Cluster-Model Calculation for Phase Shifts of α-α Scattering*

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In the framework of the binary alpha cluster model, phase shifts for s, d, g, i, and k partial waves for α-α scattering are calculated through the excitation energy of 40 MeV. These are compared with recent experments which appear to suggest the existence of the rotational states with $J^\pi=6^+$, 8^+ of the nucleus Be^8, beyond the lowest three states with $J^\pi=0^+$, 2^+, and 4^+. A reasonable agreement is obtained between the calculated phase shifts and the real part of the experimental ones. The present result is also compared with the phase shifts obtained by the phenomenological α-α two-body potential models. A detailed analysis shows that the angular-momentum-dependent but energy-independent core radii of the two-body potential models coincide with the outermost nodes of the scattering functions within the contact distance of two α nuclei. The wave functions within the contact distance do not vanish, however, and hence the repulsive core in a literal sense does not exist. This result is examined in reference to the cluster structure of the Be^8 compound nucleus.

I. INTRODUCTION

THE elastic scattering of α particles by He^4 provides important information regarding the level structure of the ground state and the low-lying excited states of the compound Be^8 nucleus. It has been shown that the cluster model calculations[1] predict the lowest three states to be $J^\pi=0^+$, 2^+, and 4^+ in agreement with experiments. In the cluster-model description, trial wave function of the Be^8 nucleus is represented by the cluster configuration of binary α particles moving relative to each other.

For collisions between composite nuclei, the formalism in which the cluster model approach is inherently incorporated is provided by the method of resonating structure.[2] The validity of the method for the α-α scattering is particularly justified in view of the strong internal binding of the α particles, and the method has been applied to the present problem by several authors.[3–5] Of these authors Schmid and Wildermuth[5] succeeded in reproducing a fair fit to the experimental phase shifts for the s, d, and g partial waves corresponding to the states with $J^\pi=0^+$, 2^+, and 4^+.

Recently Igo and his collaborators[6] and Davis and Mayer-Boricke[7] have studied the elastic scattering of α particles by He^4 with higher bombarding energies than hitherto available, and suggested the existence of broad resonances for $L=6$, and $L=8$ partial waves. The existence of such resonances of i and k partial waves is not so obvious, especially from the shell-model point of view, unless a high degree of configuration mixing is realized. No higher angular momentum states than four are obtained from the ground-state configuration $(1s)^4(1p)^4$ of the shell model. In sharp contrast to this, the cluster model description can be applied to the higher angular momentum states of α-α scattering in a straightforward manner as it has been applied to the s, d, and g waves. Consequently for the present problem the cluster-model description becomes more tractable than the corresponding shell-model description of the i and k states of the compound Be^8 nucleus.

The main purpose of the present study is therefore to investigate the highly excited states of spins 6^+ and 8^+ of the compound Be^8 nucleus based on the simplest cluster ansatz. In other words, it is our problem to examine whether the energy dependence of phase shifts for the i and k waves can be reproduced in the framework of the binary alpha-cluster model. This is done by extending the previous cluster model analysis of the α-α scattering to the i and k partial waves and for higher bombarding energies.

The restrictive and underlying assumption of the present study as well as a brief account of the resonating group method is described in Sec. II. Most of the

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[1] K. Wildermuth and Th. Kanellopoulos, Nucl. Phys. **7**, 150 (1958); K. Wildermuth and Th. Kanellopoulos, *ibid.* **9**, 449 (1958/59).

[2] J. A. Wheeler, Phys. Rev. **52**, 1038 (1937); **52**, 1107 (1937).

[3] E. van der Spuy, Nucl. Phys. **11**, 615 (1959).

[4] A. C. Butcher and J. M. McNamee, Proc. Phys. Soc. (London) **A74**, 529 (1959).

[5] E. W. Schmid and K. Wildermuth, Nucl. Phys. **26**, 463 (1961). Their calculated phase shifts do not agree with the present calculation for the s, d, and g partial waves. For the present work we made a FORTRAN program for the IBM 709 independently from Dr. Schmid's program, because of its restrictive nature. Later we borrowed his program and some examples were calculated for checking our results. It turns out that both codes give almost the same values for the phase shifts, as it should be. This means, however, that we cannot reproduce the published results in their paper. We wish to thank Dr. Schmid for allowing us to use his program and for making valuable remarks on the numerical calculations.

