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One-Particle-Exchange Model for Bootstrap

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A Cini-Fubini–like representation is assumed to describe low-energy meson-meson interaction. The spectral densities are determined by iteration from nonlinear integral equations given by unitarity. For bootstrap calculations, Breit-Wigner–type functions are proposed as a zero-order approximation. An approximate determination of the resonance parameters can be achieved through a best-fit condition between the zero and first-order approximations in the resonance region. Such calculations take into account crossing symmetry exactly but unitarity approximately. An application is given for the ρ-meson bootstrap and a symmetric-shaped amplitude is obtained with $m_\rho = 450$ MeV and $\Gamma_\rho = 40$ MeV.

I. INTRODUCTION

It has been realized that different approximations in low-energy meson-meson interactions lead to very different quantitative results. The nonuniqueness of the solutions of equations, and the use of intuitive arguments in order to retain some solutions (for which crude approximations are considered) and to reject other solutions, raise the question whether we are not practically dealing with different *mathematical models* for the same physical system.

To find the scattering amplitude, one employs the following general principles:

1. relativistic invariance,
2. analyticity,
3. unitarity,
4. crossing symmetry.

Usually, the first three principles are taken into account exactly, the last one only approximately. In this paper, another approach is proposed, in which approximations are made only in unitarity.

In Sec. II the present situation in low-energy π-π scattering is examined and the different approximations are compared. In Sec. III a crossing-symmetric Cini-Fubini-like representation [which in fact is a modified one-particle exchange (OPE) model] is postulated for the low-energy scattering amplitude. Such an amplitude is made plausible by a study of the low-energy elastic scattering of identical scalar mesons. The single spectral densities involved in the representation are determined by the unitarity condition, which leads to nonlinear integral equations. To solve these equations, an iteration program is proposed, which starts with Breit-Wigner-type functions corresponding to different partial-wave resonances. The three parameters of the Breit-Wigner functions are determined by bootstrap-like conditions.

In Sec. IV, the proposed method is applied to the ρ-meson bootstrap. Confining ourselves to the first iteration, we obtain an almost symmetric shape for the partial-wave amplitude with $m_\rho = 450$ MeV and $\Gamma_\rho = 40$ MeV.

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II. THE PRESENT SITUATION IN LOW-ENERGY π-π SCATTERING

The best studied example of meson-meson interaction is the π-π case. We shall consider this case in more detail to illustrate the actual situation. Two main directions can be discerned in this field:

(1) *A strictly low-energy point of view*, i.e. computation of low-angular-momentum phase shifts and diffusion lengths, the necessity for a resonance in the $I=1$, $J=1$ partial wave being a guide in the choice of the adequate solution. In this kind of calculation, the role of the s wave in the crossed channel is considered to be essential. The approximation methods used are, respectively:

(a) the ND^{-1} method,[1–3]
(b) the inverse-amplitude method,[4–7]
(c) the differential approximation.[8–10]

We shall not enter here into the description or criticism of these methods.[9] We should like to observe, however, that the solutions which depend on at least one parameter (the coupling constant λ) are not incompatible with the rather poor experimental data on phase shifts.[7,11] As to the ρ resonance, the results are rather contradictory. In some works the resonance emerges as a consequence of the arbitrary parameter choice,[7] with $m_\rho = 500$ MeV, $\Gamma_\rho = 70$ MeV, or "is installed in the right place by hand,[10]" its width being $\Gamma_\rho = 40$ MeV. In addition, the differential approximation leads to the conclusion that the physical mecha-

[1] G. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[2] G. Chew, S. Mandelstam, and P. Noyes, Phys. Rev. **119**, 478 (1960).
[3] J. S. Ball and D. Y. Wong, Phys. Rev. Letters **7**, 390 (1961).
[4] J. W. Moffat, Phys. Rev. **121**, 926 (1961).
[5] B. H. Bransen and J. W. Moffat, Nuovo Cimento **21**, 505 (1961); Phys. Rev. Letters **6**, 708 (1961).
[6] B. H. Bransen and J. W. Moffat, Phys. Letters **8**, 145 (1962).
[7] Kyungsik Kang, Phys. Rev. **134**, B1324 (1964); **139**, B126 (1965).
[8] A. V. Efremov, V. A. Mescheryakov, D. V. Shirkov, and H. Y. Tzu, Nucl. Phys. **22**, 202 (1961).
[9] V. V. Serebryakov and D. V. Shirkov, Fortschr. Physik **13**, 227 (1965).
[10] D. V. Shirkov, International Center for Theoretical Physics, Report No. 2/36, 1965 (unpublished).
[11] J. Hamilton, P. Menoti, G. C. Oades, and L. J. Vick, Phys. Rev. **128**, 1881 (1962); H. J. Schnitzer, *ibid.* **125**, 1058 (1962).

