The natural unit of current is

$$(c/4\pi)(\sqrt{2}H_{cb}/\lambda),$$

which ensures that $\mathbf{J}=\nabla\times\mathbf{H}$ in this system of units. Then the mean longitudinal current is given by

$$J=(j/\kappa)\langle|\Psi|^2\rangle_{\mathrm{av}}=2j[(\kappa^2-j^2)/\kappa-H_0]/(2\kappa^2-1)\beta. \quad (23)$$

We differentiate J with respect to j and set the derivative equal to zero to find the value of j for which J is a maximum:

$$\partial J/\partial j\propto\kappa^2-\kappa H_0-3j^2=0 \quad (24)$$

or

$$j^2=\tfrac{1}{3}(\kappa^2-\kappa H_0).$$

The maximum occurs at a value of j only $1/\sqrt{3}$ of the maximum possible value of j.

In ordinary cgs units, the critical current is then given by

$$\frac{4\pi J_c}{c}=(\tfrac{2}{3})^{3/2}\frac{H_{cb}}{\lambda}\frac{(1-H/H_{c2})^{3/2}}{\beta(1-1/2\kappa^2)}. \quad (25)$$

The magnetization at the transition falls to two-thirds of its equilibrium value in the absence of current. The smallness of the change in the magnetization accounts for the fact that a simple free-energy argument gives very nearly the same result for the critical current.

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Effects of Transverse and Axial Magnetic Fields on Gaseous Lasers*

W. Culshaw and J. Kannelaud
Lockheed Palo Alto Research Laboratory, Palo Alto, California
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The effects of transverse magnetic fields on a $J=1\rightarrow 0$ laser transition are considered, and expressions for the macroscopic-atomic-polarization terms are derived. The contributions from the σ modes are combined, and equations for the intensities and frequencies of the π and σ oscillations are deduced. Various coupling terms occur because of the common lower level of the transition, and a quenching of the initial π-mode oscillation with a change to the σ mode is indicated as the magnetic field increases from zero. With increased field, both modes oscillate simultaneously, and a beat occurs whose frequency depends on the operating conditions, and which may become zero again at higher magnetic fields. At line center the beat frequency should remain zero with increasing magnetic field, representing a method of laser tuning. Similar equations for the circularly polarized σ oscillations in axial magnetic fields are deduced. Here the conditions for stable two-frequency operation are more readily satisfied, and no such quenching is indicated. Again there are zero-beat-frequency regions of magnetic field in which a mutual synchronization of the oscillations should occur. Some experimental results with transverse magnetic fields on the 1.153-μ He-Ne laser are given. These display the general features indicated by the theory. Thus a quenching of the initial π-mode oscillation occurs, with more or less abrupt changes to the σ mode, depending on conditions. Similar single-beat-frequency variations with magnetic field occur, together with a region near the line center where the beat frequency, although finite, remains constant with increasing magnetic field.

1. INTRODUCTION

THE effects of relatively small magnetic fields on the operating characteristics of gaseous lasers are quite pronounced, and provide a fertile field for the investigation of the dispersive and nonlinear properties of the laser medium. Thus axial magnetic fields of a few tenths of a gauss can produce orthogonal circularly polarized oscillations and hence beat frequencies in a planar-type laser, which in zero magnetic field oscillated on a single frequency and was linearly polarized.[1] The beat frequency observed as the magnetic field increases depends on the detailed shape of the first- and third-order dispersion functions involved in the laser transition, or on the balance between frequency pulling and pushing effects. It is thus dependent on the lifetimes of the states involved, on the cavity tuning and Q value, on the applied magnetic field, and on the dispersive properties of the laser medium. When the magnetic field is zero the beat frequency is zero and the polarization is linear with a direction determined by small anisotropies, chiefly in the reflectors, of the laser cavity. The beat frequency may also pass through zero in one or more regions as the axial magnetic field increases.[1,2] In such regions the oscillations have a natural tendency to coalesce because of nonlinear effects and this again, for small anisotropy, gives rise to a linear

* Supported by the Lockheed Independent Research Funds.
[1] W. Culshaw and J. Kannelaud, Phys. Rev. **136**, A1209 (1964). See also Phys. Rev. **141**, 228 (1966); **141**, 237 (1966).

[2] R. L. Fork and M. Sargent, III, Phys. Rev. **139**, A617 (1965).

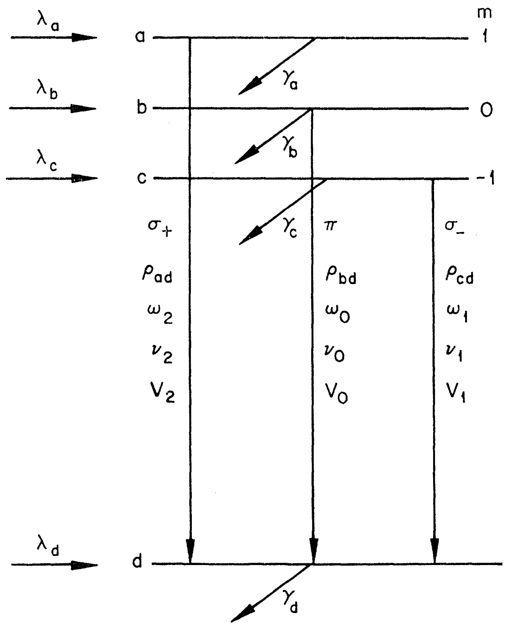


FIG. 1. Transitions and parameters used to consider transverse-magnetic-field effects in gaseous lasers.

polarization but which rotates as the magnetic field is varied within such regions.[1] Various other nonlinear oscillatory phenomena such as harmonic and subharmonic generation have also been observed when ac axial magnetic fields are also applied, both in zero dc magnetic field and in the other coherence or synchronization regions between the circularly polarized oscillations.[1] For transverse magnetic fields the behavior of a short, single mode, planar type He-Ne laser, operating on the 1.153-μ transition is found experimentally to be completely different in some respects from that pertaining to an axial magnetic field. When the magnetic field is applied along the direction of polarization in zero field, which we designate as the π mode, there is a more or less abrupt change in polarization depending on conditions, from the π mode to the σ mode of oscillation as the transverse magnetic field increases from zero. This resembles more closely a suppression of its initial π mode oscillation and an enhancement of the σ oscillation as the field increases, rather than an actual rotation as with the axial magnetic field. The laser thus oscillates in the π mode up to a value of magnetic field which depends on cavity tuning, and on the laser intensity, and then changes over into the σ mode of oscillation at a more or less specific value of magnetic field. When the magnetic field is applied perpendicular to the direction of polarization in zero magnetic field no such change in the polarization with increasing magnetic field is observed. As the magnetic field increases beyond this region, simultaneous laser oscillations in both π and σ modes occur and the usual beat frequency is observed between these orthogonal modes if an analyzer is inserted into the output beam.

The present account is mainly concerned with the theory involved in the interpretation of such effects of transverse magnetic fields on gaseous lasers in which case both $\Delta m=\pm 1$ and $\Delta m=0$ transitions are allowed. For this purpose we apply the theory of the gaseous optical maser as given by Lamb[3] to the case of a transverse magnetic field acting on a $J=1\rightarrow 0$ laser transition and develop the appropriate equations for the laser emission.[4] Any agreement between the theory thus derived and the observations on the more complicated $J=1\rightarrow 2$ transition of the He-Ne laser will be suggestive only that similar effects occur in other laser transitions. In the final analysis the actual transition involved must be considered in a way similar to that given here. The complexity, however, increases with the number of Zeeman levels involved.

2. THEORY FOR TRANSVERSE MAGNETIC FIELDS

(a) Basic Equations

The results to be derived are based on Lamb's detailed account of the theory of the optical maser,[3] and for further elucidation of the method, and of the parameters involved, this original work should be consulted. The theory essentially neglects collision effects and reduces the problem to a one-dimensional description. For an atom at position z_0 excited to state a at time t_0, the density matrix may be written as $\rho(a,z_0,t_0,v,t)$, from which the microscopic driving force or atomic polarization may be determined from the average value of the appropriate operator. Macroscopic values of atomic polarization are then determined by integration over appropriate values of t_0, and over the atomic velocity assuming a Maxwellian distribution. Similar contributions arise due to the excitation of the other atomic levels involved in the laser transition in the presence of magnetic fields, and these are summed to give the resultant macroscopic atomic polarizations, or source terms, for oscillations in the various photon polarizations.

