K^- Mesonic X Rays and the $\bar{K}$-N Scattering Lengths*

FRANK VON HIPPEL
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York

AND

JOHN H. DOUGLAS
Department of Physics, Cornell University, Ithaca, New York

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K^- mesonic x rays from helium have recently been observed and their energies and intensities measured. The strong-interaction effects on these lines appear to be an order of magnitude smaller than expected on the basis of simple considerations. We discuss a model for the strong K^--α interaction effects assuming dominance of two-body $\bar{K}$-N interactions and based upon the representation of the $\bar{K}$-N interactions by complex central Yukawa potentials adjusted to the experimental scattering lengths. For ranges associated with any known particle exchanges, the results, although very different from those obtained previously in Born approximation, still disagree by an order of magnitude with experiment. After considering various alternative explanations of this disagreement, we note that a $\bar{K}$-N scattering-length solution having a value for Im A_1 considerably smaller than that currently favored would allow agreement of our model with experiment.

I. INTRODUCTION

X-RAY lines from the de-excitation of low-lying K^--α atomic states have recently been observed and their energies and intensities have been measured.[1] The possible uses for such information had previously been the subject of some discussion.[2] Crude predictions of strong-interaction effects on line energies and intensities were made for the K^--α system among others.

The fact that these predictions[3,4] disagree with the new experimental figures by more than an order of magnitude has stimulated a recalculation which is reported in this paper.

We find that the customary direct connection of the complex energy levels to a sum of experimental $\bar{K}$-N scattering lengths is totally invalid. The strength of the $\bar{K}$-N interactions results in effects dependent upon the interaction range. We therefore consider a model in which the $\bar{K}$-N interaction is replaced by complex central Yukawa potentials for $\bar{K}$-N isospin 0 and 1. The depths of these potentials are adjusted to give the experimental $\bar{K}$-N scattering lengths and we vary their ranges between limits suggested by allowed single-particle exchanges. The strong interaction shifts and widths of the K^--α atomic-energy levels are then directly related to a sum of volume integrals of these potentials.

The predictions which this model gives bear little relation to those obtained in the previous treatments, but the order of magnitude disagreement with experiment remains. Furthermore, contrary to our expectations from any simple particle-exchange model, we find that the imaginary part of the complex potential required to fit the $I=1$ scattering lengths is almost an order of magnitude larger than that found in fitting the $I=0$ results.

After considering alternative explanations for these puzzling results, we point out that a $\bar{K}$-N scattering length solution with a smaller value for the imaginary part of the $I=1$ scattering length would remove both difficulties.

In Sec. II we review the new experimental results on the K^--α spectrum. In Sec. III we discuss the more extensive experience with π^- mesonic x-ray observations and the attempts at their theoretical interpretation. In Sec. IV we demonstrate the inadequacy of treatments relating the $\bar{K}$-N interaction in the nucleus to the $\bar{K}$-N scattering lengths by the Born approximation, and in Sec. V outline our treatment with complex Yukawa potentials. In Sec. VI we estimate the K^--α interaction resulting from the two-body $\bar{K}$-N interactions and arrive at the results quoted above. Finally, in Sec. VII, we draw our conclusions.

II. EXPERIMENTAL K^--α X RAY RESULTS

Burleson *et al.*[1] have observed two x-ray lines which they have interpreted as due to $2P \rightarrow 1S$ (K_α) and $3D \rightarrow 2P$ (L_α) transitions of the K^--α system. The energies measured were 34.7 ± 0.3 and 6.7 ± 0.2 keV, respectively. For comparison, the theoretical energies, calculated, neglecting strong interaction effects, are 34.9 and 6.5 keV.[5]

Burleson *et al.* have also obtained a rough measure of the ratio of line intensities, $K_\alpha/L_\alpha \approx 0.2\pm0.1$. Again,

* Supported in part by U. S. Office of Naval Research contract.

[1] G. R. Burleson, D. Cohen, R. C. Lamb, D. N. Michel, R. A. Schluter, and T. O. White Jr., Phys. Rev. Letters **15**, 70 (1965).

[2] See, e.g., D. H. Wilkinson, in *Proceedings of the Rutherford Jubilee International Conference*, edited by J. B. Birks (Heywood and Company, London, 1961), p. 339; and E. H. S. Burhop, D. H. Davis, and J. Zakrzewski, Progr. Nucl. Phys. **9**, 155 (1964).

[3] Y. Eisenberg and D. Kessler, Phys. Rev. **130**, 2352 (1963).

[4] T. B. Day, Nuovo Cimento **18**, 381 (1960).