[6] P. Darriulat, G. Igo, H. G. Pugh, and H. D. Holmgren, Phys. Rev. **137**, B315 (1965).

[7] R. H. Davis and C. W. Mayer-Boricke (private communication). See also *Proceedings of the International Conference on Nuclear Physics, Paris, 1964* (Editions du Centre National de la Recherche Scientifique, Paris, 1964), 3a(I)/C28.

published numerical results on the α-α scattering, with the exception of the work by Schmid and Wildermuth,[5] could not reproduce the experimental data very well although they were based on the same method of the resonating group structure. The discrepancies in some cases appear to result from the error in the numerical computation rather than from the difference in the two-nucleon forces used. For this reason, a relatively detailed account of computational technique employed in the present study is described in Sec. III. Finally, Sec. IV contains the comparison between the theoretical and experimental results of the energy dependence of the phase shifts. The present result is further compared with the ones obtained by the phenomenological models which employ the two-body local potential for α-α interaction with an L-dependent but energy-independent repulsive core. The physical significance of the repulsive cores of these models are closely examined in the light of the present theory, and a pertinent conclusion regarding the cluster structure of the Be^8 compound nucleus is drawn.

II. CLUSTER-MODEL DESCRIPTION OF THE SCATTERING

The main assumptions and restrictive nature of the present study are the following: (1) The repulsive core in the nucleon-nucleon interaction has not been included in the two nucleon potential employed here. (2) Owing to the spherical symmetry of the alpha particles, the spin-orbit and tensor forces are assumed not to contribute significantly to α-α scattering in the first approximation. (3) The explicit account of the polarization effect in the alpha particle has not been taken into consideration. (4) The collision problem is formulated in one-channel approximation. Of these four assumptions, the second one cannot be discarded without introducing a great deal of calculational complication. The third assumption implies that the alpha particles remain undistorted so that, for instance, the radii of the alpha particles remain the same as those of free alpha particles during collision. The so-called polarization effect is known to be an essential feature of the low energy deuteron-nucleus elastic scattering.[8] The neglect of this effect is assumed to be innocuous for the present problem in view of exceptionally strong internal binding of the alpha particles. The assumption whose validity is in serious doubt is the one-channel approximation especially because the analysis presented here involves high bombarding energies up to $E_\alpha=40$ MeV in the c.m. system. We will return to further discussion of these assumptions in Sec. VI.

The Hamiltonian of the eight-nucleon system is given by

$$H=-\frac{\hbar^2}{2M}\sum_{i=1}^{8}\nabla_i^2+\sum_{i>j=1}^{8}V_{ij}+\tfrac{1}{4}\sum_{i>j=1}^{8}(1+\tau_{i3})(1+\tau_{j3})\frac{e^2}{r_{ij}}, \quad (1)$$

where the first term is the kinetic-energy operator for the eight nucleons, V_{ij} represents the nucleon-nucleon interaction potential, and the last term describes the Coulomb potential between the protons in the eight-nucleon system. In the interest of obtaining an analytic form of the interaction kernel, the two-nucleon potential is assumed to be a Gaussian type given by

$$V_{ij}(r_{ij})=-V_0\exp(-\beta r_{ij}^2)\times(w+bP_{ij}^{\sigma}-hP_{ij}^{\tau}+mP_{ij}^{r}), \quad (2)$$

where P_{ij}^{σ}, P_{ij}^{τ}, and P_{ij}^{r} are a spin, an isospin, and a space exchange operator, respectively, between the ith and jth nucleons. There is some evidence[9] that the radial shapes of the two-nucleon potential are not so important so long as the two-nucleon data are described with equal accuracy. The parameters of the potential will be chosen to give the deuteron binding energy and low-energy two-nucleon scattering data as well as possible. A more ambitious program would be to employ a more realistic potential for the two-nucleon force, e.g., a Yale potential or Hamada-Johnston potential. Since the present study involves many restrictive assumptions, we decide to employ a more conservative potential represented by (2), and examine the results obtained in comparison with experiments.

Following the procedure of the resonating group method, we assume that the wave function of the eight-nucleon system is given in the form

$$\Psi=A\{\varphi(\alpha_1)\varphi(\alpha_2)\chi(\mathbf{r})\}, \quad (3)$$

where A stands for the antisymmetrizer for the eight nucleons. In the above expression $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$ represent the internal wave functions of the alpha particles. Each φ is given by the product of the spin and the isobaric spin-state vectors and a space wave function which is symmetrical in the space coordinates of the four nucleons. The function $\chi(\mathbf{r})$ stands for the scattering function describing the relative motion of the two α clusters, and depends on the intercluster distance $\mathbf{r}=\mathbf{R}_1-\mathbf{R}_2$.