nism of the ρ resonance is not a self-sustaining one, but requires a strong attraction in the crossed channel with $I=0$. The problem of the existence of the scalar σ dipion resonance, which recently has received some attention,[12,13] is still an open one,[14] and the same is true of the ABC anomaly.[15,16]

(2) *The bootstrap-type calculations* neglect completely the s wave in the crossed channels, and their main purpose is to obtain a resonant-type amplitude for the $I=1$, $J=1$ partial wave with correct values for the ρ-meson mass and width. The approximation methods involved are:

(a) the ND^{-1} method,[17,18]
(b) the determinantal method,[19]
(c) the fixed-angle method,[20]
(d) the VG^{-1} method.[21]

By all these methods, strongly asymmetric shapes for the resonance amplitude are obtained, in sharp contradiction with experiment. Moreover, the width of the ρ resonance is very large (600 MeV in the determinantal method) as compared with the experimental value $\Gamma_\rho^{\mathrm{exp}}=124$ MeV. A rather singular result has been obtained in an approximate version of the inverse amplitude,[22] where a small width $\Gamma_\rho=45$ MeV has been obtained. We shall return later to the discussion on the resonance position.

Let us discuss the approximation methods considered above, confining ourselves to the first two, which have been better studied.

The ND^{-1} and the determinantal approximations with various inputs in the left-hand cut (LHC) have been compared in nonrelativistic potential scattering with the true solution of the Schrödinger equation.[23,24] Taking the Born approximation to compute the L.H.C. (i.e., doing what we do in bootstrap calculations), the two approximations and the true solution do not generally agree, nor do they disagree in a systematic way. In addition, it may happen that in the ND^{-1} approximation the output solution is quite near the Born input, giving rise to a bootstrap *sui generis*, both values being far from the correct one.

In the relativistic case, besides the problem of uniqueness of the ND^{-1} equation for a given input,[25] we are not at all sure that in doing an iterative calculation (i.e., reintroducing the output solution as an input, and so forth) the final solution does not depend on the initial input. On the other hand, it is interesting to mention that an attempt to improve the input forces, taking into account the second order Born approximation in the L.H.C., may not give bootstrapping solutions at all.[26]

We also must stress that the aim of a bootstrap philosophy is to build a parameter-free theory. However, in the ND^{-1} method with an approximate input, the solutions are sensitive to the subtraction point in the dispersion relation for D. Also, in order to get the proper threshold behavior of the output amplitude, a cutoff parameter is necessary to assure the convergence of the integrals. The value of the cutoff parameter is determined by the condition of obtaining the correct mass of the resonance. Indeed, one can argue that such a parameter may represent, in some sense, the effect of the high-energy region, otherwise neglected. It is, however, difficult to think what physical interpretation might be given to the cutoff parameter, when it varies by an order of magnitude with small changes of the input width.[17] Such arguments have driven some people to state that the ND^{-1} approach is not an adequate mathematical device for the bootstrap concept.[27]

To obtain a better agreement with experiment, efforts have been made to consider the effect of closed inelastic channels,[28] or to solve the one-channel problem with inelastic unitarity.[29] Some improvement has indeed been obtained, the theoretical value for the resonance width being 500 MeV, or 250 MeV with a smaller asymmetry for the amplitude. One may however observe, that such a strong effect of the closed inelastic channels (especially the $\pi\pi$-$\pi\omega$ one) may be a consequence of the systematically overestimated width and of the fact that the position of the resonance is fixed with the aid of the cutoff parameter in a region where large inelastic effects are to be expected. The effect of inelasticity may be quite different, and possibly in stronger contradiction with experiment, for "strictly low-energy solutions" considered under point (1) where smaller masses and widths are involved.

Ending these considerations, we observe that the possibility of neglecting the s and d wave in computing the parameters of the ρ resonance may be a result of the determinantal approximation, so that in a better approximation their role could become essential.

With a view to getting some idea of what is really fundamental in the results mentioned above, and seeing that they do not represent merely the properties of the