[5] Klein-Gordon energies. K_α energy corrected for finite nuclear size (-0.17 keV) and vacuum polarization ($+0.27$ keV). See Ref. 3.

for comparison, electromagnetic cascade calculations would predict a ratio near unity.[6]

The reduction in K_α/L_α is most likely due to the competition of strong nuclear absorption with K_α radiation from the $2P$ state. With this interpretation the K_α/L_α ratio can be expressed as

$$\frac{K_\alpha}{L_\alpha}=\frac{\Gamma_{\rm rad}(2P)}{\Gamma_{\rm rad}(2P)+\Gamma_{\rm abs}(2P)}, \quad (1)$$

from which $\Gamma_{\rm abs}(2P)$ may be obtained in terms of the experimental K_α/L_α ratio and the theoretical value of $\Gamma_{\rm rad}(2P)$.

Adopting $\Gamma_{\rm rad}(2P)$ from the value calculated for hydrogen,[7] we have

$$\Gamma_{\rm rad}(2P)=5.6\times10^{-6}\ {\rm keV}. \quad (2)$$

Inserting this value and the experimental K_α/L_α ratio in (1) then gives

$$\Gamma_{\rm abs}(2P)=(2.2_{-1.0}{}^{+3.4})\times10^{-5}\ {\rm keV}. \quad (3)$$

In summary, our current experimental information concerning strong interaction effects on K^--α de-excitation x rays may be expressed by two numbers:

1. An upper bound on the magnitude of the strong-interaction shift of the K_α-line energy. This energy-level shift must be primarily due to a shift of the $1S$ state energy since the nuclear penetration of the $2P$ wave function is comparatively small. Therefore the upper bound may be written as

$$|\Delta E_{1S}|<0.4\ {\rm keV}. \quad (4)$$

[We do not consider the experimental upper bound on the strong interaction shift of the L_α line as significant. Because of the exclusion of the $2P$ and $3D$ wave functions from the nucleus by centrifugal barrier effects, this shift is expected to be of the order of $\Gamma_{\rm abs}(2P)$, far below the sensitivity of the experiment.]

2. Our second experimental number is the value for $\Gamma_{\rm abs}(2P)$ obtained in Eq. (3).

In the next section we discuss the relevant aspects of the experimental and theoretical situation for strong interaction effects in π^- x rays.

III. PIONIC X RAYS

Pionic x rays were first encountered more than a decade ago.[8] The strong-interaction energy shifts of the K_α x rays from the light elements[9] were measured in particular, because of their possible usefulness in checking early π-N scattering-length solutions.

These strong-interaction shifts are similar to those of kaonic x rays in that they also are primarily due to shifts of the energy of the π^- atomic $1S$ state. A brief discussion of the current status of their interpretation will therefore be useful in our attempt at understanding the same effects in K^- mesonic atoms.

The π^- $1S$ energy shifts are usually interpreted according to a model suggested by Deser, Goldberger, Baumann, and Thirring.[10] This model replaces the π^--nucleus interaction by the sum of the elastic interactions of the π^- with the individual nucleons making up a nucleus. Because of the small π-N scattering lengths, the Born approximation has been assumed applicable to these interactions. The model then gives a shift proportional to a sum of the scattering lengths:

$$\Delta E_{1S}=-(2\pi/\bar{\mu})\rho_z(0)[Z(\tfrac{1}{3}a_3+\tfrac{2}{3}a_1)+Na_3], \quad (5)$$

where $\bar{\mu}$ is the π-nucleus reduced mass, $\rho_z(0)$ is the average pion probability density in the nucleus, and a_1, a_3 are the $I=\frac{1}{2}$, $\frac{3}{2}$ π-N scattering lengths. (We take $\hbar=c=1$ throughout. A positive scattering length will have the effect of an attraction by decreasing the energy E_{1S}.)

Because π^- K_α x rays have been observed only in light nuclei ranging from Li^6 to F^{19}, for which Z and N are usually equal, most of the observed K_α energy-level shifts should, in this model, be proportional to $\bar{A}=\frac{2}{3}a_3+\frac{1}{3}a_1$. Assuming this to be the case, a value, $\bar{A}\approx-0.028$ F, has been obtained as the best fit of (5) to the experimental K_α shifts for $Z=N$ nuclei.[11]

The value of $\bar{A}$ determined by π-N scattering is also small compared to a_1 and a_3, which are about 0.2 and -0.1 F, respectively. This much has been known since

[6] The point is essentially that L_α transitions are more likely than transitions to the $2P$ state from states with $n>3$. This is a standard result for K^- cascade calculations in atoms with $Z>2$. (See, e.g., Ref. 3.) We believe that the physical reason for this result—statistical accumulation of the K^- mesons in $l=n-1$ states after a chain of many de-excitations from the initial ($n\approx30$) capture orbits—applies to the K^--α atom where external Auger de-excitation takes over the function of the ordinary Auger process. [Only one ordinary Auger de-excitation will occur, as the $(K^-\text{-}\alpha)^+$ system is energetically unable to capture another electron from a neighboring Helium atom.] Therefore the final radiative steps of the cascade must occur in steps of $n_i-n_f=1$ due to the dipole selection rule, $l_i-l_f=1$. Adapting the results calculated for hydrogen [M. Leon and H. Bethe, Phys. Rev. **127**, 636 (1962)] it is possible to show that external Auger de-excitation will not compete strongly with the final steps of the cascade, $4F\to3D(M_\alpha)$, $3D\to2P(L_\alpha)$, $2P\to1S(K_\alpha)$ which, in the absence of strong interactions, therefore have intensities roughly in the ratio 1:1:1.