The assumption of the simplest cluster-model description consists in using the wave function φ to represent the alpha particles in their *ground* states; the the explicit form of the space part of $\varphi_1(\alpha_1)\varphi(\alpha_2)$ is assumed to be

$$\exp(-\tfrac{1}{2}\alpha\sum_{i=1}^{4}(\mathbf{r}_i-\mathbf{R}_1)^2-\tfrac{1}{2}\alpha\sum_{i=5}^{8}(\mathbf{r}_i-\mathbf{R}_2)^2). \quad (4)$$

[8] B. J. Malenka, W. E. Kruse, and N. F. Ramsey, Phys. Rev. **91**, 1165 (1953).

[9] L. D. Pearlstein, Y. C. Tang, and K. Wildermuth, Phys. Rev. **18**, 23 (1960).

The range parameter α is adjusted to reproduce the experimental charge distribution of a free, unexcited alpha particle. The internal wave functions φ of the alpha particles do not depend on the intercluster distance r, which amounts to assuming that the alpha particles are not distorted during collision. Inasmuch as this neglect of the polarization effect entails an error in the total wave function ψ of (3), the error is transmitted to the scattering function $\chi(\mathbf{r})$ which is yet to be determined. This method of the resonating group consists in determining the scattering function $\chi(\mathbf{r})$ in the sense of the variational principle, such that it results in the best possible total wave function ψ in the form of (3), viz.

$$\delta\int[\psi^*(H-\mathcal{E})\psi]d\tau=0 \tag{5}$$

where $\mathcal{E}$ stands for the total energy of the system. The variation with respect to $\chi^*(\mathbf{r})$ results in an integro-differential equation for the scattering function $\chi(\mathbf{r})$.

The relative motion is resolved into partial waves:

$$\chi(\mathbf{r})=\sum_{L=\text{even}}^{\infty}(1/r)f_L(r)P_L(\cos\theta), \tag{6}$$

where the odd partial waves are missing on account of the identity of the two colliding clusters. The derived integro-differential equation for the problem reads

$$\left[\frac{\hbar^2}{4M}\left\{\frac{d^2}{dr^2}-\frac{L(L+1)}{r^2}\right\}+E-V_D(r)\right]f_L(r)=\int_0^\infty K_L(r,r')f_L(r')dr', \tag{7}$$

where E represents the bombarding energy, and $V_D(r)$ describes the direct potential originating from the identity element of the antisymmetrizer A. The self-adjoint kernel $K_L(r,r')$ describes a very complicated function, and represents all the effects of the antisymmetrization and the exchange nuclear interactions, including those of the Coulomb interaction. Although the Coulomb potential itself is of long range, the overlapping factor of the internal wave functions reduces the interaction effectively to something like a screened Coulomb interaction. The explicit expression of the kernel function is the same as that obtained by van der Spuy[3] for the part of nuclear exchange terms. The Coulomb exchange terms were calculated explicitly and confirmed to be the same as that given by Butcher and McNamee.[4] The kernel function so obtained vanishes at $r=0$ or $r'=0$, and hence does not have any significant effect on the scattering function $f_L(r)$ at very short distances.

III. NUMERICAL CALCULATION

There are six parameters to be adjusted by fitting to the two-nucleon data and matter distribution of an alpha particle. Five of these are the parameters of the two-nucleon interaction, well depth V_0, force range parameter β, and the parameters of the force mixture which are reduced to three in number owing to the normalization $w+m+b+h=1$. The parameters for the nucleon-nucleon potential are chosen to be

$$V_0=72.98\text{ MeV},\quad \beta=0.46\times10^{26}\text{ cm}^{-2},$$
$$w+m-b-h=0.63,\quad (w+m+b+h=1).$$

This potential yields an n-p singlet and triplet effective range, n-p singlet scattering length, and deuteron binding energy in a close agreement with experiments,[10] and was previously employed by Schmid and Wildermuth.[5] Two parameters for the force mixture still remain free. The main part of the two-nucleon force employed here is a Serber force ($w=m$, $b=h$) which describes the symmetry of the two-nucleon scattering angular distribution about 90° in the c.m. system. The small deviation from symmetry in the low-energy region (<50 MeV) can be reproduced in the first approximation by an admixture of a small portion of Rosenfeld force ($m=2b$, $h=2w$). In the previous analysis for the s, d, and g partial waves, 6% admixture of a Rosenfeld force was used. Thus, we have left with only one free parameter x, the ratio of Rosenfeld force to Serber force.

