Perturbation Methods in Dispersion Theory*

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A complete perturbation method for N/D calculations in potential scattering is presented. This method agrees with the Dashen-Frautschi method in first order. It is shown that in cases where resonant states are shifted by the perturbation into bound states the effect is not of first order and the Dashen-Frautschi method is therefore inadequate. Some possible utilizations of the perturbation method are outlined.

I. INTRODUCTION

DISPERSION theory calculations have given many successful results in strong interaction physics. Recently the works of Dashen and Frautschi have opened a new field of applicability of dispersion theory by introducing a first-order perturbation method to N/D calculations.[1] This method, hereafter referred to as the D.F. method, was applied to electromagnetic corrections to strong interactions, and to the theory of SU_3 symmetry breaking.[2] There are, however, drawbacks in all first-order perturbation calculations. Unless a complete perturbation expansion is available one has no way of estimating the importance of higher order terms. The convergence of a perturbation expansion can also be investigated when a complete perturbation series is available.

In this paper we will exhibit a perturbation series for N/D calculations in potential scattering. Our method is almost identical to that of Blankenbecler and Goldberger.[3] This method coincides to first order with that of Dashen and Frautschi. The importance of higher order terms will be investigated. We expand the difference between the unperturbed and perturbed partial-wave scattering amplitude as a ratio of two power series in λ, the strength of the perturbation.[4] This type of expansion will be shown to have a definite advantage over the simple power expansion which does not converge in most of the interesting cases. The reason for this is that whenever the unperturbed amplitude has a bound-state pole, which is shifted slightly by a perturbation, the change in the amplitude near the bound state is infinite. High-order terms will be shown to play a crucial role in the physically important case when the unperturbed solution has a bound state or a resonance close to the unitarity threshold. In this case even a small perturbation can change the resonance into a bound state or vice versa. This effect cannot be reproduced correctly by the D. F. method, since it is not a first-order effect in the expansion of the scattering amplitude. In our method the change of a resonance into a bound state stems from the first-order expansion of a denominator in λ. The change in energy of the bound state can therefore be of first order in λ. Physically, such a case occurs in the SU_3 breaking calculation of the decuplet, where the N^* is above threshold in the $N\pi$ channel and below threshold in the ΣK channel. Section II outlines the perturbation method. An N/D type equation is derived for the perturbation in the partial-wave scattering amplitude. In Sec. III the perturbation solution in the case where the unperturbed problem has a bound state is investigated. A Castillejo-Dalitz-Dyson (CDD) pole is shown to be necessary and an approximation based on this pole is outlined. The case where threshold effects become important is discussed in Sec. IV, a simple effective-range example is discussed as an illustration. It is shown that the first-order D.F. approximation cannot reproduce physical results when a resonance is shifted into a bound state and vice versa. In Sec. V some applications of the perturbation method are discussed. The perturbation method is employed to improve standard approximation methods used in N/D calculations.

II. THE PERTURBATION METHOD

In this section we will consider the perturbation of a partial-wave scattering amplitude caused by a change in the left-hand-cut discontinuity. We will not consider the possible changes in the right-hand phase space factor. This restricts our method to potential scattering and some relativistic cases. We assume that we have solved the $a_0=N_0/D_0$ equation for the unperturbed problem.

$$\mathrm{Im}N_0(s)=D_0(s)\,\mathrm{Im}b_0(s) \text{ on the left}, \quad (1)$$

$$\mathrm{Im}D_0(s)=-\rho(s)N_0(s) \text{ on the right}. \quad (2)$$

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[1] R. Dashen and S. Frautschi, Phys. Rev. **135**, B1190 (1964).

[2] R. Dashen, Phys. Rev. **135**, B1196 (1964). R. Dashen and S. Frautschi, Phys. Rev. Letters **13**, 497 (1964). R. Dashen, S. Frautschi, M. Gell-Mann, and Y. Hara, in *Proceedings of the 12th Annual Conference on High Energy Physics at Dubna, 1964* (Atomizdat, Moscow, 1965). R. Dashen and S. Frautschi, Phys. Rev. **137**, B1318 (1965); **137**, B1331 (1965). R. Dashen, Y. Dothan, S. Frautschi, and D. Sharp, Phys. Rev. **143**, 1185 (1966). B. Diu, H. Rubinstein, and R. Van Royen (unpublished report). B. Diu (unpublished report). H. Goldberg, Nuovo Cimento **40**, B243 (1966).