[7] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Academic Press Inc., New York, 1957), p. 266. The Bohr radius of the K^--α system is 31 F. Consequently the radiation rates between low-lying states are little affected by either the nucleus or by the surrounding electrons.

[8] See, review articles by M. B. Stearns, Progr. Nucl. Phys. **6**, 108 (1957); D. West, Rept. Progr. Phys. **21**, 271 (1958); T. Ericson, in *Proceedings of the 1963 International Conference on High Energy Physics and Nuclear Structure*, edited by T. Ericson (CERN, Geneva, 1963), p. 47.

[9] For nuclei beyond F^{19} the competition of nuclear absorption with radiation from the $2P$ state reduced the K_α intensity to a point which made observations impractical.

[10] S. Deser, M. L. Goldberger, K. Baumann, and W. Thirring, Phys. Rev. **96**, 774 (1954). See also H. A. Bethe and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, Illinois, 1955), Vol. 2, p. 103 ff. See also Sec. IV of the present paper.

[11] M. Stearns and M. B. Stearns, Phys. Rev. **103**, 1534 (1956).

1956, but only recently, with the aid of forward dispersion relations, have π-N scattering-length determinations become sufficiently precise to allow a quantitative check of (5). The current value $\bar{A}=-0.012 \pm 0.004$ F from these analyses[12] corresponds to energy level shifts of the correct order of magnitude and sign. If the theoretical error estimates are realistic, however, it would appear that the experimental data are *not* fitted[13] by the single-nucleon effects as described by Eq. (5).

Another contribution to the observed $\pi^- K_\alpha$ shifts in $Z=N$ nuclei is suggested by the model, originally due to Brueckner, in which they are interpreted as an indirect effect from the absorption process of π^- mesons on nucleon pairs. It has not been possible, however, to make more than very crude estimates of this effect.[14]

It is because of an accidental cancellation in $Z=N$ nuclei, that the single nucleon contributions to the $\pi^- K_\alpha$ shifts are not clearly observed. A more sensitive test for this contribution would be through the measurement of the difference in strong-interaction shifts of K_α x rays from several pairs of stable nuclei, each having one $Z=N$ nucleus paired with a nucleus with an additional neutron or with one less proton. Since the single nucleon effect is intrinsically about four to five times larger, in the absence of cancellation, than one-half the average effect observed from a proton-neutron pair, the differences should be large and relatively well described by (5).

We take the four pairs of nuclei for which the K_α line energies have been measured[8] and estimate the effect of the paired nucleons from the shift measured for the $N=Z$ nucleus. By adding or subtracting the effect of one-half a pair from the shift measured for the other nucleus (which has one more neutron or one less proton), it is then possible to obtain a measure of the single-neutron effect. The ratios of the experimental effects to those predicted by Eq. (5) are: from (Li^6, Li^7), 2.2 ± 0.5; from (Be^9, B^{10}), 0.5 ± 0.2; from (B^{10}, B^{11}), 1.2 ± 0.4; and from (B^{11}, C^{12}), 1.5 ± 0.5—giving an average of 1.35 ± 0.4. The effects are of the correct magnitude but little more can be concluded until the experiments are improved.

For the L_α-energy shifts and widths, it is possible to make a very similar model to that described by Eq. (5) using p-wave π-N scattering and π-production data. (Most of the s-wave effects cancel again.) This has been done by Ericson and Ericson[13] who point out that such models give agreement, within experimental errors of 20–30%, with a recent measurement of the L_α-energy-level shift in Al^{27}, and with $2P$ and $3D$ absorption widths deduced from $\pi^- K_\alpha$ and L_α x-ray line intensities.

The purpose of this discussion has been to obtain as much insight as possible into the K^- x-ray problem from our partial understanding of strong interaction effects in π^- x rays.

Since our model for the K^- x-ray-line energy shifts is based on the assumed dominance of the $\bar{K}$ interactions with single nucleons, we have discussed primarily the evidence for π^- interactions with single nucleons from π^- x-ray spectra. We have seen that there is some experimental evidence that the π^-–single-nucleon effects, although masked by cancellation in the $1S$ state, are present and are estimated reasonably well in Born approximation. In the next section we show that the much stronger K^-–single-nucleon effects cannot be so easily handled.