One of the six parameters is the range parameter α connected with the matter probability distribution of an alpha particle. We have assumed that the alpha cluster has approximately the same radius as the experimental one of a free alpha particle. The present choice $\alpha=0.535$ F^{-2} corresponds to the cluster radius of 1.45 F. This value is the proper radius of the alpha particle[11] if the correction is taken into account of the finite charge distribution of the proton, which has been reported to be 0.72 F for an assumed Gaussian charge distribution.[11] The calculated binding energy of an alpha cluster is 28.4 MeV using the values of the parameters chosen here. This is in a good agreement with the experimental binding energy of an alpha particle.

The kernel function $K_L(r,r')$ on the right side of Eq. (7) decreases sufficiently rapidly as r or r' increases to enable us to replace the infinite upper limit in the integral by some finite value r_c. The range of integration can now be divided into many intervals of equal distance h with pivotal points at $r_0=0$, r_1, r_2, $\cdots$, r_c. The integral involving the kernel function is then evaluated by use of the Simpson formula for numerical integration. The numerical integration of the entire equation (7) is then performed by employing the Fox

[10] L. Hulthén and M. Sugawara, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39.
[11] R. Hofstadter, Ann. Rev. Nucl. Sci. 7, 231 (1957).

and Goodwin method VII[12] with errors of an order $O(\delta^6 f)$, where δ is a central difference operator. This consists in replacing the derivative of the second order by its central difference equivalent to start with, and the operator $(1+\delta^2/12)$ is then applied on both sides of Eq. (7). Since the kernel function is negligibly small beyond the point r_c, compared with the contribution from the nuclear interaction $V_n(r)$ in the direct potential $V_D(r)$, the contribution from the nonlocal part can be neglected for values of r larger than r_c. In this way, the integro-differential equation can be converted into a set of simultaneous linear algebraic equations, expressed in the recursion formula.[13] The recursion formula was used repeatedly to obtain the function $f_L(r)$ through the point r_m, at which $V_n(r)$ in $V_D(r)$ becomes negligibly small (see below for criterion) in comparison with the Coulomb potential in $V_D(r)$. At this point, the logarithmic derivative of $f_L(r)$ is computed by use of central difference formula, and then is matched to the logarithmic derivative of the exterior solution which consists of a linear combination of the regular and irregular Coulomb wave functions. This procedure results in the determination of the nuclear phase shifts.

In the present code the numerical values r_c and r_m are automatically computed and reset in the following way. The Coulomb contribution to the direct potential $V_D(r)$ has been obtained in the form

$$(4e^2/r)\Phi[(2\alpha/3)^{1/2}r],$$

where

$$\Phi(x)=\frac{2}{\pi^{1/2}}\int_0^x \exp(-t^2)dt.$$

The function $\Phi(x)$ tends to unity as x goes to infinity and thus the Coulomb contribution reduces to a point charge Coulomb potential $4e^2/r$, which indicates that, under such a condition, the two alpha particles are well separated. Consequently, a criterion $1-\Phi[(2\alpha/3)^{1/2}r_c] \leqslant \epsilon_1$ serves to determine a value of r_c. In order to see how much the final results are sensitive to the value of r_c, two values 4.8 F and 5.6 F of r_c were used in the test runs of the code corresponding to the values 10^{-4} and 10^{-5} of ϵ_1. The two sets of phase shifts obtained were found to be same within 1%. For the matching point r_m, the criterion $|V_n(r_m)/V_c(r_m)| \leqslant \epsilon_2$, where ϵ_2 is a preassigned number, has been used. In the production run of the code, the same value of 10^{-5} is used for both ϵ_1 and ϵ_2. With the values of the parameters assigned in this way, the numerical values of r_c and r_m turned out to 5.6 and 9.2 F, respectively, when the mesh interval h was chosen to be 0.2 F.