[3] R. Blankenbecler and M. L. Goldberger (unpublished), private communication from R. Blankenbecler.

[4] The method of expanding D as a power series in λ has been utilized by Gilman and Abarbanel. F. J. Gilman, Ph.D. thesis, Princeton University (unpublished). H. D. I. Abarbanel and F. J. Gilman, Phys. Rev. **144**, 1355 (1966). This method suffers from a drawback since in order to obtain higher order terms in λ, the solution of an integral equation is necessary. The kernel of this integral equation is, however, identical to that of the unperturbed problem.

Now introduce a small change in the discontinuity on the left-hand-cut: $\lambda \operatorname{Im} b_1$. We can write the perturbed scattering amplitude $a(s)$ as a sum of two terms, and define a_1 by

$$a(s)=a_0(s)+a_1(s)\,. \tag{3}$$

On the left-hand-cut $a(s)$ satisfies

$$\operatorname{Im} a(s)=\operatorname{Im} b_0(s)+\lambda \operatorname{Im} b_1(s)\,. \tag{4}$$

On the right-hand-cut the unitarity relation holds. Unitarity can be written for a_1

$$\operatorname{Im} a_1(s)=\rho(s)[a_0^*(s)a_1(s)+a_1^*(s)a_0(s)+|a_1(s)|^2]\,. \tag{5}$$

On the left-hand-cut, the discontinuity of a_1 is given by

$$\operatorname{Im} a_1(s)=\lambda \operatorname{Im} b_1(s)\,. \tag{6}$$

Equations (5) and (6) can now be used as a basis for finding a_1. This is, however, rather inconvenient since the discontinuity of a_1 is connected through equation (5) to both the real and imaginary part of a_1. In order to overcome this difficulty we will consider the function f defined by

$$f(s)=D_0^2(s)a_1(s)\,. \tag{7}$$

We now observe that as a result of unitarity

$$a_0(s)=(1/\rho)e^{i\delta(s)}\sin\delta(s) \tag{8}$$

and that the phase of D_0^2 is given by

$$D_0^2(s)=|D_0(s)|^2e^{-2i\delta(s)}. \tag{9}$$

These equations hold on the right-hand-cut. Substituting Eqs. (7), (8), and (9) into (5) and (6) we obtain for f:

$$\operatorname{Im} f(s)=\frac{\rho(s)}{|D_0(s)|^2}|f(s)|^2 \text{ on the right} \tag{10}$$

and

$$\operatorname{Im} f(s)=\lambda D_0^2(s) \operatorname{Im} b_1(s) \text{ on the left.} \tag{11}$$

We will use these fundamental equations in order to calculate $f(s)$. We notice that these equations are identical in form to the standard partial-wave N/D equations, and can therefore be treated by the N/D method. The fact that the left-hand discontinuity is presumably small will enable us to treat the problem by some approximation method or as a power series in λ. We notice also that the function ρ got replaced by $\rho|D_0|^{-2}$ which is large in the neighborhood of a resonance of the unperturbed problem, we will return and investigate the physical significance of this point in Sec. IV.

The simplest approximation procedure which comes to mind is based on the fact that the right-hand side of Eq. (10) is quadratic, whereas the right-hand side of Eq. (11) is linear in λ. An expansion of f as a power series in λ is therefore natural. The first term in the power series would result from the left-hand-cut, and higher terms would have only a right-hand-cut. Moreover the discontinuity on the nth term in this power expansion of $f(s)$ can be expressed in terms of the first $n-1$ terms. This approach is troubled by lack of convergence whenever the unperturbed problem has a bound state, as one can easily convince oneself.[5]