Figure 1 shows the atomic-level scheme with the excitation rates λ_a, and decay constants γ_a, etc., together with the atomic frequencies ω_2, ω_0, and ω_1 of the transitions involved. We assume for the present that the single axial mode of the laser is split by the transverse magnetic field to give laser oscillations at frequencies ν_2, ν_0, and ν_1 in the π and σ modes as shown.[5]

[3] W. E. Lamb, Jr., Phys. Rev. **134**, A1429 (1964).

[4] A recent paper by C. V. Heer and R. D. Graft, Phys. Rev. **140**, A1088 (1965) is also significant for these Zeeman studies, particularly as regards arbitrary directions of magnetic field and polarization.

[5] Actually the results also apply when three axial modes of a short laser operate with well resolved Zeeman components, but we consider throughout a single axial mode and values of magnetic field such that the Doppler broadened Zeeman transitions overlap.

We require the values of the off-diagonal matrix elements ρ_{ad}, ρ_{bd}, and ρ_{cd} to third order in the respective perturbations V_2, V_0, and V_1. Because of the common lower atomic level d involved in these transitions, there will be a coupling between the oscillations in the π and σ modes which will depend on the applied magnetic field.

The equation of motion for the density matrix may be written as

$$\dot{\rho}=-i[\mathcal{H},\rho]-\tfrac{1}{2}(\Gamma\rho+\rho\Gamma), \tag{1}$$

where $\mathcal{H}$ is the appropriate Hamiltonian, which in the present case is given by

$$\mathcal{H}=\begin{bmatrix} \omega_a & 0 & 0 & V_2(t) \\ 0 & \omega_b & 0 & V_0(t) \\ 0 & 0 & \omega_c & V_1(t) \\ V_2^*(t) & V_0^*(t) & V_1^*(t) & \omega_d \end{bmatrix}, \tag{2}$$

and Γ is the diagonal matrix representing the phenomenological decay of the atomic states. Here the energies of the atomic levels and the perturbations $V(t)$ are expressed in angular-frequency units. Hence we write

$$\hbar V_j(t)=-\tfrac{1}{2}\Delta_j^* E_j[z-v(t-t_0),t]e^{-i(\nu_j t+\phi_j)}, \tag{3}$$

where $j=2$, 0, or 1 for the transitions $\Delta m=\mp 1$ and $\Delta m=0$, respectively, and Δj represents the matrix element involved. These may be written in reduced form as[6]

$$\Delta_2=-\mathbf{j}i/\sqrt{2},\quad \Delta_1=\mathbf{j}i/\sqrt{2},\quad \Delta_0=\mathbf{i}1, \tag{4}$$

for the respective transitions, where the orthogonal unit vectors indicate the polarization of the emitted photons. Here $\mathbf{k}$ is along the laser axis, and we assume that the magnetic field is along the $\mathbf{i}$ direction.

Equations for the various components of the density matrix are now readily written down using Eqs. (1) and (2), and we require the values of ρ_{ad}, ρ_{bd}, and ρ_{cd} given by

$$\dot{\rho}_{ad}=-(i\omega_2+\gamma_{ad})\rho_{ad}+iV_2(\rho_{aa}-\rho_{dd})+i(V_0\rho_{ab}+V_1\rho_{ac}), \tag{5}$$

$$\dot{\rho}_{bd}=-(i\omega_0+\gamma_{bd})\rho_{bd}+iV_0(\rho_{bb}-\rho_{dd})+i(V_2\rho_{ba}+V_1\rho_{bc}), \tag{6}$$

$$\dot{\rho}_{cd}=-(i\omega_1+\gamma_{cd})\rho_{cd}+iV_1(\rho_{cc}-\rho_{dd})+i(V_2\rho_{ca}+V_0\rho_{cb}), \tag{7}$$

where $\omega_2=\omega_a-\omega_d$, $\omega_0=\omega_b-\omega_d$, and $\omega_1=\omega_c-\omega_d$.

(b) Linear Approximation

Referring to Eq. (5) we see that ρ_{ad} is not affected by the coupling terms $V_0\rho_{ab}$ and $V_1\rho_{ac}$ in the first-order approximation. This also applies for ρ_{bd} and ρ_{cd}, and the solutions for the respective macroscopic polarizations in the nth axial mode may be written down directly[3] as

$$P_{ad}^{(1)}(t)=-\tfrac{1}{2}(|\Delta_2|^2/\hbar Ku)\bar{N}E_2e^{-i(\nu_2t+\phi_2)}Z(\nu_2-\omega_2)+\text{c.c.}, \tag{8}$$

where

$$P_{ad}(t)=\Delta_2\rho_{ad}+\Delta_2^*\rho_{da}. \tag{9}$$

Here $\bar{N}$ is the excitation density, which is assumed to be the same for all transitions, Z is the dispersion function,[3] and there are analogous equations for $P_{bd}^{(1)}(t)$ and $P_{cd}^{(1)}(t)$.

Expressing the equation of condition for laser oscillations at frequency ν_n and polarization $\mathbf{e}$ in the form

$$\tfrac{1}{2}\mathbf{e}[-i(2\nu_n\dot{E}_n+\nu\nu_nE_n/Q_n)+E_n(\Omega_n^2-\nu_n^2)]e^{-i(\nu_nt+\phi_n)}+\text{c.c.}=\tfrac{1}{2}\mathbf{e}(\nu^2/\epsilon_0)P_n(t)e^{-i(\nu_nt+\phi_n)}+\text{c.c.}, \tag{10}$$

and using Eq. (8) for $P_{ad}^{(1)}(t)$, and its counterpart for $P_{cd}^{(1)}(t)$ and $P_{bd}^{(1)}(t)$, we find that the parameter α_n in the linear approximation

$$\dot{E}_n=\alpha_nE_n, \tag{11}$$

is given by

$$\alpha_0=\tfrac{1}{2}\nu[(|\Delta_0|^2/\epsilon_0\hbar Ku)\bar{N}Z_i(\nu_n-\omega_0)-Q_0^{-1}] \tag{12}$$

$$\alpha_1=\tfrac{1}{2}\nu\{(|\Delta_1|^2/\epsilon_0\hbar Ku)\bar{N}[Z_i(\nu_n-\omega_2)+Z_i(\nu_n-\omega_1)]-Q_1^{-1}\} \tag{13}$$

for the π and σ polarizations, where Z_i represents the imaginary part, and where, in the present instance, we have combined the σ contributions for reasons given later. The respective threshold parameters defined for zero magnetic field are thus given by

$$\begin{aligned}\bar{N}_0{}^t&=\epsilon_0\hbar Ku/|\Delta_0|^2Z_i(0)Q_0\\ \bar{N}_1{}^t&=\epsilon_0\hbar Ku/2|\Delta_1|^2Z_i(0)Q_1,\end{aligned} \tag{14}$$

and since $|\Delta_0|^2=2|\Delta_1|^2$ from Eq. (4), and we assume that Q_0 is slightly greater than Q_1, the oscillation will build up in this preferred direction, which we term the π mode when the magnetic field is also applied along this direction.