The analytic form of the kernel $K_L(r,r')$ is rather complicated, and therefore considerable care was taken to program this part correctly. For instance, the kernel function can be expressed in terms of a combination of the modified Bessel functions of an imaginary argument, $I_{L+1/2}(\rho)/\sqrt{\rho}$. The evaluation of this function was performed in three intervals of ρ, in each of which different formulas and prescriptions were used to evaluate them. The demarcation points of three intervals were chosen and reset automatically in the program. An independent test which was made to ascertain whether the kernel function was correctly programmed was the following. It is known[14] that under complete antisymmetrization the redundant solutions should appear if and only if the equation were correctly programmed. In order to investigate this problem, the different mesh intervals $h=0.15$, 0.20, and 0.25 F were chosen for the s wave at $E_\alpha=5$ MeV, where the kernel function included both nuclear and Coulomb exchange terms. In this case of the complete kernel function, there indeed appear three different solutions for the scattering functions corresponding to the use of the three different mesh intervals. The result is shown in Fig. 1. It is noticed that all three different solutions converge to the same scattering function in the exterior region, as they should be since they should not affect the asymptotic behavior of the solution. If we leave out the exchange part of the Coulomb interaction from the kernel function, thereby deliberately making an incorrect kernel, then we find that no such redundant solution appears. Consequently we are led to believe that the kernel function has been correctly expressed in our code.

Numerical calculation of phase shifts was performed from 0.25 to 40 MeV for the incident-energy alpha particles in the c.m. system and for each partial wave of $L=0$, 2, 4, 6, and 8. Two sets of phase shifts are shown in Fig. 2 corresponding to the two values of x: $x=0$ (a pure Seber force) and $x=0.06$ (94% Seber plus 6% Rosenfield force). In comparison with these

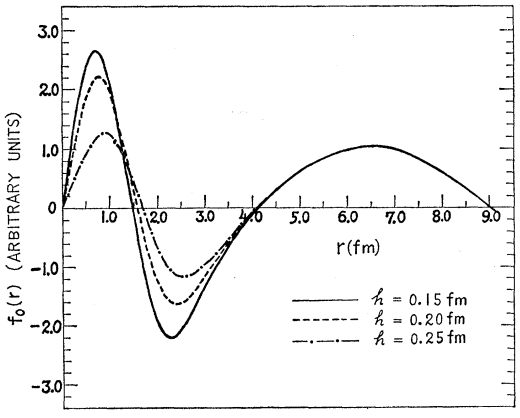

FIG. 1. Demonstration of the existence of redundant solutions for the s wave at the incident energy of 5 MeV in the c.m. system.

[12] L. Fox and E. T. Goodwin, Proc. Cambridge Phil. Soc. **45**, 373 (1949).

[13] H. H. Robertson, Proc. Cambridge Phil. Soc. **52**, 538 (1956).

[14] S. Hochberg, H. S. Massey, and L. H. Underhill, Proc. Phys. Soc. (London) **A67**, 975 (1954). Y. C. Tang, E. W. Schmid, and K. Wildermuth, Phys. Rev. **131**, 2631 (1963).

calculated results, the real parts of the experimental phase shifts[15,16] are also represented by the solid curves in the same figure. Some fluctuations have been observed in the real part of the experimental phase shifts due to sharp resonances within the energy range investigated here. Their effects on the phase shifts are confined to a narrow energy range just around the resonance energies and are left out of the consideration.

The contribution from the Coulomb potential was tentatively included in the kernel function, but it did not change the values of the phase shifts very much as compared with the case where the kernel function was free from the Coulomb exchange terms. The slight shift to the larger values of the phase shifts due to the Coulomb exchange contribution was almost reproduced by the calculation in which the Coulomb exchange terms were dropped off the kernel, but the value of x was decreased slightly. The fact that the inclusion of the Coulomb exchange terms lends itself to an increase in the phase shifts, i.e. to a more attractive interaction, can be readily understood. The major portion of the Coulomb interaction is repulsive and contained in the direct potential $V_D(r)$. The Coulomb exchange terms contained in the nonlocal kernel thus possess opposite signs relative to the direct Coulomb repulsive potential. Consequently, the inclusion of the Coulomb exchange effects enhances the attractive character of the effective interaction between the alpha clusters. In the actual production run the Coulomb exchange terms were switched off. For this reason our conclusion regarding the basic two-nucleon interaction slightly overestimates the Seber component relative to the Rosenfeld component.