[12] L. M. Brown and P. Singer, Phys. Rev. **133**, B812 (1964).
[13] S. H. Patil, Phys. Rev. Letters **13**, 261 (1964); L. Durand III and Y. T. Chiu, *ibid.* **14**, 329 (1965).
[14] J. Fischer, International Center for Theoretical Physics, Report No. 2/25 1965 (unpublished).
[15] N. E. Booth and A. Abashian, Phys. Rev. **132**, 2314 (1963).
[16] D. Atkinson, Phys. Letters **9**, 69 (1964). S. Ciulli and G. Ghika, *ibid.* **11**, 336 (1964).
[17] M. Bander and G. L. Shaw, Ann. Phys. (N. Y.) **31**, 506 (1964).
[18] A. Morel, Nuovo Cimento **38**, 1649 (1965).
[19] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).
[20] H. Burkhardt and G. P. McCauley, Nuovo Cimento **38**, 872 (1965).
[21] J. Franklin, University of California Radiation Laboratory Report No. UCRL-12482-T, 1965 (unpublished).
[22] G. Q. Hassoun and K. Kang, Phys. Rev. **137**, B955 (1965).
[23] M. Luming, Phys. Rev. **136**, B1120 (1964).
[24] P. B. Kantor, Ann. Phys. (N. Y.) **33**, 196 (1965).
[25] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).
[26] J. L. Gervais, Nuovo Cimento **34**, 1347 (1964).
[27] P. S. Isaev, International Center for Theoretical Physics Report No. 65/62, 1965 (unpublished).
[28] J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. **137**, B1242 (1965).
[29] P. W. Coulter and G. L. Shaw, Phys. Rev. **138**, 1273 (1965).

different approximations, we shall examine another type of calculation in which relativity, analyticity, and crossing are taking into account correctly, and unitarity only approximately. In this direction, some efforts have already been made.[30]

To find such a model, we still had in mind the idea that low-energy physics is strictly governed by exchange of particles, resonances, and perhaps antibound states and the belief that this idea must be included in the model itself. It should be mentioned that an analogous point of view is adopted in the new strip approximation, where one assumes that Regge trajectories determine the scattering amplitude for not too high-momentum transfers,[31] and writes down equations for the Regge trajectories.

To achieve such a program, we postulate a low-energy expression for the amplitude, which is crossing-symmetric and possesses the proper analytical properties, but which implies only spectral densities of a single variable. With the aid of unitarity we obtain nonlinear integral equations for spectral densities. These equations are then solved by iteration. We believe essentially that the iteration must start with Breit-Wigner-type functions, which, even corrected by unitarity, will perhaps maintain a symmetric form near the resonance position. It is indeed possible that if one began the iteration with some other functions, owing to the nonlinearity of the equations, other solutions would be possible.

III. USE OF CINI-FUBINI REPRESENTATION FOR A BOOTSTRAP PROGRAM

In order to obtain an approximate expression for the scattering amplitude with the desired properties of crossing and analyticity, we shall use a Cini-Fubini representation.[32] Assuming that for low energies the scattering process take place through the exchange of particles, resonances, and perhaps antibound states, the expression for the scattering amplitude is obtained by writing down only exchange Feynman graphs, including all the resonances we know from experience to exist. The rules for calculating the exchange of unstable particles are analogous to those for stable-particle exchange, the only difference being that the pole term $1/(m^2-s-i\epsilon)$ must be replaced by $\int[\rho(s')/(s'-s-i\epsilon)]ds'$. The spectral density $\rho(s')$ must be determined by the general principle not used until now (i.e., unitarity), and is assumed to have a Breit-Wigner shape near the resonance positions.

We shall develop the method for the simplest case of elastic scattering of identical neutral scalar mesons of unit mass, and shall return to the π-π interaction in the following section. Of course, the method can be usefully extended to the general two-particle interaction, including spin and internal symmetries.

One can easily see that the exchange of $l=0, 2,\cdots$ spin resonances gives the following expression for the invariant amplitude (for simplicity, we shall not consider the exchange of stable particles):

$$A(s,t,u)=\sum_l \nu_s{}^l P_l(z_s)\frac{1}{\pi}\int_4^\infty \frac{\rho_l(s')}{s'-s}ds' +\sum_l \nu_t{}^l P_l(z_t)\frac{1}{\pi}\int_4^\infty \frac{\rho_l(t')}{t'-t}dt' +\sum_l \nu_u{}^l P_l(z_u)\frac{1}{\pi}\int_4^\infty \frac{\rho_l(u')}{u'-u}du', \quad (1)$$

where s, t, u are the usual Mandelstam variables, and z_i and ν_i are, respectively, the cosine of the c.m. scattering angle and the c.m. squared momentum in the i channel [for instance $s=4(\nu_s+1)$; $z_s=1+t/2\nu_s=-(1+u/2\nu_s)$].