We will therefore utilize the usual N/D approach and put $f=nd^{-1}$ where n and d satisfy the equations:

$$\operatorname{Im} n(s)=d(s)\lambda D_0^2(s) \operatorname{Im} b_1(s) \text{ on the left,} \tag{12}$$

$$\operatorname{Im} d(s)=-n(s)\mu(s) \text{ on the right,} \tag{13}$$

where

$$\mu(s)=\rho(s)|D_0(s)|^{-2}.$$

Since the right-hand side of Eq. (12) is small, one can use the determinantal method in solving for $n(s)$ and $d(s)$ in Eqs. (12) and (13). The method gives n and d as power series in λ. The function f will therefore be a ratio of two power series in λ, and will converge in cases when the simple power series diverges.

III. PERTURBATION IN THE PRESENCE OF BOUND STATES

Equations (12) and (13) are not sufficient in general to determine the functions n and d because of the CDD ambiguity.[6] When investigating the solutions of our equations in cases where the unperturbed solution a_0 has bound-state poles, we will show that introduction of CDD poles naturally arises.

Assume therefore that a_0 has a bound state at $s=s_0$.

$$D(s_0)=0. \tag{14}$$

We know that a_1 should have a pole at $s=s_0$, with a residue of opposite sign to that of a_0, so that the two poles cancel and the perturbed problem solution $a(s)$ has no pole at s_0. This means that

$$f(s_0)=0. \tag{15}$$

The zero in $f(s)$ should be simple so that upon dividing by $D_0^2(s)$, a_1 will have a simple pole at s_0. The easiest way of introducing a zero into $f(s)$ is by introducing a pole into d, at $s=s_0$. Such a CDD pole would have the desired effect on f without introducing additional singularities into the scattering amplitude. Physical consideration require thus, a particular way of resolving the CDD ambiguity for Eqs. (12) and (13). Whenever the unperturbed solution has several bound states, several CDD poles can be added to $d(s)$ without additional complication. We now exhibit the integral equations for n and d, assuming that only one CDD pole is

[5] When $a_0(s)$ has a pole at $s=s_0$, $a(s)$ will have a pole at a slightly shifted location, such a pole will come about from the nonconvergence of the power series for f. In addition, the nth term in this expansion will have a pole of order $n-1$ at $s=s_0$.

[6] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

necessary.

$$n(s)=\lambda\int_L \frac{D_0^2(x)\,\mathrm{Im}b_1(x)d(x)}{x-s}\frac{dx}{\pi}, \tag{16}$$

$$d(s)=1+\frac{\alpha}{s-s_0}-\int_R \frac{\mu(x)n(x)}{x-s}\frac{dx}{\pi}. \tag{17}$$

The parameter α is fixed by the conditions on the residue of a_1

$$\mathrm{Res}a_1(s)=(n(s_0))/([D_0'(s_0)]^2\alpha)=-R, \tag{18}$$

where R is the residue of a_0 and $D'(s_0)$ is the derivative of $D_0(s)$ at $s=s_0$. This gives for α

$$\alpha=-(n(s_0))/(R[D_0'(s_0)]^2). \tag{19}$$

We are now in a position to recover the Dashen-Frautschi results. This is done by using a first-order iteration procedure for $n(s)$. Substituting $d(s)=1$ in Eq. (16) and neglecting the integral term in Eq. (17), we obtain

$$d(s)=1-\frac{1}{R[D_0'(s_0)]^2}\frac{1}{s-s_0}\times\int_L \frac{\lambda D_0^2(x)\,\mathrm{Im}b_1(x)}{x-s_0}\frac{dx}{\pi}. \tag{20}$$

The solution for the existence of a bound state at $s=s_1$ is $d(s_1)=0$, giving

$$s_1-s_0=\frac{1}{R[D_0'(s_0)]^2}\int_L \frac{\lambda D_0^2(x)\,\mathrm{Im}b_1(x)}{x-s_0}\frac{dx}{\pi}, \tag{21}$$

which is identical to the D.F. result. Using the conventional terminology we can say that the D. F. result is obtained by the "one-half-order" determinantal approximation. We can proceed and iterate Eqs. (16) and (17) as is usually done in the determinantal procedure the number of iterations depending on the accuracy required of our calculation. If both α and the integral on the right hand side of Eq. (17) turn out to be small we note that $d(s)$ will be close to one except at the neighborhood of $(s=s_0)$ where the denominator $(s-s_0)$ causes a large change in d. This denominator will cause $d(s)$ to have a zero near $s=s_0$; this zero in $d(s)$ corresponds to the shifted bound state. The smaller α is, the smaller is the energy shift of the bound state. The sign of α determines the direction of the energy shift $\alpha>0$ corresponding to tighter binding.