IV. DISCUSSION AND CONCLUSION

The present study is concerned with the analysis of α-α scattering with range of the bombarding energies up to 40 MeV in the c.m. system. Within this bombarding-energy range there are several other inelastic and reaction channels, beside the one considered here, for instance, the reaction channel, $p+Li^7$, whose threshold energy is 17.25 MeV. The collision matrix involved in the elastic channel will then be necessarily represented by phase shifts with nonvanishing imaginary parts. For this reason, as noted in Sec. II, the one-channel approximation adopted here is admittedly unrealistic. Nevertheless, it is hoped that an important insight would be gained by treating the complex problem in the simplest cluster model assumption, and at the same time some light would be shed regarding the nature of the improvement that a more complete theory must tackle with. We proceed therefore to assume that the transition probabilities to channels other than those considered here are still relatively small even at the excitation energy of 40 MeV. Since the phase shifts calculated here are real quantities, we compare them with the real parts of experimental phase shifts. This is a somewhat arbitrary procedure, since the opening of inelastic and reaction channels not only introduces imaginary parts of phase shifts but also affects the real parts of the phase shifts for the entrance channel. Again, we assume that the effect is negligible.

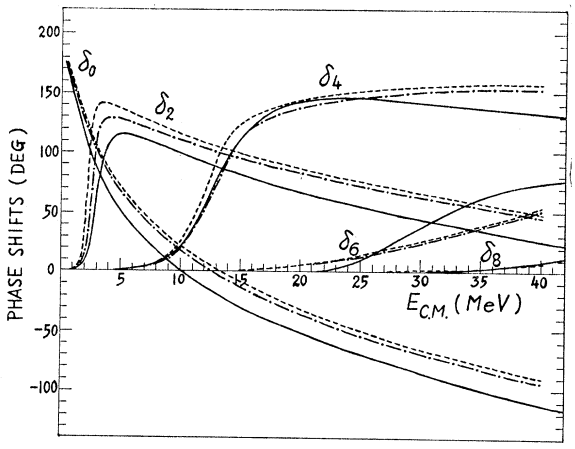

FIG. 2. The energy dependence of the calculated phase shifts is shown by the dashed curves for a pure Serber force and by the dash-and-dot curves for the mixed force, respectively. The real parts of the experimental phase shifts are represented by the solid curves.

It is clear from Fig. 2 that as far as the general trend of the energy dependence of the phase shifts is concerned the calculation based on the simplest cluster-model assumption reproduces the experimental result fairly well. There appear, however, some systematic discrepancies between the calculated and the experimental phase shifts, which will now be discussed. For the s and d partial waves, according to Fig. 2, the calculated phase shifts have almost the same energy dependence as the experimental ones, although the values of the calculated phase shifts are systematically larger by about 20 deg. than those of the experiment. For the g wave a better agreement with experiment is obtained, but still the calculated phase shifts assume larger values than the experimental ones in the high-energy region (>25 MeV). For the $L=6$ partial wave, the discrepancy is considerable, while for $L=8$ the agreement is good.

As it is illustrated in Fig. 2, the calculation by use of

[15] N. P. Heydenburg and G. M. Temmer, Phys. Rev. **104**, 123 (1956); J. L. Russel, G. C. Phillips, and C. W. Reich, *ibid.* **108**, 135 (1956); C. M. Jones, G. C. Phillips, and P. D. Miller, *ibid.* **117**, 525 (1960); N. Berk, F. E. Steigert, and G. L. Salinger, *ibid.* **117**, 531 (1960); J. R. Dunning, A. M. Smith, and F. E. Steigert, *ibid.* **121**, 580 (1961); R. Nilson, R. O. Kerman, G. R. Briggs, and W. K. Jentschke, *ibid.* **104**, 1673 (1956); *Proceedings of the International Conference on Nuclear Physics* (Editions du Centre National de la Recherche Scientifique, Paris, 1964), 1 Bis/C260; T. A. Tombrello and L. S. Senhouse, Phys. Rev. **129**, 2252 (1963); D. J. Bredin, W. E. Burcham, D. Evans, W. E. Gibson, J. S. C. McKee, D. J. Prowse, J. Rotblat and J. N. Snyder, Proc. Roy. Soc. (London) **A251**, 143 (1959).