(c) Third-Order Approximations

These are deduced from Eqs. (5), (6), and (7), together with other pertinent equations derived from Eqs. (1) and (2), following the procedure developed by Lamb,[3] and adding the various third order terms. The process may be simplified by comparing the various equations involved, and thus deducing the appropriate interchange of the subscripts 2, 1, and 0. It is then found that the third-order contributions to the macro-

[6] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, New York, 1935), p. 63.

scopic atomic polarizations are given by

$$
\begin{aligned}
P_{cd}^{(3)}(t)=&\tfrac{1}{16}i|\Delta_1|^4A\bar{N}E_1^3e^{-i(\nu_1t+\phi_1)}[\gamma_{ad}(\gamma_{ad}-i(\omega_1-\nu_1))\mathfrak{L}_{ad}(\omega_1-\nu_1)+1]\\
&+\tfrac{1}{32}i|\Delta_1|^2\gamma_a\gamma_dA\bar{N}E_1e^{-i(\nu_1t+\phi_1)}\{|\Delta_2|^2E_2^2[\gamma_d^{-1}\{(\gamma_{ad}-i(\omega_{21}-\nu_{21}))\mathfrak{L}_{ad}(\omega_{21}-\nu_{21})+(\gamma_{ad}+i\delta_{21})\mathfrak{L}_{ad}(\delta_{21})\}\\
&+(\gamma_a+i2\delta_{21})\mathfrak{L}_a(2\delta_{21})\{(\gamma_{ad}+i\delta_{21})\mathfrak{L}_{ad}(\delta_{21})+(\gamma_{ad}-i(\omega_1-\nu_1))\mathfrak{L}_{ad}(\omega_1-\nu_1)\}]\\
&+|\Delta_0|^2E_0^2[\gamma_d^{-1}\{(\gamma_{ad}-i(\omega_{10}-\nu_{10}))\mathfrak{L}_{ad}(\omega_{10}-\nu_{10})+(\gamma_{ad}-i\delta_{10})\mathfrak{L}_{ad}(\delta_{10})\}\\
&\qquad+(\gamma_a-i2\delta_{10})\mathfrak{L}_a(2\delta_{10})\{(\gamma_{ad}-i(\omega_1-\nu_1))\mathfrak{L}_{ad}(\omega_1-\nu_1)+(\gamma_{ad}-i\delta_{10})\mathfrak{L}_{ad}(\delta_{10})\}]\}+\text{c.c.}
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
P_{bd}^{(3)}(t)=&\tfrac{1}{16}i|\Delta_0|^4A\bar{N}E_0^3e^{-i(\nu_0t+\phi_0)}[\gamma_{ad}(\gamma_{ad}-i(\omega_0-\nu_0))\mathfrak{L}_{ad}(\omega_0-\nu_0)+1]\\
&+\tfrac{1}{32}i|\Delta_0|^2\gamma_a\gamma_dA\bar{N}E_0e^{-i(\nu_0t+\phi_0)}\{|\Delta_1|^2E_1^2[\gamma_d^{-1}\{(\gamma_{ad}-i(\omega_{01}-\nu_{01}))\mathfrak{L}_{ad}(\omega_{01}-\nu_{01})+(\gamma_{ad}+i\delta_{10})\mathfrak{L}_{ad}(\delta_{10})\}\\
&+(\gamma_a+i2\delta_{10})\mathfrak{L}_a(2\delta_{10})\{(\gamma_{ad}+i\delta_{10})\mathfrak{L}_{ad}(\delta_{10})+(\gamma_{ad}-i(\omega_0-\nu_0))\mathfrak{L}_{ad}(\omega_0-\nu_0)\}]\\
&+|\Delta_2|^2E_2^2[\gamma_d^{-1}\{(\gamma_{ad}-i(\omega_{20}-\nu_{20}))\mathfrak{L}_{ad}(\omega_{20}-\nu_{20})+(\gamma_{ad}+i\delta_{20})\mathfrak{L}_{ad}(\delta_{20})\}\\
&\qquad+(\gamma_a+i2\delta_{20})\mathfrak{L}_a(2\delta_{20})\{(\gamma_{ad}-i(\omega_0-\nu_0))\mathfrak{L}_{ad}(\omega_0-\nu_0)+(\gamma_{ad}+i\delta_{20})\mathfrak{L}_{ad}(\delta_{20})\}]\}+\text{c.c.},
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
A&=\pi^{1/2}(\hbar^3\gamma_a\gamma_dKu)^{-1},\\
\delta_{mn}&=\tfrac{1}{2}[\omega_m-\omega_n-(\nu_m-\nu_n)],\\
\omega_{mn}&=\tfrac{1}{2}(\omega_m+\omega_n),\\
\nu_{mn}&=\tfrac{1}{2}(\nu_m+\nu_n),\\
\mathfrak{L}_\alpha(\omega)&=(\gamma_\alpha^2+\omega^2)^{-1},\\
\gamma_{ad}&=\tfrac{1}{2}(\gamma_a+\gamma_d),\\
m,n&=2,1,0.
\end{aligned}
\tag{17}
$$

The expression for $P_{ad}^{(3)}(t)$ follows from Eq. (15) by interchanging the subscripts 1 and 2. We assume here that Ku, the Doppler linewidth parameter, is somewhat larger than γ_{ad}, which is usually the case in gaseous lasers.

These results for the various third-order macroscopic-polarization terms would now be substituted into Eq. (10), together with the first-order contributions given by Eq. (8) and similar equations, to give the steady state intensities and frequencies of the laser oscillations in the π and σ modes. We defer the discussion of these results, however, in order to discuss briefly the effect of an axial magnetic field on this same laser transition.

3. AXIAL MAGNETIC FIELD

In this case the magnetic field is along the **k** direction, which is also the laser axis. The transition $\Delta m=0$ is not allowed, and the transitions $\Delta m=\mp1$ will now correspond to laser oscillations in left- and right-handed circular polarizations, respectively, provided the cavity losses are relatively isotropic as regards the polarization of the laser emission. This is generally so for a planar, or internal optics laser, in contrast to the polarization constraint imposed by Brewster angle windows in a confocal laser.

The equations for the required off-diagonal matrix elements ρ_{ad} and ρ_{cd} of the density matrix are deduced exactly as before from Eq. (1), together with the Hamiltonian

$$
\mathcal{H}=\begin{bmatrix}\omega_a & 0 & 0 & V_2(t)\\ 0 & \omega_b & 0 & 0\\ 0 & 0 & \omega_c & V_1(t)\\ V_2^*(t) & 0 & V_1^*(t) & \omega_d\end{bmatrix},
\tag{18}
$$

where the reduced matrix elements Δ_1 and Δ_2 now have the numerical values $\mp\sqrt{2}$, respectively. However, the required results may also be deduced directly from Eq. (8) for $P_{ad}^{(1)}(t)$ and the similar equation for $P_{cd}^{(1)}(t)$, together with Eq. (15) for $P_{cd}^{(3)}(t)$ and the similar equation for $P_{ad}^{(3)}(t)$. Here we simply put $E_0=0$, and change the numerical value of the matrix elements Δ_1 and Δ_2. These macroscopic polarization terms are then substituted into Eq. (10), and the equations for the laser emission are deduced. The results are in agreement with the equations given by Fork and Sargent for the effect of an axial magnetic field on a $J=1\rightarrow0$ laser transition.[2]

4. DISCUSSION OF THE RESULTS

(a) Transverse Magnetic Field

The equations governing the steady-state intensities and frequencies of the laser oscillations for this geometry are now obtained by substituting the first- and third-order macroscopic polarizations given by Eqs. (8), (15), and (16), and analogous equations, into Eq. (10) and equating the imaginary and real parts, respectively. However, we have developed these equations assuming two σ-mode and one π-mode oscillation with frequencies ν_1, ν_2, and ν_0, and since it is found experimentally, at least for the He-Ne 1.153-μ laser transition, that only one low-frequency beat actually occurs instead of the two which this would imply, we now combine the two σ modes into one oscillation at the frequency ν_1.[7] This is

[7] It is conceivable that the two σ-mode frequencies ν_1 and ν_2 could be different even for a single axial mode due to frequency pulling and pushing effects, and hence would not combine in general even though the polarizations are in the same direction. The experiments, however, indicate that the σ components do combine when the Zeeman transitions overlap. They would not combine, of course, when three axial modes act on well resolved Zeeman transitions, although coupling effects are possible between them due to the common lower level of the laser transition. See also R. L. Fork and C. K. N. Patel, Appl. Phys. Letters **2**, 180 (1963), and Ref. 2.

done by putting $E_2=E_1$ and $\nu_2=\nu_1$ into the equations for the σ components and adding the results.