In some perturbation problems the integral on the right-hand side of Eq. (17) can be small and therefore negligible with respect to 1. The pole term near $s=s_0$ can however be important even when α is very small. This term is important, for small α, mainly near $s=s_0$. One should therefore include the pole term in $d(s)$, especially in cases where the left-hand cut due to the perturbation extends to the neighborhood of $s=s_0$, and use as a first approximation

$$d(s)=1+\alpha/(s-s_0). \tag{22}$$

This causes no difficulty with the analyticity because of the vanishing of $\mathrm{Im}n(s)$ at the pole of $d(s)$. Substituting this into Eq. (16), we obtain

$$n(s)=I(s)+\alpha[I(s)-I(s_0)](s-s_0)^{-1}, \tag{23}$$

where

$$I(s)=\int_L \frac{\lambda D_0^2(x)\,\mathrm{Im}b_1(x)}{x-s}\frac{dx}{\pi}. \tag{24}$$

The value of α is now fixed by considering Eq. (23) at $s=s_0$ and using Eq. (19) for α. This gives

$$\alpha=-I(s_0)/(R[D_0'(s_0)]^2+I'(s_0)). \tag{25}$$

$I'(s_0)$ is the derivative of I at the point s_0. Substituting this value of α into Eq. (22), we obtain for the bound state energy s_1

$$s_1-s_0=I(s_0)/(R[D_0'(s_0)]^2+I'(s_0)). \tag{26}$$

The importance of this CDD pole approximation depends on the size of $I'(s_0)$ in comparison with $RD_0'^2(s_0)$. The CDD pole approximation outlined above may be an important improvement to the D.F. approximation; its validity is restricted however to cases where the right-hand-cut in $d(s)$ is unimportant. There are cases where such a right-hand-cut can be of importance even for small perturbations; these will be the subject of the next section.

IV. INFLUENCE OF THE UNITARITY CUT

In the last section we have treated the perturbation expansion in the cases when the unperturbed solution has a bound state, neglecting the right hand unitarity cut in $d(s)$. It is clear that $d(s)$ will play a more important role when the unperturbed solution has a resonance, and especially if this resonance is close to the elastic threshold.[7] When dealing with resonances where no bound states are present the introduction of CDD poles into $d(s)$ is unnecessary and one can solve the n/d equations in a standard iteration manner. One should note that the function $\mu(s)$ defined in Eq. (13) which replaces the standard unitarity discontinuity will be large at the point where the unperturbed problem has a resonance because of the appearance of $|D_0(s)|^2$ in the denominator. The D.F. method disregards completely the right-hand-cut in $d(s)$ because its effect on $f(s)$ is only through terms higher than first orders in λ.

In order to exhibit qualitatively some features of the case when a resonance exists near threshold we will treat the extremely simple example of an effective range

[7] This point was noted by Dashen and Frautschi, Phys. Rev. 137, 1318 (1965), footnote 9.

approximation.[8] The advantage of this example is that in spite of its simplicity it is well suited to the description of phenomena near threshold. We will consider the case where the left-hand-cut consists of a single pole at $s=-1$ having the residue λ. In this case $N_0(s)$ and $D_0(s)$ are given by:

$$N_0(s)=2\lambda(2+\lambda)^{-1}(s+1)^{-1},$$
$$D_0(s)=1-2\lambda(2+\lambda)^{-1}(1-i\sqrt{s})^{-1}. \tag{27}$$