We must stress that the low-energy amplitude (1) postulated here, may in fact be justified for very low energy[33] by an argument which for $l=0$ coincides with that of Cini and Fubini.[32] From the fact that the double spectral functions have the property:

$$\rho(x,y)=0 \quad \text{if} \quad 4\leq x,\ y\leq 16,$$

the Mandelstam representation for the amplitude may be written as

$$A(s,t,u)=\alpha_s(s,t,u)+\alpha_t(s,t,u)+\alpha_u(s,t,u)\,, \quad (2)$$

where

$$\alpha_s(s,t,u)=\frac{1}{\pi^2}\int_4^\infty \frac{dx}{x-s}\int_{16}^\infty dy\,\tilde{\rho}(x,y)\left(\frac{1}{y-t}+\frac{1}{y-u}\right). \quad (3)$$

Using the Heine expansion in Legendre polynomials, we have:

$$\frac{1}{y-t}=\frac{1}{2\nu_s}\sum_l P_l(z_s)Q_l\left(1+\frac{y}{2\nu_s}\right) \quad (4)$$

which is valid for $|z_s|\leq|1+y/2\nu_s|$. Taking into account that $y\geqslant 16$, one can see that the range of validity of the expansion (4) is $u, t\leq 16$ or $u, t\geqslant 16$. An analogous expansion may be used for $1/(y-u)$. Then for

$$|s|,|t|,|u|\leq 16 \quad (5)$$

the expression (3) for $\alpha_s(s,t,u)$ may be written as

$$\alpha_s(s,t,u)=\frac{1}{2\pi^2\nu_s}\sum_l[1+(-1)^l]P_l(z_s) \times\int_4^\infty \frac{dx}{x-s}\int_{16}^\infty dy\tilde{\rho}(x,y)Q_l\left(1+\frac{y}{2\nu_s}\right) \quad (6)$$

[30] J. Fischer, Czech. Academy of Sciences, 1964 (unpublished).

[31] G. F. Chew, Phys. Rev. **129**, 2363 (1963); G. F. Chew and C. E. Jones, *ibid.* **135**, B208 (1964).

[32] M. Cini and S. Fubini, Ann Phys. (N. Y.) **10**, 352 (1960); D. Amati and S. Fubini, Ann. Rev. Nucl. Sci. **12**, 359 (1962).

[33] By "very low energy" we mean the elastic region, while by "low energy" we mean the range of energies where the scattering takes place through the exchange of resonances.

and we have analogous expansions for $\alpha_t(s,t,u)$ and $\alpha_u(s,t,u)$.

We shall now assume that the spectral density $\tilde{\rho}(x,y)$ is such that the convergence domain of the expansion (6) is larger than the elastic region (5). We shall not make any concrete assumption about the domain of convergence, but we expect it to be valid only in the resonance region. In other words, we assume that in the integral (3) regions with $y\gg 16$ are important.

We express the Legendre function of the second kind by the hypergeometric function

$$Q_l(1/z)=[\pi^{1/2}(l+1)!/\Gamma(l+\tfrac{3}{2})](z/2)^{l+1}\times F(\tfrac{1}{2}l+1,\tfrac{1}{2}l+\tfrac{1}{2},l+\tfrac{3}{2},z^2). \quad (7)$$

For

$$z^{-2}=[1+2y/(s-4)]^2\gg(l+1)(l+2)/2(2l+3), \quad (8)$$

which is fulfilled in the range (5) for not too high angular momentum, one can retain the first term from the expansion in z^2 of the hypergeometric function and obtain

$$Q_l(1+y/2\nu_s)\approx[\pi^{1/2}(l+1)!/\Gamma(l+\tfrac{3}{2})]\times(\nu_s/y)^{l+1}(1+2\nu_s/y)^{-l-1}. \quad (9)$$

Assuming now that

$$(s-4)/2y\ll(l+1)^{-1} \quad (10)$$

we get from (9) and (6):

$$Q_l(1+y/2\nu_s)\approx[\pi^{1/2}(l+1)!/\Gamma(l+\tfrac{3}{2})](\nu_s/y)^{l+1}, \quad (11)$$

$$\alpha_s(s,t,u)=\frac{1}{2\pi^{3/2}}\sum_l[1+(-1)^l]\frac{(l+1)!}{\Gamma(l+\tfrac{3}{2})}\times\nu_s{}^lP_l(z_s)\int_4^\infty\frac{dx}{x-s}\int_{16}^\infty dy\frac{\tilde{\rho}(x,y)}{y^{l+1}}, \quad (12)$$

and analogous expansions for $\alpha_t(s,t,u)$ and $\alpha_u(s,t,u)$. Denoting

$$\rho_l(x)=\frac{1}{(2\pi)^{1/2}}\frac{(l+1)!}{\Gamma(l+\tfrac{3}{2})}[1+(-1)^l]\int_4^\infty dy\frac{\tilde{\rho}(x,y)}{y^{l+1}} \quad (13)$$

and using the expressions (2) and (12), we find the postulated amplitude (1). From the above considerations we expect that the amplitude (1) is valid in the resonance region where not too high angular momenta are involved.