[16] E. Shield, H. E. Conzett, P. Darriulat, H. G. Pugh, and R. J. Slobodrian, Bull. Am. Phys. Soc. **9**, 703 (1964); H. E. Conzett, E. Shield, R. J. Slobodrian, and S. Yamabe, Bull. Am. Phys. Soc. **9**, 704 (1964). See also *Proceedings of the International Conference on Nuclear Physics* (Editions du Centre National de la Recherche Scientifique, Paris, 1964), 1 Bis/C173.

a pure Serber force gives consistently larger values of the phase shifts for every partial wave than those resulting from an admixture of Rosenfeld force. Since a pure Serber force is known to be contradictory to the two-nucleon scattering data, and since the inclusion of the Rosenfeld component secures a better agreement with the α-α scattering phase shifts, the trend discussed previously proves to be a favorable feature. The fact that the inclusion of the Rosenfeld force renders the values of the phase shifts smaller is due mainly to the increasing repulsive effect of the nonlocal interaction when the value of the force mixture parameter x is increased. It is also noticed that the direct nuclear potential $V_n(r)$ becomes less attractive when x is increased, and vanishes in the extreme case of a pure Rosenfeld force. The general expectation discussed above has been confirmed by the calculation where 10% of Rosenfeld is mixed up with a Serber force, and the resulting phase shifts assumed systematically smaller values than the results obtained from using 6% admixture of a Rosenfeld force.

We would like to make the following observation regarding the constant shifts for $L=0$ and 2 partial waves. It has been pointed out that the exchange interactions through the kernel function simulate a repulsive potential of *short range*. In other words, such a short-range repulsive interaction is to be mainly ascribed to the antisymmetrization of the wave function. Our calculation shows, however, that for $L=0$ and 2 partial waves, this type of a repulsive interaction is still too small to reproduce the experimental values quantitatively. Only an inordinate amount of admixture of Rosenfeld force will give rise to an appreciable decrease in the phase shifts of $L=0$ and 2 partial waves. It seems therefore to call for the explicit inclusion of a repulsive core in the two-nucleon forces.

There have been several revealing phenomenological models for α-α scattering. The outstanding feature of the conclusion of these studies is the following: (1) The α-α interaction is angular-momentum (L)-dependent, (2) An energy-independent repulsive core is called for. It is noticed that the first feature is realized in the present theory in that the nonlocal kernel function $K_L(r,r')$ in Eq. (7) is L-dependent. The repulsive character of the α-α interaction at short distances is due, in the present theory, partly to the nonlocal term $K_L(r,r')$, which is there on account of the exclusion principle.[17] Had one explicitly included a repulsive core in the nucleon-nucleon interaction, the repulsive character of the α-α interaction would have been strengthened. A preliminary estimate, calculated by use of a soft-core two-nucleon force,[18] shows that there indeed appears a soft repulsive core for the α-α direct nuclear potential whose magnitude is at least 1.6 times greater than the maximum value of the attractive potential $V_n(r)$ at zero internuclear distance. The main feature of the phenomenological potential models thus appear to be reproduced in the present theory based on the method of resonating group structure.

There is, however, one aspect of the phenomenological potential models that is hard to comprehend on general theoretical grounds, namely *the energy independence* of the two-body α-α potential. According to Endo, Shimodaya, and Hiura,[19] the α-α local potential determined to fit the scattering data is characterized by the core radius and other features, which are L-dependent, but it is *independent* of incident energy through 25 MeV in the c.m. system for a given partial wave. This latter feature, namely, the energy independence of the α-α local potential, is rather surprising. For the interaction between complex nuclei can be shown to be highly nonlocal and becomes energy-dependent if the interaction is translated into a local two-body potential. In fact, this is reflected in most of the current phenomenological studies of interaction between complex nuclei, which employ energy-dependent and angular-momentum-dependent optical potentials.

Below we will examine the so-called energy independence of α-α potentials from the standpoint of the present theory and discuss in what sense the hard-core potential can schematically represent the α-α potential. The use of the hard-core potential was justified by Shimodaya, Tamagaki, and Tanaka[20] for the reason that it will attenuate the scattering function at short distances between alpha particles, thus simulating the repulsive character of antisymmetrization. The kernel function $K_L(r,r')$ of Eq. (7) contains indeed a considerable amount of repulsion, which, however, is not sufficiently strong enough to be described by a hard-core local potential. In order to see this in a straightforward way, we plot the radial scattering functions for different values of the incident energy. In Fig. 3, the calculated wave functions of s and d waves are plotted against the intercluster distance r for different values of the incident energy.

As can be seen from the figure there are two zero points for the s wave and one zero point for the d wave within the distance of 3.0 F, which is very close to twice the root-mean-square radius of a free alpha particle; and hence $r=3.0$ F is roughly the contact distance of two free alpha particles. A truly remarkable feature of the plots is that within the contact distance the zero points of the s and d wave functions remain practically unchanged over the wide range of incident energies. For the g wave we found no zero point within the contact distance and therefore the corresponding plot is not shown.