IV. LIMITATIONS OF THE BORN APPROXIMATION

In order to estimate ΔE_{1S} and $\Gamma_{\rm abs}(2P)$ for the K^--α system, it is necessary to have a description of the $\bar{K}$-N strong interactions. In Section III we noted that, if the Born approximation holds for the meson-single nucleon interaction, it is only necessary to know the meson-nucleon scattering lengths in order to calculate ΔE_{1S}. As the Born approximation has been used in most treatments to date of the mesonic x-ray line shifts, we now discuss the limits of its applicability in a rather general manner. The validity of approximations using low energy p-wave $\bar{K}$-N parameters in calculating $\Delta E_{L\alpha}$ and $\Gamma_{\rm abs}(2P)$, can be discussed in similar terms.

Our conclusion, which we prove for the example of a complex central potential in the Schrödinger equation, is that the Born approximation is only valid when $|A|/R \ll 1$, where $A=a+ib$ is the complex scattering length and R is the range of the interaction. This condition is not satisfied by the s-wave $\bar{K}$-N interactions, although it may be valid for π-N interactions.

The s-wave radial Schrödinger equation for $u(r)\equiv r\psi(r)$ may be written

$$-u''(r)+2mU(r)u(r)=2mEu(r)\,, \tag{6}$$

where $U(r)$ is a complex potential with range R. If $u(r)$ is normalized to have the asymptotic form

$$u(r) \underset{r\to\infty}{\longrightarrow} k^{-1}[\sin kr-\tan\delta\cos kr]\,, \tag{7}$$

Eq. (6) yields[15] an integral equation for the complex phase shift:

$$\tan\delta=-2m\int_0^\infty \sin kr'\,U(r')u(r')dr'\,. \tag{8}$$

[12] V. K. Samaranayake and W. S. Woolcock, Phys. Rev. Letters **15**, 936 (1965).

[13] This fact has also been observed recently by M. Ericson and T. E. O. Ericson, CERN Report 65/509/5-Th. 539, 1965 (unpublished).

[14] K. A. Brueckner, Phys. Rev. **98**, 769 (1955); R. Karplus, in *Proceedings of Sixth Annual Rochester Conference on High Energy Physics* (Interscience Publishers, Inc., New York, 1956), pp. IX-33; R. Karplus and F. Halpern, Bull. Am. Phys. Soc. **2**, 5 (1957); D. J. Thouless, Proc. Phys. Soc. **A69**, 280 (1956).

[15] See, e.g., L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), p. 181.

At threshold $u(r)$ in (7) assumes the form

$$u(r) \underset{r\to\infty}{\longrightarrow} r+A\,, \tag{9}$$

where A is, by definition, the complex scattering length. Then (8) goes over into an equation for A,

$$A=-2m\int_0^\infty dr'U(r')u(r')r'\,. \tag{10}$$

This form is convenient for analyzing the circumstances of validity for the Born approximation scattering length obtained by the zeroth-order approximation $u(r')=r'$:

$$A_B=-2m\int_0^\infty r'^2dr'U(r')\,. \tag{11}$$

In the absence of bound states (when the Born approximation is obviously invalid), the magnitude of error resulting from this approximation may be estimated by substitution of the first-order form $u(r')=r'+A$ into (10). The correction term will clearly be of the order of A/R, where R is the range of the potential. Obviously, only if

$$A/R\ll 1\,, \tag{12}$$

will the Born approximation hold. [In (11) A_B is proportional to the potential-volume integral. This is also true for the expression for the $1S$-state energy shift of the K^--α mesonic atom which we will obtain in Sec. VI. The Born approximation derivation of Eq. (5) for the energy-level shift in terms of the scattering lengths will therefore become obvious.]

The value of A/R for the $\bar{K}$-N interaction is by no means small. The complex $\bar{K}$-N scattering lengths suggested by the most recent analyses[16] of low-energy $\bar{K}$-N interactions are (in fermis),

$$-1.77<a_0<-1.57\,,\quad 0.45<b_0<0.75 \tag{13a}$$

$$-0.3\ <a_1<0.3\,,\qquad 0.4\ <b_1<0.7\,, \tag{13b}$$

where the subscripts stand for isospin 0 and 1. As no particle exchange can be expected to yield a $\bar{K}$-N interaction with range greater than 1 F (one π exchange is not allowed), the Born approximation is obviously inapplicable.

For later comparison, however, we list here the volume integrals of the complex $\bar{K}$-N potentials calculated in Born approximation, i.e. according to (11):

$$v_0+iw_0=(16.4+i2.3)\mathrm{F}^2 \tag{14a}$$

$$v_1+iw_1=(0.0+i2.1)\mathrm{F}^2. \tag{14b}$$

The central values of the scattering lengths given in (13) have been used here. Note particularly that, because of the $Y_0^*(1405)$ resonance below $\bar{K}$-N threshold, a_0 and therefore v_0 have signs appropriate to a *repulsion*.

V. THE COMPLEX $\bar{K}$-N POTENTIALS

We now discuss our procedure for translating the complex $\bar{K}$-N scattering lengths of (13) into effective central potentials in the $\bar{K}$-N coordinate.

