimulated further exploitation of these relations in a number of sum rules. When certain assumptions concerning the dominant contributions to the sum rules are made, relations between different matrix elements are obtained. In all of the calculations published so far, little use has been made of the symmetry between the vector currents and the axial currents. Thus, in the calculation of the axial-current renormalization constant only matrix elements of the pion field operator appear, with the commutation relations providing a scale. Similarly, the calculations of the magnetic moment of the nucleons[4,5] relate these to matrix elements of the isovector current alone. What has been missing so far is the analog of the old photoproduction or form-factor[6] calculations of dispersion theory, in which, using certain assumptions concerning the intermediate states, photonic matrix elements were calculated in terms of pionic ones. It is the purpose of this paper to suggest a method of using the commutation relations for such problems, and to illustrate it by a calculation of the rate for the process $\omega^0 \rightarrow \pi^0+\gamma$.

THE SUM RULE

We begin by considering two matrix elements, one involving a commutator of vector isospin currents, the other a commutator of axial isospin currents:

$$V_{\mu\rho}{}^{\alpha\beta}(p',a,k;\,p,b,q)=-i\int dx\, e^{ikx}\theta(x_0) \times\langle p',a|[\mathfrak{F}_\mu{}^\alpha(x),\mathfrak{F}_\rho{}^\beta(0)]|p,b\rangle \quad (1)$$

and

$$A_{\mu\rho}{}^{\alpha\beta}(p',a,k;\,p,b,q)=-i\int dx\, e^{ikx}\theta(x_0) \times\langle p',a|[\mathfrak{F}_\mu{}^{5\alpha}(x),\mathfrak{F}_\rho{}^{5\beta}(0)]|p,b\rangle . \quad (2)$$

The indices a, b, α, and β refer to the isospin, and we con-

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