The resulting equations for the steady-state intensities of the two oscillations may then be written in the form

$$\begin{aligned}\dot{E}_1&=\alpha_1E_1-\beta_1E_1^3-\theta_{10}E_0^2E_1\\ \dot{E}_0&=\alpha_0E_0-\beta_0E_0^3-\theta_{01}E_1^2E_0,\end{aligned}\qquad(19)$$

where

$$\alpha_1=\tfrac{1}{2}\nu[\tfrac{1}{2}C_1\bar{N}(Z_i(\nu_n-\omega_2)+Z_i(\nu_n-\omega_1))-Q_1^{-1}],\qquad(20)$$

$$\begin{aligned}\beta_1=\tfrac{1}{32}\nu C_2\bar{N}\tfrac{1}{4}\{&2[\gamma_{ad}^2(\mathfrak{L}_{ad}(\omega_2-\nu_n)+\mathfrak{L}_{ad}(\omega_1-\nu_n))+2]+2\gamma_a\gamma_{ad}[\mathfrak{L}_{ad}(\omega_0-\nu_n)+\mathfrak{L}_{ad}(\delta)]\\ &+\gamma_a\gamma_d\mathfrak{L}_a(2\delta)[\gamma_a\gamma_{ad}(\mathfrak{L}_{ad}(\omega_2-\nu_n)+\mathfrak{L}_{ad}(\omega_1-\nu_n))-2\delta((\omega_2-\nu_n)\mathfrak{L}_{ad}(\omega_2-\nu_n)-(\omega_1-\nu_n)\mathfrak{L}_{ad}(\omega_1-\nu_n))]\\ &+\gamma_a\gamma_d[2(\gamma_a\gamma_d-2\delta^2)\mathfrak{L}_a(2\delta)\mathfrak{L}_{ad}(\delta)]\},\end{aligned}\qquad(21)$$

$$\begin{aligned}\theta_{10}=\tfrac{1}{32}\nu C_2\bar{N}\tfrac{1}{2}\{&\gamma_a\gamma_{ad}[2\mathfrak{L}_{ad}(\tfrac{1}{2}\delta)+\mathfrak{L}_{ad}(\omega_{20}-\nu_n)+\mathfrak{L}_{ad}(\omega_{10}-\nu_n)]\\ &+\gamma_a\gamma_d\mathfrak{L}_a(\delta)[\gamma_a\gamma_{ad}(\mathfrak{L}_{ad}(\omega_2-\nu_n)+\mathfrak{L}_{ad}(\omega_1-\nu_n))-\delta((\omega_2-\nu_n)\mathfrak{L}_{ad}(\omega_2-\nu_n)-(\omega_1-\nu_n)\mathfrak{L}_{ad}(\omega_1-\nu_n))]\\ &+\gamma_a\gamma_d\mathfrak{L}_a(\delta)\mathfrak{L}_{ad}(\delta/2)(2\gamma_a\gamma_{ad}-\delta^2)\},\end{aligned}\qquad(22)$$

$$\alpha_0=\tfrac{1}{2}\nu[C_1\bar{N}Z_i(\nu_n-\omega_0)-Q_0^{-1}],\qquad(23)$$

$$\beta_0=\tfrac{1}{32}\nu C_2\bar{N}\{2[\gamma_{ad}^2\mathfrak{L}_{ad}(\omega_0-\nu_0)+1]\},\qquad(24)$$

$$\begin{aligned}\theta_{01}=\tfrac{1}{32}\nu C_2\bar{N}\tfrac{1}{2}\{&\gamma_a\gamma_{ad}[2\mathfrak{L}_{ad}(\tfrac{1}{2}\delta)+\mathfrak{L}_{ad}(\omega_{01}-\nu_n)+\mathfrak{L}_{ad}(\omega_{02}-\nu_n)]+2\gamma_a^2\gamma_d\gamma_{ad}\mathfrak{L}_a(\delta)\mathfrak{L}_{ad}(\omega_0-\nu_n)\\ &+\gamma_a\gamma_d\mathfrak{L}_a(\delta)\mathfrak{L}_{ad}(\delta/2)(2\gamma_a\gamma_{ad}-\delta^2)\},\end{aligned}\qquad(25)$$

where $C_1=|\Delta_0|^2/(\epsilon_0\hbar Ku)$, $C_2=|\Delta_0|^4A/\epsilon_0$, and we have put $\delta_{21}=\delta=\gamma H$, $\delta_{20}=\delta/2=-\delta_{10}$, in the equations since the differences between ν_0, ν_1, and ν_2 will be small, and we may replace these frequencies by ν_n corresponding to the eigenfrequency of the cavity for the single axial mode considered.

The oscillations with electric field strength E_0 would correspond to the π mode if this is the direction of polarization in zero magnetic field and the field is applied along this direction. E_1 would then correspond to oscillations in the σ mode of orthogonal polarization. It is apparent that either a single oscillation in π or in the σ mode may occur, i.e., one oscillation may suppress the other, or both oscillations may occur simultaneously and low-frequency beats will be observed. The behavior will be governed by the coefficients in the nonlinear equations and by their variation with the magnetic field.

In order to investigate these possibilities we write Eqs. (19) in the form

$$\begin{aligned}\dot{y}&=2y(\alpha_1-\beta_1y-\theta_{10}x)=Q(x,y)\\ \dot{x}&=2x(\alpha_0-\beta_0x-\theta_{01}y)=P(x,y),\end{aligned}\qquad(26)$$

where $x=E_0^2$ and $y=E_1^2$. The singular points (x_0,y_0) of these equations are then given by $Q(x_0,y_0)=0=P(x_0,y_0)$, and we proceed to investigate the stability of these points in the phase plane by making a Taylor expansion around them and retaining only the linear approximation.[8] This gives

$$\begin{aligned}\dot{y}&=ax+by,\\ \dot{x}&=cx+dy,\end{aligned}\qquad(27)$$

where

$$\begin{aligned}a&=Q_x(x_0,y_0),\quad b=Q_y(x_0,y_0),\\ c&=P_x(x_0,y_0),\quad d=P_y(x_0,y_0),\end{aligned}\qquad(28)$$

x_0, y_0 are the coordinates of the singular point, and the subscripts indicate differentiation with respect to the indicated variable. The stability at the singular point may then be determined from the character of the roots of the characteristic equation[9]

$$S^2-(b+c)S+bc-ad=0.\qquad(29)$$

For the case under consideration, and with the condition

$$(b-c)^2+4ad>0,\qquad(30)$$

we have a node if $ad-bc<0$, and the node is stable if $b+c<0$, and unstable if $b+c>0$. If $ad-bc>0$ we have a saddle point.

On applying these criteria to the singular points of

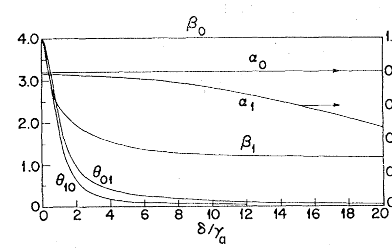


FIG. 2. Variation of the intensity and coupling parameters for π and σ oscillations with transverse magnetic field. Laser tuned to line center; $\eta_0=2$, $\eta_1=1.98$.

[8] A. A. Andronow and C. E. Chaikin, *Theory of Oscillations* (Princeton University Press, Princeton, New Jersey, 1949), pp. 182–199.

[9] J. J. Stoker, *Nonlinear Vibrations in Mechanical and Electrical Systems* (Interscience Publishers, Inc., New York, 1950), pp. 40–44.

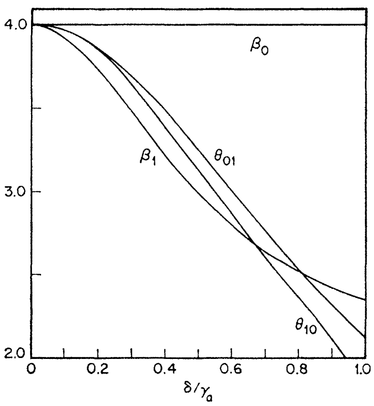


FIG. 3. Expanded plot of the small magnetic-field region of Fig. 2 in which θ_{01} and θ_{10} are greater than β_1.