Note that we start with two approximations here:

1. We have replaced an essentially multi-channel relativistic interaction with a phenomenological potential.

This we regard as a necessary simplification due to the well-known difficulties inherent in any approach based directly on particle exchange diagrams. Our belief is that there is enough flexibility in the phenomenological potentials used to reproduce the $\bar{K}$-N interactions adequately for the order of magnitude comparisons made in this paper. Similar representations of the Λ-N interactions in hypernuclei have had considerable success.[17]

2. We have assumed a central potential.

This is an unnecessary simplification made only for convenience. In general there should be $\mathbf{L}\cdot\mathbf{S}$ and other momentum-dependent terms in the potential. Such gradient terms should not affect our results for ΔE_{1S} significantly, as the unperturbed $1S$ state K^- wave function is nearly a constant over the nucleus. A term proportional to $\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}$ would affect $\Gamma_{\mathrm{abs}}(2P)$, however,[18] and our calculation for $\Gamma_{\mathrm{abs}}(2P)$, assuming only a central absorptive interaction, must therefore be considered as a lower limit.

We begin then by assuming that the $\bar{K}$-N interaction is due to the sum of two complex central potentials multiplied by projection operators in the $I=0,1$ states,

$$U(\bar{K},N)=U_0(r)\tfrac{1}{4}(1-\boldsymbol{\tau}_{\bar{K}}\cdot\boldsymbol{\tau}_N) +U_1(r)\tfrac{1}{4}(3+\boldsymbol{\tau}_K\cdot\boldsymbol{\tau}_N)\,, \tag{15}$$

where we take the $U_k(r)$ to be of Yukawa form with ranges appropriate to the nature of the forces involved:

$$U_k(r)=V_k(e^{-\mu_k r}/r)-iW_k(e^{-\nu_k r}/r)\,. \tag{16}$$

With these potentials the threshold solutions of the Schrödinger equation with relativistic kinematics have been obtained as a function of the real and imaginary potential-volume integrals v and w. A search has been made for points in the v,w space corresponding to scattering-length solutions falling in the intervals (13).

For definiteness we discuss the results obtained for $\mu_{0,1}=763$ MeV appropriate to the static ρ exchange

[16] W. E. Humphrey and R. R. Ross, Phys. Rev. 139, B719 (1965); J. K. Kim, Phys. Rev. Letters 14, 29 (1965). The limits quoted in Eq. (13) are based on a comparison of these two solutions—not on their quoted errors.

[17] See, e.g., R. H. Dalitz, *Nuclear Interactions of the Hyperons* (Oxford University Press, London, 1965).

[18] See, e.g., Ref. 13 for an analysis of the effect of such terms on pionic x-ray spectra.

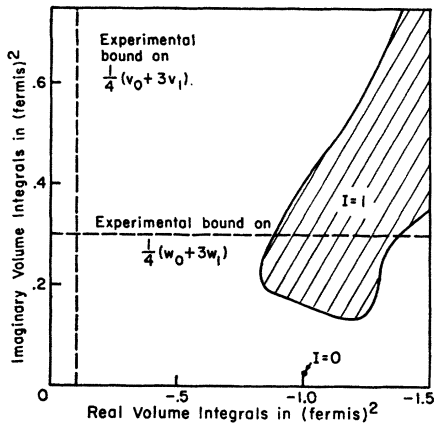

FIG. 1. Regions in the complex potential-volume integral plane corresponding to the experimentally allowed $I=0$ and 1 $\bar{K}$-N scattering-length solutions. Yukawa range parameters for the real and imaginary parts of the potential are 763 and 891 MeV, respectively.

potential which plays an important role in most models for the $Y_0^*(1405)$ resonance just below $\bar{K}$-N threshold,[19] and we take $\nu_{0,1}=891$ MeV, as an approximate range for the inelastic reactions, $\bar{K}+N\rightarrow\pi+(\Sigma$ or $\Lambda)$, proceeding through K^* (or Λ or Σ) exchange. Our conclusions are independent of variations of μ and ν up to several BeV, however.

The results are displayed in Fig. 1. Although quite unexpected from Born approximation considerations [see Eq. (14)], they have rather simple explanations. We discuss the $I=0$ and $I=1$ results separately.[20]

$I=0$: These results are characterized by being localized in a small neighborhood of the v,w plane and by a small value of w_0.

The reason for the sensitivity of the scattering lengths to v and w is because of the $Y_0^*(1405)$ resonance strongly coupled to the $\bar{K}$-N channel 30 MeV below threshold. The complex distance of this resonance below threshold is sensitive to v_0+iw_0, and a_0+ib_0 is directly related to this distance.[21] The small value of w_0 required to produce the large absorptive cross section (large b_0) is due to an enhancement of the threshold wave function at small $\bar{K}$-N separations by the large value of a_0. The comparatively weak imaginary potential acting on this large wave function yields an amplified result.