For the determination of the spectral densities $\rho_l(x)$ we shall use the partial-wave unitarity in one of the channels. As usual, we shall consider the s channel:

$$\mathrm{Im}A_l(s)=\frac{1}{16\pi}[\nu_s/(\nu_s+1)]^{1/2}|A_l(s)|^2. \quad (14)$$

From the expression (1) we have

$$\mathrm{Im}A_l(s)=\nu_s{}^l\rho_l(s),$$
$$\mathrm{Re}A_l(s)=\nu_s{}^l\left(\frac{1}{\pi}P\int_4^\infty\frac{\rho_l(s')}{s'-s}ds'+\sum_{l'}\int_4^\infty K_{ll'}(s,x)\rho_{l'}(x)dx\right), \quad (15)$$

where

$$K_{ll'}(s,x)=\frac{1}{\pi\nu_s{}^l}\int_{-1}^1\frac{P_l(z_s)P_{l'}((4\nu_s+4)/\nu_s(1-z_s))[1+\tfrac{1}{2}\nu_s(1-z_s)]^{l'}}{x+2\nu_s(1-z_s)}dz_s. \quad (16)$$

We can easily see that

$$\lim_{\nu_s\to 0}K_{ll'}(s,x)=\mathrm{const}. \quad (17)$$

The condition of unitarity gives then the following system of nonlinear integral equations for $\rho_l(s)$:

$$\rho_l(s)=\frac{\nu_s{}^l}{16\pi}\left(\frac{\nu_s}{\nu_s+1}\right)^{1/2}\left\{\rho_l{}^2(s)+\left[\frac{1}{\pi}P\int_4^\infty\frac{\rho_l(s')}{s'-s}ds'+\sum_{l'}\int_4^\infty K_{ll'}(s,x)\rho_{l'}(x)dx\right]^2\right\}. \quad (18)$$

Let us assume that we know the solution of the system (18) and consider some properties of this solution. Taking into account the relation (17) for $\nu_s\to 0$, we have

$$\rho_l(s)\approx(1/16\pi)\nu_s{}^{l+1/2}[\rho_l{}^2(s)+c_l{}^2], \quad (19)$$

where c_l has the form

$$c_l=\frac{1}{\pi}P\int_4^\infty\frac{\rho_l(s')}{s'}ds'+\sum_{l'}\int_4^\infty K_{ll'}(\nu_s=0,\,x)\rho_{l'}(x)dx. \quad (20)$$

The relation (19) is satisfied by

$$\rho_l(s)\approx(c_l{}^2/16\pi)\nu_s{}^{l+1/2}. \quad (21)$$

From the relations (15) we then obtain

$$\mathrm{Im}A_l(s)\simeq(c_l{}^2/16\pi)\nu_s{}^{2l+1/2},$$
$$\mathrm{Re}A_l(s)\simeq c_l\nu_s{}^l, \quad (22)$$

giving the correct threshold behavior.

On the other hand, considering the limit for $\nu\to\infty$ in Eq. (18), we can see that for $l>0$ we have no solutions. We have come to the known result that the peripheral model is not compatible with unitarity for high energies.[34] In fact, for high energies, the approximation (11) is surely wrong and the amplitude (1) is not valid. Thus, the integral Eq. (18) cannot give $\rho_l^{(s)}$ for high energies, where the ρ_l must be given empirically or a cutoff in the integrals must be made.

To solve Eq. (18), an iteration program must be constructed. The zero-order approximation is chosen to assure a pole in the second Riemann sheet. Two alternatives appear to be simplest:

$$\rho_l^{(0)}(s)=g_l/[(s-m_l^2)^2+\gamma_l^2]\,, \tag{23a}$$

$$\rho_l^{(0)}(s)=\frac{\tilde{g}_l[(s-4)^{2l+1}/s]^{1/2}}{(s-m_l^2)^2+\tilde{\gamma}_l^2[(s-4)^{2l+1}/s]}\,. \tag{23b}$$

The starting functions are to be introduced in the right-hand side of Eq. (18) to compute $\rho_l^{(1)}(s)$ and so on. Since we are interested in obtaining a correct solution especially in the resonance region, we determine the free parameters g_l, γ_l, and m_l from the conditions

$$\mathrm{Re}A_l^{(0)}(\mu_l^2)=0 \tag{24}$$

[where $\mathrm{Re}A_l^{(0)}$ is obtained by inserting $\rho_l^{(0)}$ in (15)],

$$\rho_l^{(1)}(\mu_l^2)=\rho_l^{(0)}(\mu_l^2)\,, \tag{25}$$

and

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho_l^{(1)}(s)}\right)_{s=\mu_l^2}=\frac{d^2}{ds^2}\left(\frac{1}{\rho_l^{(0)}(s)}\right)_{s=\mu_l^2}, \tag{26}$$

where μ_l^2 is the solution of the equation

$$\frac{d}{ds}\left[\left(\frac{\nu_s+1}{\nu_s}\right)^{1/2}\mathrm{Im}A_l^{(0)}(s)\right]=0\,. \tag{27}$$