Eqs. (45) we obtain the following results:

$$\left.\begin{array}{l}\theta_{10}\alpha_0>\alpha_1\beta_0\\ \alpha_0>0\end{array}\right\}\begin{array}{l}\text{Stable node at}\\ x=\alpha_0/\beta_0,\quad y=0\end{array} \tag{31}$$

$$\left.\begin{array}{l}\theta_{01}\alpha_1>\alpha_0\beta_1\\ \alpha_1>0\end{array}\right\}\begin{array}{l}\text{Stable node at}\\ x=0,\qquad y=\alpha_1/\beta_1.\end{array} \tag{32}$$

We note that, as far as these oscillations are concerned, there is no condition on $\beta_1\beta_0$ as compared with $\theta_{01}\theta_{10}$.

$$\left.\begin{array}{l}\beta_1\beta_0>\theta_{10}\theta_{01}\\ \theta_{10}\alpha_0<\alpha_1\beta_0\\ \theta_{01}\alpha_1<\alpha_0\beta_1\\ \alpha_0,\,\alpha_1>0\end{array}\right\}\begin{array}{l}\text{Stable node at}\\ x=\dfrac{\alpha_1\theta_{01}-\alpha_0\beta_1}{\theta_{10}\theta_{01}-\beta_1\beta_0}\\ y=\dfrac{\alpha_0\theta_{10}-\alpha_1\beta_0}{\theta_{10}\theta_{01}-\beta_1\beta_0}.\end{array} \tag{33}$$

If $\beta_1\beta_0<\theta_{01}\theta_{10}$ there is a saddle point at the values of x and y in Eq. (51), and no stable two frequency operation is possible. The system reverts to a stable oscillation at $x=\alpha_0/\beta_0$ or $y=\alpha_1/\beta_1$ depending on conditions. The conditions $\beta_1\beta_0\gtrless\theta_{01}\theta_{10}$ are analogous to those given by Lamb[3] in his discussion of the interaction between axial modes of a gas laser, and are referred to as weak and strong coupling respectively.

Assuming that the laser is initially tuned to the line center, and expressing α_0 and α_1 in the form

$$\begin{aligned}\alpha_0&=\tfrac{1}{2}\nu C_1\bar{N}[Z_i(\nu_n-\omega_0)-\eta_{0t}^{-1}Z_i(0)]\\ \alpha_1&=\tfrac{1}{2}\nu C_1\bar{N}\tfrac{1}{2}[Z_i(\nu_n-\omega_2)+Z_i(\nu_n-\omega_1)\\ &\qquad-\eta_{1t}^{-1}2Z_i(0)],\end{aligned} \tag{34}$$

where

$$\eta_{0t}=\bar{N}/\bar{N}_0{}^t,\quad \eta_{1t}=\bar{N}/\bar{N}_1{}^t \tag{35}$$

are the relative excitation parameters, curves of the variable parts of the expressions given by Eqs. (20) through (25) are plotted in Fig. 2 against the applied magnetic field expressed as δ/γ_a. The constant terms $\frac{1}{2}\nu C_1\bar{N}$ and $\frac{1}{32}\nu C_2\bar{N}$ are omitted in these plots, and the parameters $\gamma_d=7\gamma_a$, $\gamma_{ad}=4\gamma_a$, $Ku=10\gamma_{ad}$ together with $\eta_{0t}=2$ and $\eta_{0t}=1.01\eta_{1t}$ have been used. It is clear that the effect of the parameters θ_{01},θ_{10} will be greatest within the natural line width of the transition, and that they will be negligible at the higher values of magnetic field. Thus, referring to Eq. (33), at the value of magnetic field where $\theta_{10}<\beta_0$, and $\theta_{01}<\beta_1$, stable oscillations in both π and σ modes are possible and beat frequencies should occur. The situation is quite different in near zero magnetic fields, however, since although θ_{01}, θ_{10}, β_1, and β_0 are equal in zero field, we see that there is a region of magnetic field in which $\beta_1<\theta_{10}$ and θ_{01}. This is more clearly shown in the expanded plot of this region given in Fig. 3. Hence, for the conditions assumed here we have $\alpha_0=1.009\alpha_1$ in zero magnetic field and Eq. (31) is satisfied, and the initial oscillation is in the π mode. However, as the field increases from zero the condition $\theta_{01}\alpha_1>\alpha_0\beta_1$ may be satisfied in the region where $\theta_{01}>\beta_1$, whilst the condition $\theta_{10}\alpha_0>\alpha_1\beta_0$ is no longer satisfied due to the decrease in θ_{10}. Hence the oscillation will change from the π mode to the σ mode as the magnetic field increases from zero. This will occur when $\theta_{01}/\beta_1>\alpha_0/\alpha_1$, and hence the magnetic field required will depend on the laser intensity, on the anisotropy in the cavity Q, and on the cavity tuning. For $\eta_{0t}=1.2$, or operation closer to threshold, we have $\alpha_0/\alpha_1=1.059$ for the same conditions, and a higher magnetic field will be required to change from π to σ operation due to the larger ratio of θ_{01}/β_1 which is then necessary. If Eq. (32) cannot be satisfied for some specific conditions, then the π mode oscillation will be suppressed to some extent at a specific value of magnetic field until the conditions in Eq. (33) are satisfied and simultaneous oscillations occur in both π and σ modes.

Figure 4 shows similar curves of α_0, α_1, β_0, etc., for a cavity-tuning position of $\omega_0-\nu_n=8\gamma_a$. The same characteristics are apparent, but in addition there are some resonance effects in β_1 and in θ_{01} at $\delta=8\gamma_a$ and $\delta=16\gamma_a$, respectively. Conclusions similar to those above apply in near zero magnetic field, and a higher field will be necessary to change from the π to the σ oscillation due to the larger value of $\alpha_0/\alpha_1=1.072$ which is now involved. Figure 5 shows the laser intensities

$$\frac{E_0{}^2\pi^{1/2}|\Delta_0|^2}{\hbar^2\gamma_a\gamma_d}=16\left[\frac{\alpha_1\theta_{01}-\alpha_0\beta_1}{\theta_{10}\theta_{01}-\beta_1\beta_0}\right], \tag{36}$$

and the similar equation for $E_1{}^2$, plotted as a function

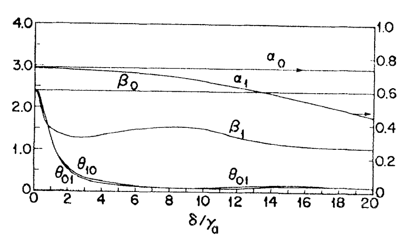


FIG. 4. Variation of the intensity and coupling parameters for π and σ oscillations with transverse magnetic field. Laser detuning $\omega_0-\nu_n=8\gamma_a$; $\eta_0=2$, $\eta_1=1.98$.

of δ/γ_a for $(\omega_0-\nu_n)$ equal to zero and $8\gamma_a$, respectively. The interaction between the oscillations and the dip in the intensity E_1^2 when Zeeman shift equals the cavity detuning are apparent, as well as the fact that the σ oscillation is the strongest at these values of magnetic field.