$I=1$: If the $I=0$ results display the indirect effects of a on b, the $I=1$ interaction shows the converse effects. Again, as in the $I=0$ case, the imaginary scattering length b_1 in (14) is of the order of the range of the absorptive potential. This time, however, there is no large real scattering length a_1 to produce an enhanced threshold wave function for small $\bar{K}$-N separations. Therefore absorption must occur through the brute force of an imaginary potential large enough to act as a "dark grey" sphere. Such a potential is similar to a "hard sphere" potential in that it generates by itself a real scattering length characteristic of a repulsion. To neutralize this effect on a_1 and enhance the wave function for small separations, a large attractive potential is necessary.

In the next section we discuss the effective K^--α interaction resulting from these $\bar{K}$-N potentials.

VI. THE K^--α INTERACTION

It appears from the smallness of the experimental K_α line shift and width that the strong interactions in the atomic K^--α system can be treated by first-order perturbation theory.

We therefore begin by writing down a wave function for this system in an unperturbed Coulomb state,

$$|\bar{K};1,2,3,4\rangle=\psi_{nlm}(\mathbf{r}_{\bar{K}})\phi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)\chi(1,2,3,4). \quad (17)$$

All spatial coordinates in this wave function are referred to the center of mass of the α particle. It has been assumed that the wave function factors into: a real Coulomb wave function ψ_{nlm} in the relative K^--α coordinate, where nlm are the principal, orbital and magnetic quantum numbers; a real normalized completely symmetric space wave function of the four nucleons, ϕ; and a normalized $I=0$, $S=0$ spin-isospin wave function, χ.

The complex shift of the (n,l) state energy will then be, in first order,

$$\Delta(E+i\Gamma)=\langle\bar{K};1,2,3,4|\sum_{l=1}^{4}U(\bar{K},N_l)|\bar{K};1,2,3,4\rangle, \quad (18)$$

where $U(\bar{K},N)$ has been defined in (15).

Upon summing over spin and isospin indices (18) reduces to

$$\Delta(E+i\Gamma)=\int d\mathbf{r}_{\bar{K}}d\mathbf{r}_1d\mathbf{r}_2d\mathbf{r}_3d\mathbf{r}_4 \times[\psi_{nlm}(\mathbf{r}_{\bar{K}})\phi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)]^2 \times\sum_{l=1}^{4}[\tfrac{1}{4}U_0(r_K)+\tfrac{3}{4}U_1(r_K)]. \quad (19)$$

If we take advantage of the symmetry of ϕ, it is possible to substitute $U_k(r_{\bar{K}1})$ for $U_k(r_{\bar{K}l})$, $(l=2,3,4)$. Equation (19) may then be written as

$$\Delta(E+i\Gamma)=\int d\mathbf{r}_{\bar{K}}d\mathbf{r}_1[\psi_{nlm}(\mathbf{r}_{\bar{K}})]^2\rho(\mathbf{r}_1) \times[U_0(\mathbf{r}_{\bar{K}1})+3U_1(\mathbf{r}_{\bar{K}1})], \quad (20)$$

[19] See, e.g., A. W. Martin and K. C. Wali, Nuovo Cimento **31**, 1324 (1964). For a nonrelativistic treatment with Yukawa potentials due to vector meson exchange see R. H. Dalitz, Proc. Roy. Soc. (London) **A288**, 183 (1965).

[20] *Footnote added in proof.* If the nonrelativistic Schrödinger equation had been used to obtain the potentials associated with the experimental scattering lengths, the values of the real volume integrals would be roughly twice as large as those displayed in Fig. 1.

[21] See, e. g., R. H. Dalitz, Rev. Mod. Phys. **33**, 471 (1961) or *Strange Particles and Strong Interactions* (Oxford University Press, London, 1962).

where

$$\rho(\mathbf{r}_1)=\int d\mathbf{r}_2 d\mathbf{r}_3 d\mathbf{r}_4[\phi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)]^2 \tag{21}$$

is the density distribution of a nucleon in the α particle.

In evaluating (20) we use a form of ρ suggested by electron-scattering experiments. These show that the charge distributions of the α particle[22] and proton[23] may, for our purposes, be approximated by Gaussians with rms radii of 1.61 and 0.805 F, respectively. The nucleon density then has the form

$$\rho(r_1)=(\alpha/\pi)^{3/2}e^{-\alpha r^2}, \tag{22}$$

where $\alpha=0.772$ F^{-2}.