For narrow widths, we have, of course, $\mu_l^2\approx m_l^2$. These conditions correspond to the usual bootstrap conditions, i.e., equality of the input and output masses, coupling constants, and widths. We mention that, instead of the conditions (26), we could consider

$$\rho_l^{(1)}(\mu_l^2+\gamma_l)=\rho_l^{(0)}(\mu_l^2\oplus\gamma_l) \tag{26'}$$

or

$$\rho_l^{(1)}(\mu_l^2-\gamma_l)=\rho_l^{(0)}(\mu_l^2-\gamma_l)\,, \tag{26''}$$

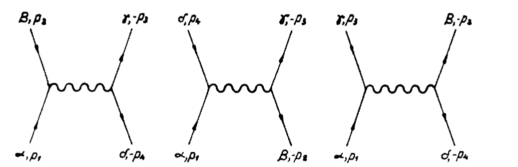

FIG. 1. ρ-meson exchange diagrams for the process $\pi\pi\to\pi\pi$.

which, with the idea that the solution is symmetric, would give almost the same result as (26).

We may also observe that if we confine ourselves to the first iteration, as in usual bootstrap calculations, the integrals in (18) are convergent and no cutoff parameters are necessary. At the same time we mention that for any starting function which vanishes at $\nu_s=0$, the function $\rho_l^{(1)}$ has the correct threshold behavior [if $\rho_l^{(0)}(0)\neq 0$, this assertion is valid for $\rho_l^{(2)}$].

The eventual advantage of the input (23b), for which $\rho_l^{(0)}(0)=0$, is that we can expect that on fixing the parameters in the resonance region, $\rho_l^{(1)}$ will not be too far from $\rho_l^{(0)}$, even at the threshold.

To illustrate the method proposed in this section, we shall consider the problem of the ρ meson.

IV. APPLICATION TO THE ρ-MESON BOOTSTRAP

We now consider the application of the present method to the simple one-channel example of a self-consistent ρ-meson bootstrap in the p wave of the π-π system.

Following Chew and Mandelstam, we express the elastic scattering amplitude in terms of invariant amplitudes:

$$\langle p_1,\alpha;\,p_2,\beta|T|-p_3,\gamma;\,-p_4,\delta\rangle=\delta_{\alpha,\beta}\delta_{\gamma,\delta}A(s,t,u)+\delta_{\alpha,\gamma}\delta_{\beta,\delta}B(s,t,u)+\delta_{\alpha,\delta}\delta_{\beta,\gamma}C(s,t,u)\,. \tag{28}$$

Considering the exchange diagrams of Fig. 1 and using the Sec. III recipe, i.e., replacing pole terms by spectral representations:

$$\frac{1}{m^2-x-i\epsilon}\to\int_4^\infty dx'\frac{\rho(x')}{x'-x-i\epsilon}$$

we obtain

$$\begin{aligned}
A(s,t,u)&=(s-u)\int_4^\infty dt'\frac{\rho(t')}{t'-t-i\epsilon}+(s-t)\int_4^\infty du'\frac{\rho(u')}{u'-u-i\epsilon}\,,\\
B(s,t,u)&=(t-s)\int_4^\infty du'\frac{\rho(u')}{u'-u-i\epsilon}+(t-u)\int_4^\infty ds'\frac{\rho(s')}{s'-s-i\epsilon}\,,\\
C(s,t,u)&=(u-s)\int_4^\infty dt'\frac{\rho(t')}{t'-t-i\epsilon}+(u-t)\int_4^\infty ds'\frac{\rho(s')}{s'-s-i\epsilon}\,.
\end{aligned} \tag{29}$$

[34] F. Salzman and G. Salzman, Phys. Rev. **125**, 1703 (1962).