When simultaneous oscillation occurs in both π and σ modes the respective frequencies are given by

$$\begin{aligned}\nu_0&=\nu_n+\sigma_0-\rho_0E_0^2-\tau_{01}E_1^2\\ \nu_1&=\nu_n+\sigma_1-\rho_1E_1^2-\tau_{10}E_0^2,\end{aligned}\tag{37}$$

where

$$\sigma_0=\tfrac{1}{2}\nu C_1\bar{N}Z_r(\nu_n-\omega_0),\tag{38}$$

$$\rho_0=\tfrac{1}{32}\nu C_2\bar{N}[2\gamma_{ad}(\omega_0-\nu_n)\mathcal{L}_{ad}(\omega_0-\nu_n)],\tag{39}$$

$$\tau_{01}=\tfrac{1}{32}\nu C_2\bar{N}\tfrac{1}{2}\{\gamma_a[(\omega_{10}-\nu_n)\mathcal{L}_{ad}(\omega_{10}-\nu_n)+(\omega_{20}-\nu_n)\mathcal{L}_{ad}(\omega_{20}-\nu_n)]+\gamma_a\gamma_d[2\gamma_a(\omega_0-\nu_n)\mathcal{L}_a(\delta)\mathcal{L}_{ad}(\omega_0-\nu_n)]\},\tag{40}$$

$$\sigma_1=\tfrac{1}{2}\nu C_1\bar{N}\tfrac{1}{2}[Z_r(\nu_n-\omega_2)+Z_r(\nu_n-\omega_1)],\tag{41}$$

$$\begin{aligned}\rho_1=\tfrac{1}{32}\nu C_2\bar{N}\tfrac{1}{4}\{&2\gamma_{ad}[(\omega_2-\nu_n)\mathcal{L}_{ad}(\omega_2-\nu_n)+(\omega_1-\nu_n)\mathcal{L}_{ad}(\omega_1-\nu_n)]\\ &+\gamma_a[2(\omega_0-\nu_n)\mathcal{L}_{ad}(\omega_0-\nu_n)]+\gamma_a\gamma_d\mathcal{L}_a(2\delta)[2\delta\gamma_{ad}(\mathcal{L}_{ad}(\omega_2-\nu_n)-\mathcal{L}_{ad}(\omega_1-\nu_n))\\ &+\gamma_a((\omega_2-\nu_n)\mathcal{L}_{ad}(\omega_2-\nu_n)+(\omega_1-\nu_n)\mathcal{L}_{ad}(\omega_1-\nu_n))]\},\end{aligned}\tag{42}$$

$$\begin{aligned}\tau_{10}=\tfrac{1}{32}\nu C_2\bar{N}\tfrac{1}{2}\{&\gamma_a[(\omega_{20}-\nu_n)\mathcal{L}_{ad}(\omega_{20}-\nu_n)+(\omega_{10}-\nu_n)\mathcal{L}_{ad}(\omega_{10}-\nu_n)]\\ &+\gamma_a\gamma_d\mathcal{L}_a(\delta)[\gamma_{ad}\delta(\mathcal{L}_{ad}(\omega_2-\nu_n)-\mathcal{L}_{ad}(\omega_1-\nu_n))+\gamma_a((\omega_2-\nu_n)\mathcal{L}_{ad}(\omega_2-\nu_n)+(\omega_1-\nu_n)\mathcal{L}_{ad}(\omega_1-\nu_n))]\}.\end{aligned}\tag{43}$$

It is clear from an inspection of these equations that all these coefficients are zero when the laser cavity is tuned to the line center, and hence the beat frequency $\nu_1-\nu_0$ will be zero for all values of transverse magnetic field and laser intensity. As the cavity is tuned from the line center a beat frequency will appear, and this may be a reasonably sensitive way of tuning the laser to the line center. Omitting the constant terms $\frac{1}{2}\nu C_1\bar{N}$ and $\frac{1}{32}\nu C_2\bar{N}$, Fig. 6 gives the variation of the parameters given by Eqs. (38) through (43) as a function of δ/γ_a for a cavity detuning $(\omega_0-\nu_n)$ equal to $8\gamma_a$, when it is apparent that they all now assume finite values and that there are marked variations with magnetic field, particularly in the value of ρ_1 at Zeeman shifts near the cavity tuning position. This will cause a similar rapid variation in the beat frequency given by

$$\begin{aligned}\nu_1-\nu_0=\tfrac{1}{2}(\nu/Q_n)(\eta/Z_i(0))\{&\sigma_1-\sigma_0\\ &-\tfrac{1}{16}(\rho_1E_1^2-\rho_0E_0^2+\tau_{10}E_0^2-\tau_{01}E_1^2)\},\end{aligned}\tag{44}$$

where E_1^2 and E_0^2 are as given in Fig. 5. Equation (44) is plotted as a function of δ/γ_a in Fig. 7, taking $\eta=2$, and $\nu/Q=10^6$ and shows this variation, the beat frequency being positive initially due to frequency pushing

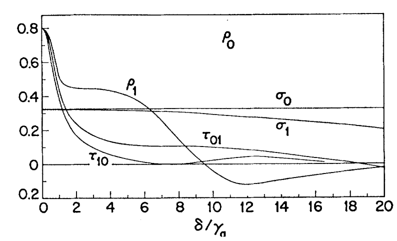


FIG. 6. Variation of the frequency coefficients of the π and σ modes with transverse magnetic fields. Laser detuning $\omega_0-\nu_n=8\gamma_a$.

effects, after which it begins to decrease and would change sign around a value of magnetic field corresponding to $\delta/\gamma_a=30$. In the region of zero beat frequency at this higher value of magnetic field we may expect phenomena such as a frequency locking of the π and σ modes together with a rotation of the polarization with magnetic field as occurs with axial magnetic fields,[1] but we shall not discuss this any further at present.

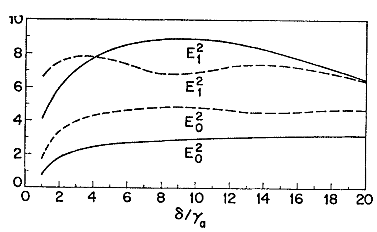


FIG. 5. Intensity parameters $\pi^{1/2}E^2|\Delta_0|^2/(\hbar^2\gamma_a\gamma_d)$ of the π and σ modes versus transverse magnetic field. Full curves—laser tuned to line center. Dashed curves—laser detuning $\omega_0-\nu_n=8\gamma_a$. $\eta_0=2$, $\eta_1=1.98$.

(b) Axial Magnetic Field

Using Eqs. (8), (15), and (10), together with the results of Sec. 3, the equations governing the oscillation intensities when an axial magnetic field is applied to a $J=1\rightarrow 0$ laser transition, may be written as

$$\begin{aligned}\dot{E}_1&=\alpha_1E_1-\beta_1E_1^3-\theta_{12}E_2^2E_1\\ \dot{E}_2&=\alpha_2E_2-\beta_2E_2^3-\theta_{21}E_1^2E_2,\end{aligned}\tag{45}$$

where the parameters are given by

$$\alpha_1=\tfrac{1}{2}\nu[C_1'\bar{N}Z_i(\nu_n-\omega_1)-Q_1^{-1}],\tag{46}$$

$$\beta_1=\tfrac{1}{32}\nu C_2'\bar{N}[2(\gamma_{ad}^2\mathcal{L}_{ad}(\omega_1-\nu_n)+1)],\tag{47}$$

$$\begin{aligned}\theta_{12}=\tfrac{1}{32}\nu C_2'\bar{N}\{&\gamma_a\gamma_{ad}[\mathcal{L}_{ad}(\omega_0-\nu_n)+\mathcal{L}_{ad}(\delta)]\\ &+\gamma_a\gamma_d\mathcal{L}_a(2\delta)[\mathcal{L}_{ad}(\delta)(\gamma_a\gamma_{ad}-2\delta^2)\\ &+\mathcal{L}_{ad}(\omega_1-\nu_n)(\gamma_a\gamma_{ad}+2\delta(\omega_1-\nu_n))]\},\end{aligned}\tag{48}$$

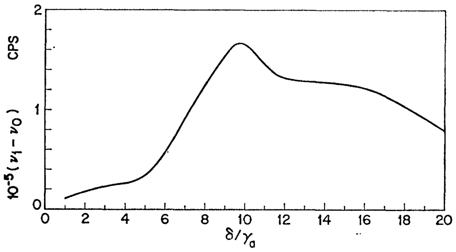


FIG. 7. Beat frequency variation $\nu_1-\nu_0$ between π and σ modes with transverse magnetic field. Laser detuning $\omega_0-\nu_n=8\gamma_a$; $\eta=2$; $\nu/Q=10^6$.

where

$$C_1'=|\Delta_1|^2/(\epsilon_0\hbar Ku),\quad C_2'=\pi^{1/2}|\Delta_1|^4/(\epsilon_0\hbar^3\gamma_a\gamma_d Ku),$$

and $|\Delta_1|^2$ now equals $2|\Delta_0|^2$. There are similar equations for α_2, β_2, and θ_{21} obtained by interchanging ω_1 and ω_2 etc., and by changing the sign of δ in Eq. (48). Figure 8 shows the variation of these parameters for the case when the cavity is initially tuned to the line center. Here the constant terms $\frac{1}{2}\nu C_1'\bar{N}$, and $\frac{1}{32}\nu C_2'\bar{N}$, have been omitted, and the same ratios of γ_a, γ_d, and γ_{ad} used as before.