Equation (20) may now be cast into a form describing a K^--α interaction,

$$\Delta(E+i\Gamma)=\int d\mathbf{r}_{\bar{K}}[\psi_{nlm}(\mathbf{r})]^2 U(\mathbf{r}_K), \tag{23}$$

where

$$U(r_{\bar{K}})=\int d\mathbf{r}_1\rho(\mathbf{r}_1)[U_0(r_{\bar{K}1})+3U_1(r_{\bar{K}1})]. \tag{24}$$

We now proceed to evaluate (23) for $nl=1S$, and $2P$. As the Bohr radius (r_B) of the K^--α system is 31 F, we may approximate ψ_{1S} and ψ_{2P} by their values of small r

$$\psi_{100}(r)\approx\left(\frac{1}{\pi r_B}\right)^{1/2}\frac{1}{r_B},\ \psi_{210}\approx\left(\frac{1}{2\pi r_B}\right)^{1/2}\frac{\mathbf{r}\cdot\hat{Z}}{4r_B^2}. \tag{25}$$

It is then apparent that in order to evaluate ΔE_{1S} and $\Gamma_{\rm abs}(2P)$ only the integral properties: Re$[\langle U\rangle]$ for the $1S$ state and Im$[\langle Ur^2\rangle]$=Im$[\langle U\rangle\langle r^2\rangle]$ for the $2P$ state are required. As $\langle U\rangle$ is directly expressible as

$$\langle U\rangle=\langle U_0\rangle+3\langle U_1\rangle, \tag{26}$$

it only remains to obtain a rough measure of $\langle r^2\rangle$. It will be adequate for our purposes to take

$$\langle r^2\rangle=\langle r_\alpha^2\rangle+6\lambda^{-2}, \tag{27}$$

as would be approprate if the Yukawa potential with $\lambda=\mu_i$ or ν_i were replaced by a Gaussian potential with the same rms radius. For λ ranging from 800 MeV to ∞, $\langle r^2\rangle$ ranges from 2.3 F^2 to 1.9 F^2. In the arguments below it will be adequate to use the intermediate value 2.2 F^2 corresponding to $\lambda=900$ MeV.

We then find, using Eq. (25) in (23), that

$$\Delta(E+i\Gamma)_{1S}=2.1[\langle U_0\rangle+3\langle U_1\rangle]\ \text{keV} \tag{28a}$$

$$\Delta(E+i\Gamma)_{2P}=5.1\times10^{-5}[\langle U_0\rangle+3\langle U_1\rangle]\ \text{keV} \tag{28b}$$

when the potential volume integrals are expressed in (fermis)^2 as in Fig. 1. Comparing Eqs. (28) to the experimental upper limits in Eqs. (3) and (4), we find the upper bounds

$$|v_0+3v_1|<0.2\ \text{F}^2 \tag{29a}$$

$$w_0+3w_1<1.0\ \text{F}^2, \tag{29b}$$

respectively.[24] These bounds are shown in the figure where it may be seen that the bound on v_0+3v_1 is hopelessly low. An independent bound on w_0+3w_1 may be obtained from the estimated[25] upper bound on the K_α x-ray linewidth of 1.5 keV. This would imply an upper bound,

$$w_0+3w_1<1.4\ \text{F}^2. \tag{29c}$$

In Sec. VII we discuss alternative explanations for this mismatch between theory and experiment.

VII. CONCLUSIONS

If we neglect the possibility of extremely short-range potentials ($\lambda=3$–4 BeV) dominating $\bar{K}$-N interactions, we are left with the following alternative explanations of our model's failure:

1. It is inadequate to represent the $\bar{K}$-N s-wave interaction by a complex potential of reasonable range.

Considerable investigation would be required to calrify this point. In our view, however, the magnitude of the discrepancy in Fig. 1 tends to rule this out as the sole explanation.

2. Many nucleon effects may be important.

Experimentally the (2 nucleon)/(1 nucleon) capture ratio for K^- mesons in helium[26] is 0.2+0.05. The experiment of Burleson *et al.*[1] appears to indicate that this absorption occurs primarily from the $2P$ state. Thus we have evidence that many nucleon interactions do not determine the value of $\Gamma_{\rm abs}(2P)$. There is no equivalent information about the many-nucleon effects on ΔE_{1S}, however.

[22] R. Hofstadter, Rev. Mod. Phys. **28**, 214 (1956).

[23] L. N. Hand, D. G. Miller, and R. Wilson, Rev. Mod. Phys. **35**, 335 (1963).

[24] *Footnote added in proof.* For an attractive potential the bound in Eq. (29a) can be doubled if account is taken of the repulsive effect of an imaginary K^--α potential with a volume integral as large as 1 F^2. We have also considered the possibility that the K^--α scattering length is small (as implied by the small K_α energy shift and width) not because of the weakness of the K^--α interaction, but because it is so strong that the K^--α wave function has more than a half wavelength inside the potential, i.e., there is a K^--α bound state. We have examined this possibility by solving the Schrödinger equation in the K^--α relative coordinate; approximating the K^--α potential by one of Gaussian form with r.m.s. radius 1.5 F. The real potential depth was increased until the magnitude of the real scattering length became less than 0.07 F again for an imaginary scattering length less than 1 F. These conditions were satisfied when the real volume integral fell in the interval $(-18.5, -18.8)$ F^2. These values of the real volume integral are about four times larger than the value which one would estimate on the basis of the results displayed in Fig. 1 and the allowed range is quite small. We have therefore not considered this possibility further.