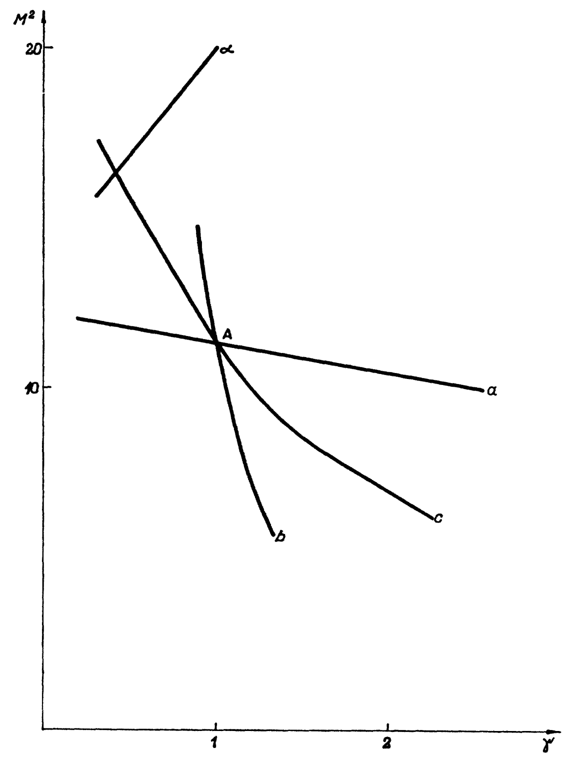

FIG. 2. Solutions of the equations: (a) $\mathrm{Re}A_1{}^{1(0)}(\mu^2)=0$; (b) $(d^2/ds^2)(1/\rho^{(1)})_{s=\mu^2}=(d^2/ds^2)(1/\rho^{(0)})_{s=\mu^2}$; (c) $\rho^{(1)}(\mu^2+\gamma)=\rho^{(0)}(\mu^2+\gamma)$; and (d) $\rho^{(1)}(\mu^2-\gamma)=\rho^{(0)}(\mu^2-\gamma)$. The point A corresponds to $m_\rho=450$ MeV, $\Gamma_\rho=40$ MeV.

We then compute the partial wave with $J=1$, $I=1$ corresponding to the ρ quantum numbers:

$$A_1{}^1(\nu_s)=\frac{1}{2}\int_{-1}^{1}dz_s z_s[B(\nu_s,z_s)-C(\nu_s,z_s)]. \quad (30)$$

Introducing the expressions (29) into (30) we obtain:

$$\mathrm{Im}A_1{}^1(\nu_s)=\frac{8\pi}{3}\nu_s\rho(4+4\nu_s),$$
$$\mathrm{Re}A_1{}^1(\nu_s)=\int_4^\infty dx\rho(x)L(x,\nu_s), \quad (31)$$

where

$$L(x,\nu_s)=(4+8\nu_s+x)\nu_s{}^{-1}\times\left[\frac{x+2\nu_s}{4\nu_s}\ln\left(1+\frac{4\nu_s}{x}\right)-1\right]+\frac{8\nu_s}{3}P\frac{1}{x-4-4\nu_s}. \quad (32)$$

Using the elastic-unitarity condition:

$$\mathrm{Im}A_1{}^1(\nu_s)=(1/16\pi)[\nu_s/(\nu_s+1)]^{1/2}|A_1{}^1(\nu_s)|^2 \quad (33)$$

and denoting

$$\bar{\rho}(\nu_s)=\rho(4+4\nu_s);\quad \bar{L}(\nu',\nu_s)=L(4+4\nu',\nu_s), \quad (34)$$

we get the nonlinear integral equation:

$$\bar{\rho}(\nu_s)=\frac{\nu_s}{6}\left(\frac{\nu_s}{\nu_s+1}\right)^{1/2}\bar{\rho}^2(\nu_s)+\frac{3}{8\pi^2}\left(\frac{\nu_s}{\nu_s+1}\right)^{1/2}\left[\int_0^\infty d\nu'\bar{\rho}(\nu')\bar{L}(\nu',\nu_s)\right]^2. \quad (35)$$

To solve Eq. (35) by iteration, we have used the input (23a). The equation (25) gives the known relation between the coupling constant and the width,[19] while Eqs. (24) and (26), which are solved numerically, determine the values of the ρ-meson mass m_ρ and width $\Gamma_\rho=\gamma_\rho/m_\rho$. For completeness we have also tried replacing Eq. (26) with Eqs. (26′) and (26″); the results are shown in Fig. 2. The usual bootstrap equations (24), (26) have the following solution: $m_\rho=450$ MeV, $\Gamma_\rho=40$ MeV. Equation (26′) gives the same result, while the curve representing (26″) is near the solution point, so that a near-symmetric solution for $\bar{\rho}^{(1)}$ is to be expected. The input $\bar{\rho}^{(0)}(\nu_s)$ and the output $\bar{\rho}^{(1)}(\nu_s)$ solutions are shown in Fig. 3, while the modulus of the partial-wave amplitude and $\mathrm{Re}A_1{}^{1(0)}$ are shown in Fig. 4.