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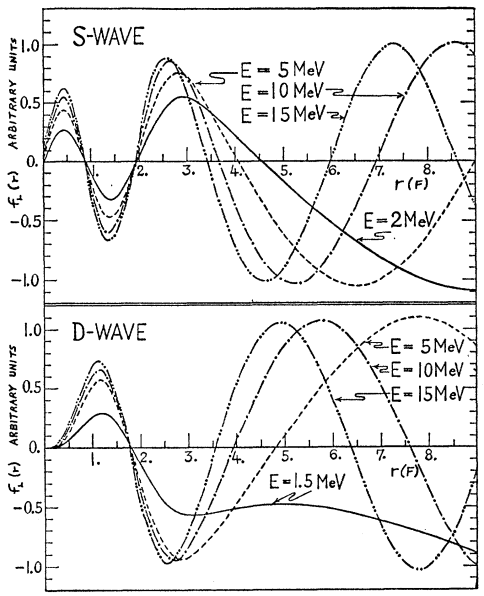

FIG. 3. The calculated wave functions for the s and d partial waves are plotted against the distance (in F) between two alpha particles. The normalization of the wave functions is determined to conform with their asymptotic functions whose maximum amplitude is chosen to be unity.

We would like to make the following observation regarding the result shown in Fig. 3. First, the fact that the wave functions are not completely washed out at small distances argues against the existence of a hard-core α-α local potential in a rigorous sense. Second, suppose we place hard cores at the outermost zero points, that is, a hard core $r_c^{(0)}=1.9$ F for the s wave and $r_c^{(2)}=1.8$ F for the d wave, and arrange the local potential in the usual Schrödinger equation so as to yield the same radial dependence of the figure beyond the hypothetical hard-core radii. Such s and d wave functions would reproduce the same scattering data as the original solutions.

At this point it is extremely interesting to compare the zero points $r_c^{(0)}$ and $r_c^{(2)}$ with the values of the hard-core radii determined by a phenomenological analysis. According to Endo, Shimodaya, and Hirura,[19] a satisfactory fit to the scattering data can be obtained by placing a hard core in the range $1.4\text{ F}\leqslant r_c^{(0)}\leqslant 2.1$ F for the s wave and in the range $1.2\text{ F}\leqslant r_c^{(2)}\leqslant 1.8$ F for the d wave. For the g wave they found it difficult to ascertain the existence of a hard-core potential. Thus, it is seen that the values of the zero point, $r_c^{(0)}$ and $r_c^{(2)}$, provided by the present calculation are within the range of values of the hard-core radii found in the phenomenological analysis. Since our resonating group calculation shows that there exists no hard core in a strict sense, it is clear that the ansatz of a hard core in a phenomenological analysis should be interpreted as only a schematic representation of the really more intricate α-α interaction.

We have observed in Fig. 3 that there are two zero points for the s wave and one zero point for the d wave within the contact distance. This can be readily understood if we assume the harmonic oscillator wave functions of three states 0^+, 2^+, and 4^+ for the nucleus Be^8 in agreement with the cluster-model description of Wildermuth and Kanellopoulos.[1] As a matter of fact, the zero points of these harmonic-oscillator wave functions turn out, according to the present calculation, to be 0.93 and 1.95 F for the $3s$ state, and 1.81 F for the $2d$ state. The fact that these values agree very closely with the zero points shown in Fig. 3 indicates therefore that, at least within their outermost zero points, the scattering wave functions behave like the corresponding harmonic-oscillator wave functions. Based on this feature of the wave functions, Wildermuth and Kanellopoulos[1] have claimed success in reproducing the excitation energies of the lowest three states of the nucleus Be^8.

In conclusion, we observe that it is possible to reproduce a fair agreement with the experimental data without explicitly assuming the energy-independent α-α local potentials. From the point of view of the present study, the hard-core α-α potentials of the phenomenological models are the substitutes for the outermost zero point of the scattering functions within the contact distance of two alpha clusters. The present calculation in addition brings out explicitly the connection between the zero points of the scattering functions and the nodes of the harmonic oscillator "bound" state functions which can be identified with the lowest three excited virtual states 0^+, 2^+, and 4^+ for the Be^8 compound nucleus.

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