[25] The measured full width of the K_α line is not observably larger than the experimental resolution of 3 KeV (R. L. Schluter, private communication). We have therefore taken the experimental upper bound on the "natural" linewidth to be 3 KeV.

[26] The Helium Bubble Chamber Collaboration Group, in *Proceedings of the 1960 High Energy Physics Conference at Rochester* (Interscience Publishers, Inc., New York, 1960), p. 426.

3. The $\bar{K}$-N–scattering-length solution is not correct: We believe that this alternative is worthy of consideration. Aside from the poor fit of our model to the K^--α x-ray experiment, we find the order of magnitude difference in imaginary volume integrals in the $I=0$ and 1 states difficult to understand. For such a strong isospin dependence to occur some delicate cancellation would have to occur in the crossed isospin channels.[27]

Independent support for the experimental $I=0$ scattering length may be found in the existence of the $Y_0^*(1405)$ whose mass and width are correctly predicted from A_0 in the Dalitz-Tuan formalism.[20] No such independent support seems to exist for the $I=1$ scattering-length solution, however. An $I=1$ scattering length corresponding to a relatively weak repulsive potential ($v_1 \approx -\frac{1}{3}v_0$) and to an absorptive potential of the order of that found in Sec. V for the $I=0$ state would make possible the agreement of our model with the K^--α x-ray experiments.

[27] See, e.g., the crossing matrices listed for this case by P. A. Carruthers and J. P. Krisch, Ann. Phys. (N.Y.) 33, 1 (1965).

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Note added in proof. After this work was completed, we learned of a related calculation by J. Uretsky (to be published), using square-well potentials rather than Yukawa potentials to fit the $\bar{K}$-N scattering length. We would like to thank Dr. Uretsky for a stimulating discussion of the two calculations.

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Unsubtracted Dispersion Relations in Weak Interactions and Reggeization*

TAKESI SAITO†
Lawrence Radiation Laboratory, University of California, Berkeley, California
(Received 24 January 1966)

In a previous paper it was shown that in field theory there are two possible conditions under which an elementary particle lies on a Regge trajectory. The first is that the proper vertex function vanishes and that the proper vertex poles are not the poles of scattering amplitudes; the second, due to Kaus and Zachariasen, is that the form factor and Z_3 both vanish. In the present paper it is shown that under the second condition, the pology approach (due to Bernstein *et al.*) and the original approach (due to Goldberger and Treiman) of obtaining the Goldberger-Treiman relation both fail. Therefore, this condition may be inadequate as a condition for Reggeization.

I. INTRODUCTION

IN a previous paper[1] we explored the connections between the elementary pion and the Regge pion. We found that if in field theory the proper vertex function with the elementary pion off the mass shell vanishes and if the proper vertex poles are not the poles of scattering amplitudes, then the elementary pion disappears completely but the bootstrapped pion takes its place, lying on the Regge trajectory. However, we also found a different condition for Reggeization, namely, that the form factor $\mathbf{K}(s)$ with the pion off the mass shell and Z_3 (the wave-function renormalization constant of the pion) should both vanish:

$$\mathbf{K}(s) \to 0, \quad \text{and} \quad Z_3 \to 0. \tag{1.1}$$

(When bound-state poles exist, the vanishing of the form factor does not always mean the vanishing of the coupling constant,[2] as was shown in a previous paper.) This latter condition (1.1) is essentially the same as that derived by Kaus and Zachariasen[3] (see Sec. II in their paper).

In this paper we apply the condition (1.1) to the π-μ decay process. It is then shown that the pology approach, due to Bernstein *et al.*,[4] and the original approach, due to Goldberger and Treiman,[5] of obtaining the Goldberger-Treiman relation both fail. These approaches are essentially based on the assumption that the divergence of the axial-vector current is a highly convergent operator whose matrix elements satisfy unsubtracted dispersion relations in the momentum-transfer squared. The validity of this assumption, known as that of "partially conserved axial-vector current," has been established by experiments.[4–6] There-

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† Formerly Takesi Ogimoto. On leave of absence from Department of Physics, Osaka University, Osaka, Japan.

[1] T. Saito, Phys. Rev. 145, 1302 (1966).

[2] The argument will be repeated later on. See footnote 11.

[3] P. E. Kaus and F. Zachariasen, Phys. Rev. 138, B1304 (1965).

[4] J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento 17, 757 (1960).

[5] M. L. Goldberger and S. B. Treiman, Phys. Rev. 110, 1178 (1958).

[6] For example, S. L. Adler, Phys. Rev. Letters 14, 1051 (1965); W. I. Weisberger, *ibid.* 14, 1047 (1965).