We can see that the amplitude has the required property of symmetry near the resonance. The values

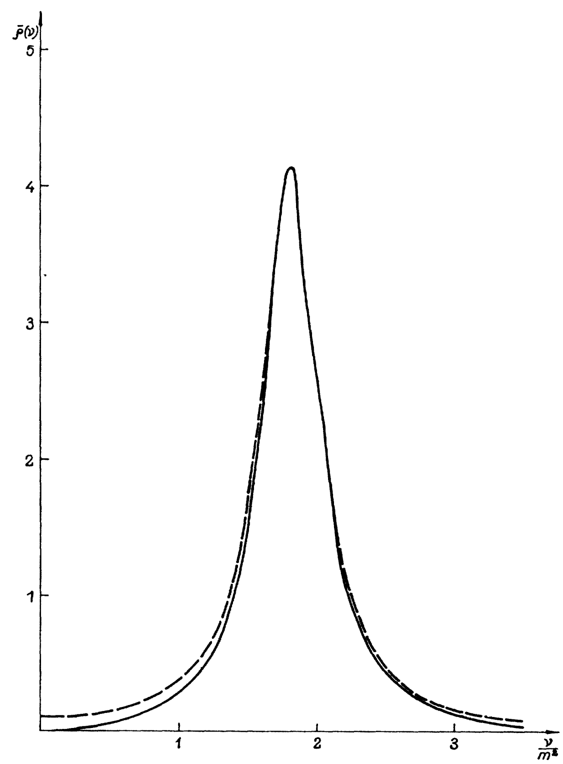

FIG. 3. The zeroth- and the first-order approximation for the spectral density ($m_\rho=450$ MeV, $\Gamma_\rho=40$ MeV). The solid curve represents $\rho^{-(1)}(\nu)$; the dashed curve, $\rho^{-(0)}(\nu)$.

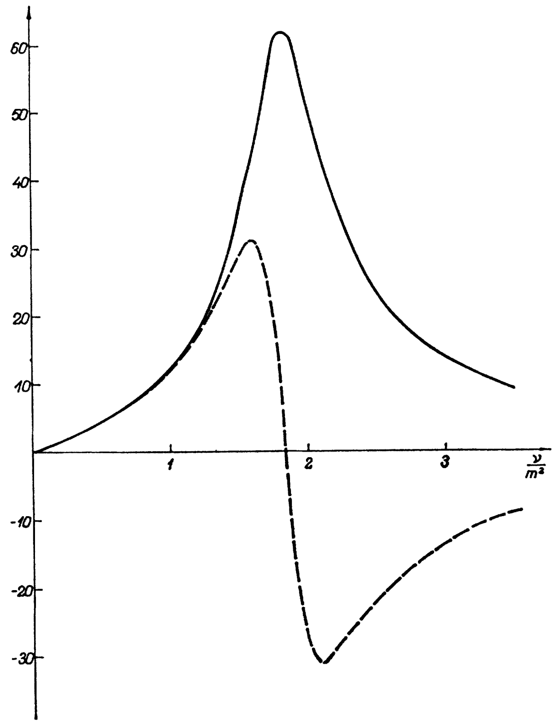

FIG. 4. The zeroth-order solution for the modulus and the real part of the partial-wave amplitude. The solid curve represents $|A_1^{1(0)}(\nu)|$; the dashed curve, $\mathrm{Re}A_1^{1(0)}(\nu)$.

of the mass and of the width are near those of Kang[22] ($m_\rho=350$ MeV, $\Gamma_\rho=45$ MeV) and not very far from those of Balázs[35] ($m_\rho=560$ MeV, $\Gamma_\rho=126$ MeV). Now if we are optimistic, we can assume that, if such different methods of approximation give about the same results, they must express something about physics, but we can be pessimistic and think that it is a numerical accident. Our hope is that the latter is not the case.

[35] L. Balázs, Phys. Rev. 134, B1315 (1964).

V. CONCLUDING REMARKS

We have assumed a Cini-Fubini–like representation for the low-energy scattering amplitudes, implying single-variable spectral densities. These spectral densities must be determined by iteration from nonlinear integral equations given by unitarity. In these calculations analyticity and crossing symmetry are treated exactly, while unitarity is treated only approximately through an iteration program. The iteration starts with Breit-Wigner expressions which assure the convergence of the integrals in computing the first iteration, giving use to a *sui generis* cutoff. In computing the parameters of the resonances or studying different models, we can probably confine ourselves to this first iteration, as in usual bootstrap calculations. Further iterations would make it necessary to consider a cutoff procedure more seriously.

Applied to a crude model of the ρ resonance, a symmetric form for the amplitude was obtained with $m_\rho=450$ MeV, $\Gamma_\rho=40$ MeV. Consequently, we can conjecture that if the present calculations have something to do with reality, and if small widths are obtained, an even more approximate one-particle-exchange form for the amplitude can be taken into consideration: the sum of the exchange terms with complex propagators whose parameters are to be determined from the partial-wave unitarity in the resonance region through equations of the type (25)–(27).

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