We note that, for $\lambda=2$, $D_0(0)$ vanishes and therefore the unperturbed problem has a pole at $s=0$. Consider a case $\lambda<2$ and introduce a perturbation $\lambda\rightarrow\lambda+\eta$. The D.F. approximation will give

$$f_{\text{D.F.}}(s)=4\eta/(2+\lambda)^2(s+1)^{-1}. \tag{28}$$

This function clearly has no pole near $s=0$ and therefore all it can do is shift the location of the resonance. The first determinantal approximation for n and d can be easily obtained in this case. Since $\lambda<2$ and the unperturbed problem has no bound state, no CDD pole is necessary. In this first determinantal approximation

$$n(s)=f_{\text{D.F.}}(s),$$
$$d(s)=1-4\eta(2+\lambda)^{-1}[(2-\lambda)-(2+\lambda)i\sqrt{s}]^{-1}. \tag{29}$$

It is easy to see that the right-hand-cut can become important indeed for $\lambda\approx 2$, even when η is small.

The smallness of the perturbation is therefore not an assurance that we can use first-order perturbation theory for $f(s)$. The function $d(s)$ can have a zero corresponding to a bound state when

$$\eta\geqslant(2-\lambda)+\tfrac{1}{4}(\lambda-2)^2,$$

which can be very small for $\lambda\approx 2$. Only when $\lambda\approx 2$ is this approximation expected to be valid. In this region the bound state energy is

$$-i\sqrt{s_1}=+[4\eta+4(\lambda-2)+(\lambda-2)^2](2+\lambda)^{-2}. \tag{30}$$

The D.F. method cannot reproduce this bound state when it starts moving out of the unitarity cut since it is not of first order in λ in the expansion of f. It is also interesting to note that in this case, $\sqrt{|s|}$, for the bound-state energy is of first order in η and our perturbation method reproduces this feature correctly.

The case outlined above has an exact correspondence to Schrödinger perturbation theory. There, the presence of the continuum near a bound state makes first-order perturbation inadequate.

When partial waves higher than the s wave are treated, the function $\rho(s)$ will behave as $s^{l+1/2}$ near threshold; this can reduce the importance of resonances near threshold.

The qualitative discussion presented above and the simple effective-range perturbation example lead us to the conclusion that in cases where threshold effects may be important, the D.F. method is insufficient for treating perturbations in spite of their apparent smallness.

V. SOME APPLICATIONS

In this section we outline possible utilizations of the perturbation method in N/D calculations. The procedure we adopt is that of solving the original equations for N and D in some approximation and then making use of the perturbation method for improving the approximation. We are using the fact that many approximate methods for solving the N/D equations retain the main virtue of these by producing a scattering amplitude which obeys two particle unitarity exactly at the expense of mistreating the left-hand singularities.

The original N/D integral equations are

$$N(s)=\int_L \frac{D(x)\,\text{Im}b(x)}{x-s}\frac{dx}{\pi}, \tag{31}$$

$$D(s)=1-\int_R \frac{\rho(x)N(x)}{x-s}\frac{dx}{\pi}. \tag{32}$$

The approximation of these equations is usually obtained by substituting some crude approximation for D, $\bar{D}(x)$, into the right-hand side of Eq. (31). This gives as a first approximation

$$N_0(s)=\int_L \frac{\bar{D}(x)\,\text{Im}b(x)}{x-s}\frac{dx}{\pi}. \tag{33}$$

$D_0(s)$ is then obtained by substituting N_0 into the right-hand side of Eq. (32). In the determinantal approximation one uses the asymptotic form of $D(x)$

$$\bar{D}(x)=1. \tag{34}$$

The improved methods of Roy and Blankenbecler[9] substitute a more realistic function for $\bar{D}(x)$. Their method for potential scattering uses

$$\bar{D}(x)=(s^{1/2}-b)/(s^{1/2}+c), \tag{35}$$

where b is the location of the bound state and c is an adjustable parameter. Thus $\bar{D}(s)$ has the advantage of having the same behavior as $D(s)$ both near the bound state and at infinity, and of also having the same analyticity properties as $D(s)$. For further details on this method, especially on its relativistic version, the reader is referred to the work of Roy and Blankenbecler.