Again the effects of the coupling constants θ_{21}, θ_{12} are greatest within the natural linewidth of the transition, and tend to a limit of 0.25 as $\delta/\gamma_a\to\infty$. The symmetry which exists with axial magnetic fields is apparent, and the conditions given by Eq. (33) for two frequency operation, and the appearance of circularly polarized beats, are readily satisfied as the magnetic field increases from zero. Near zero magnetic field, however, we have $\theta_{12}=\theta_{21}=\beta_1=\beta_2$, and since $\alpha_1=\alpha_2$ in this case, none of the conditions given by Eqs. (31) through (33) can be satisfied in this region. Here a mutual synchronization, or a locking of the two frequencies may occur, giving rise to linear polarization and to a rotation of the plane of polarization as the magnetic field increases from zero, until two frequency operation occurs. Thus in near zero magnetic fields a single frequency and polarization may be effective and the problem must be formulated in a different way.[1] Figure 9 shows the variation in the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$, etc., for a cavity tuning position given by $\omega_0-\nu_n=8\gamma_a$. A resonance then occurs in β_1 when the Zeeman shift is equal to the cavity detuning, and there are changes in all the other parameters. However, the remarks made above still apply in near zero magnetic fields. Figure 10 shows the

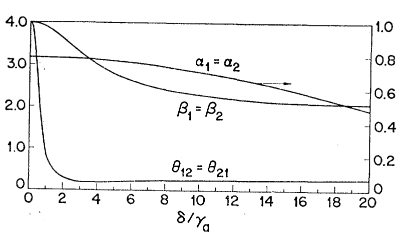


FIG. 8. Variation of the intensity and coupling parameters of the σ modes with axial magnetic field. Laser tuned to line center; $\eta=2$.

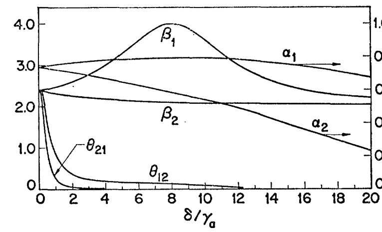


FIG. 9. Variation of the intensity and coupling parameters of the σ modes with axial magnetic field. Laser detuning $\omega_0-\nu_n=8\gamma_a$; $\eta=2$.

intensity parameter $E^2\pi^{1/2}|\Delta_1|^2/(\hbar^2\gamma_a\gamma_d)$, given by the equation analogous to Eq. (36), plotted as a function of δ/γ_a for the cases $\omega_0-\nu_n=0$ and $8\gamma_a$, respectively. The dip in the curve for E_1^2, analogous to the Lamb dip, when the Zeeman shift equals the cavity detuning should be noted.

When simultaneous oscillations occur in right and left-handed circular polarizations, the respective frequencies are again given by equations similar to Eqs. (37) where the parameters are now given by

$$\sigma_1=\tfrac{1}{2}\nu C_1'\bar{N}Z_r(\nu_n-\omega_1),\tag{49}$$

$$\rho_1=\tfrac{1}{32}\nu C_2'\bar{N}[2\gamma_{ad}(\omega_1-\nu_n)\mathcal{L}_{ad}(\omega_1-\nu_n)],\tag{50}$$

$$\begin{aligned}\tau_{12}=\tfrac{1}{32}\nu C_2'\bar{N}\{&\gamma_a[(\omega_0-\nu_n)\mathcal{L}_{ad}(\omega_0-\nu_n)-\delta\mathcal{L}_{ad}(\delta)]\\&+\gamma_a\gamma_d\mathcal{L}_a(2\delta)[\mathcal{L}_{ad}(\omega_1-\nu_n)(\gamma_a(\omega_1-\nu_n)-2\delta\gamma_{ad})\\&-\mathcal{L}_{ad}(\delta)(2\delta\gamma_{ad}+\delta\gamma_a)]\},\end{aligned}\tag{51}$$

with similar equations for σ_2, ρ_2, and τ_{21} obtained by interchanging ω_1 and ω_2 and changing the sign of δ. Figures 11 and 12 show the variation of these parameters with magnetic field for $\omega_0-\nu_n=0$ and $8\gamma_a$, respectively. It is clear that the effects of τ_{12} and τ_{21} on the beat frequency will be greatest at low values of magnetic field where the transitions overlap, and that various resonances in τ_{12}, ρ_1, etc., occur, together with various shifts in the curves due to cavity detuning. Finally, Fig. 13 shows the beat frequency variation with magnetic field for these two cavity tuning positions. The effect of the rapid variation in ρ_1 at $\delta/\gamma_a=8$ shown in Fig. 12, is apparent in curve 2 in Fig. 13 for the beat frequency variation. The apparently large values of beat frequencies shown in Fig. 13 for small values of δ/γ_a are due to the rapid increase in the coupling parameters τ_{12}, and τ_{21}. The beat frequency must, however, be zero magnetic field and we indicate this by the

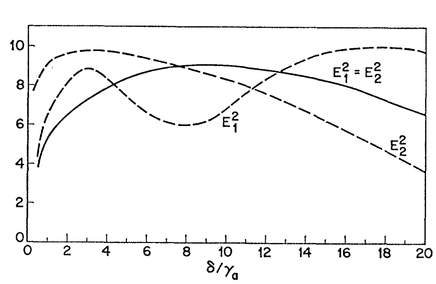


FIG. 10. Intensity parameters $\pi^{1/2}E^2|\Delta_1|^2/(\hbar^2\gamma_a\gamma_d)$ of the σ modes versus axial magnetic field. Full curve—laser tuned to line center. Dashed curves—laser detuning $\omega_0-\nu_n=8\gamma_a$; $\eta=2$.

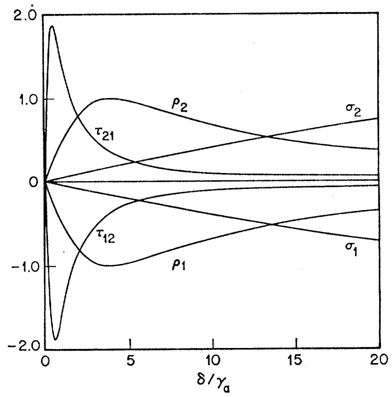


FIG. 11. Variation of the frequency coefficients of the σ modes with axial magnetic field. Laser tuned to line center.

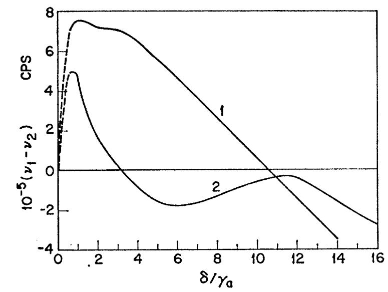


FIG. 13. Beat frequency variation $\nu_1-\nu_2$ between the σ modes with axial magnetic field. Curve 1—Laser tuned to line center. Curve 2—Laser detuning $\omega_0-\nu_n=8\gamma_a=2$; $\nu/Q=10^6$.

dashed portion of the curves. Since frequency-locking effects occur in this region of magnetic field it must be investigated somewhat differently, and this rapid increase in beat frequency may not be observed in practice. Similar remarks as regards a locking or mutual synchronization of the oscillations will also apply for any other value of magnetic field at which the beat frequency passes through zero.