[8] H. Goldberg [Nuovo Cimento **40**, B243 (1966)] has considered the difference between the nn and np ^1S scattering length due to electromagnetic corrections. He used the D. F. method, and an effective range approximation. His results do not agree with those of H. P. Noyes who used a Bargmann potential which reproduces the same analyticity properties. These calculations show deviation from linearity in the input even when the perturbation is less than 1% of the unperturbed potential. This can be understood in the light of the importance of unitarity cuts in this case, where a virtual state appears near threshold. I am grateful to H. P. Noyes for pointing this out to me and for informing me about his results.

[9] S. M. Roy and R. Blankenbecler, Ann. Phys. (N. Y.) **35**, 314 (1964).

$N_0(s)$ and $D_0(s)$ obtained from the above method satisfies unitarity exactly. On the left hand cut the discontinuity of $a_0=N_0/D_0$ is given by

$$\mathrm{Im}a_0(s)=\bar{D}(s)D_0^{-1}(s)\ \mathrm{Im}b(s)\ . \tag{36}$$

We can therefore define the perturbation $\lambda\mathrm{Im}b_1$ as the difference between $\mathrm{Im}b(s)$ and $\mathrm{Im}a_0(s)$ on the left-hand cut.

$$\lambda\ \mathrm{Im}b_1(s)=[1-\bar{D}(s)D_0^{-1}(s)]\ \mathrm{Im}b(s)\ . \tag{37}$$

If $\bar{D}(s)$ is close to $D_0(s)$, this will be a small perturbation and we can therefore proceed to use a perturbation method where the left-hand-cut input is determined by Eq. (37). The use of the C.D.D. pole approximation may be advantageous in this case.

We can also use this expression in order to introduce a variational method for solving the N/D equations. Since we have an arbitrary function $\bar{D}(s)$ at our disposal, we can fit $\bar{D}(s)$ so that the first-order shift in bound states is small. Our purpose is to minimize the function $I(s)$ defined by Eq. (24). In our case

$$I(s)=\int_L \frac{D_0(x)[D_0(x)-\bar{D}(x)]\ \mathrm{Im}b(x)}{x-s}\frac{dx}{\pi}\ . \tag{38}$$

This function which determines the lowest order corrections to bound state energies can be minimized by making $\bar{D}(x)$ close to $D_0(x)$ in the region where $\mathrm{Im}b(x)$ is large. The function $I(s)$ is important also in cases discussed in Sec. IV where the right-hand-cut in $d(s)$ is crucial, since it is essentially the first determinantal approximation to $n(s)$.

The above approximation method could be used in conjunction with our perturbation method even in relativistic calculations. The reason for this being that no change in the right-hand side discontinuity is necessary for this approximation method and thus the N/D perturbation problem is identical to the nonrelativistic problem which we have discussed.

VI. CONCLUSION

We have exhibited a complete perturbation theory for the N/D equations in nonrelativistic potential scattering. This method works as well in relativistic cases when the right-hand-cut discontinuity $\rho(s)$ is unchanged by the perturbation. First-order calculations are identical to the D.F. method but in some cases higher order corrections are essential. These include most of the SU_3-breaking calculations where thresholds play an important role in the calculation of mass shifts and coupling constant shifts. The method outlined above could be used as it is in calculating the effects of faraway discontinuities on the left-hand-cut, such as contributions from high-order Born approximations. Some other uses for practical N/D calculations are given in Sec. V.

One drawback of our method is that we have assumed that $\mathrm{Im}a_1$ on the left-hand-cut is given. Generally, $\mathrm{Im}a_1$ is given by the distorted-wave Born-approximation of the perturbing potential and is not easily calculated. This point is related to the criticism of the D.F. method by Kim.[10] In practice, however, one assumes the left-hand-cut to be given by a first-order Born-approximation, and in spite of the bound states this may be a reasonable approximation. The generalizations of our method to many channels and to relativistic calculations are essential before the octet enhancement calculations can be treated with the inclusion of right-hand singularities. These will be the subject of future research.

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[10] Y. S. Kim, Phys. Rev. 142, 1150 (1966).