5. EXPERIMENTAL RESULTS

No experimental results on the specific $J=1\rightarrow 0$ transition, to which the theory strictly applies, are available at present. However, the effects of both axial and transverse magnetic fields on the 1.153μ ($J=1\rightarrow 2$) He-Ne laser transition have been investigated experimentally. The results for axial magnetic fields have been given already in some detail and display the general features of the theory derived using the simpler energy-level schemes.[1] Thus the beat frequency between the circularly polarized oscillations as a function of magnetic field displays the main characteristics shown in Fig. 13, and the mutual synchronization of these oscillations and the rotation of the plane of polarization in regions of magnetic field, where the beat frequency passes through zero, have been observed. Such general agreement with results derived using simpler energy levels schemes are, however, suggestive only that similar effects occur in this more complicated laser transition. Any detailed comparison of these experiments with theory will require the consideration of the more complicated $J=1\rightarrow 2$ transition in a similar way. The results given here on the effects of transverse magnetic fields on the $1.153\,\mu$ He-Ne laser transition did indicate that the phenomena involved were different from those applicable to axial magnetic fields, especially in near zero fields, and provided the impetus for the theory given above.

A short, planar-type laser oscillating essentially in a single axial mode was used in the experiments, and transverse magnetic fields were applied with a suitable coil giving about 11.5 G/A. The uniformity of the field was around 4% over the complete length of the discharge and better than 1% up to within 1 cm from the end of the discharge. Inadvertently, it also produced an axial field component which in some places appeared as high as one tenth of the transverse field. As in previous experiments the discharge and the coil were placed inside a magnetic-shielding box.

Figure 14 shows the variation in laser intensity when a transverse magnetic field is applied along the direction of polarization in zero magnetic field. This then corresponds to the variation in the intensity of the π mode of oscillation. Here an analyzer was oriented so as to pass this direction of polarization. The constancy of the intensity of the π mode of oscillation up to certain

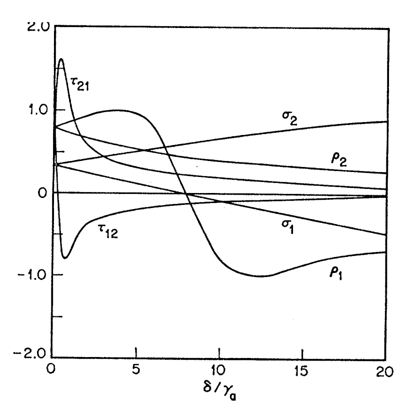


FIG. 12. Variation of the frequency coefficients of the σ modes with axial magnetic field. Laser detuning $\omega_0-\nu_n=8\gamma_a$.

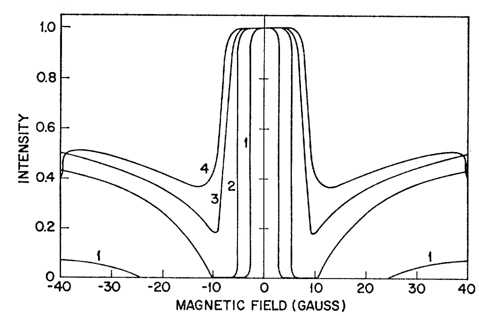


FIG. 14. Variation of π mode intensity with transverse magnetic field. Laser tuning $\Delta f=(\omega_0-\nu_n)/2\pi$. Curve No. 1, $\Delta f\approx 0$ (in Lamb dip). No. 2, $\Delta f=120$. No. 3, $\Delta f=185$. No. 4, $\Delta f=220$ Mc/sec. Rf excitation 9 W. Relative laser intensity 1.9.

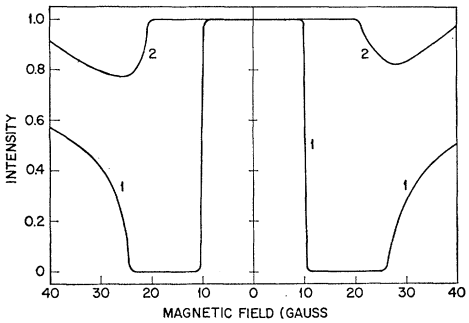
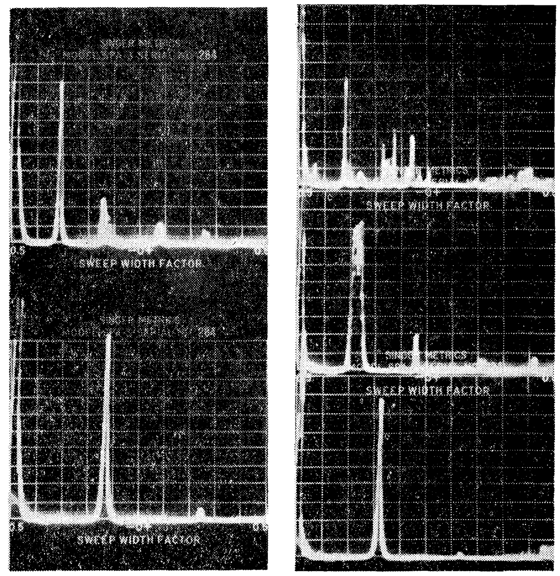


FIG. 15. Variation of π mode intensity with transverse magnetic field. Laser tuning—Curve No. 1, at line center. No. 2, $\Delta f=235$ Mc/sec. rf excitation 5 W. Relative laser intensity 0.4.



(a) (b)

FIG. 17. Beat spectra in the transition between the finite and zero beat frequency regions with transverse magnetic fields. (a) Top trace—Magnetic field 50 G. Bottom trace—Increased to 65 G. (b) Slightly different cavity setting. Magnetic field 39, 41, and 50 G from top to bottom. Spectrum analyzer sweep 20 kc/sec/cm. Zero frequency on left of all traces.

values of magnetic field, depending on the cavity tuning, is apparent. At these values of magnetic field, a more or less abrupt change to the σ mode of oscillations occurs for some cavity tuning positions. This is the zero intensity gap in these curves, since the analyzer is at 90° to the σ polarization. On increasing the magnetic further both π and σ modes begin to oscillate and low-frequency beats appear at magnetic fields around 30 G. For the curves of Fig. 14 which do not show a region of zero intensity there are indications that both π and σ oscillations occur on the slope portions of these curves, but which are synchronized to a single frequency and give rotation effects similar to those obtained with axial magnetic fields. At higher values of magnetic field, low-frequency beats finally appear in all cases. The effects observed thus depend on the operating conditions, and vary with cavity tuning and anisotropy in the cavity Q values for the two polarizations, although the general features shown in Fig. 14 persist. Figure 15 shows similar curves for a lower value of laser intensity in zero magnetic field, where it is apparent the magnetic field at which the oscillation changes from the π mode to the σ mode is increased. Low-frequency beats again occur at magnetic fields around 30 G. For these results the magnetic field was applied in the direction of polarization in zero field.

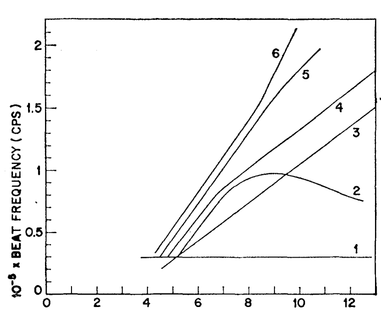


FIG. 16. Experimental curves of beat frequency versus transverse magnetic field. Curve No. 1. In Lamb dip. No. 2 Laser detuning $\Delta f=150$, No. 3, $\Delta f=200$. No. 4, $\Delta f=250$. No. 5, $\Delta f=300$. No. 6, $\Delta f=340$ Mc/sec. Transverse magnetic field 11.5 G/A.

When it is applied perpendicular to this direction no 90° change in polarization, or rotation, was observed, though low-frequency beats occurred at higher values of magnetic field.[10]

Figure 16 shows the beat frequency versus magnetic field for the π mode of operation and for various cavity tuning positions. In contrast to previous results with an axial magnetic field, when the cavity is tuned on or near line center we see that the beat frequency between the π and σ modes is independent of magnetic field, at least up to fields around 100 G. Slightly off, but near line center, a region occurs where the beat frequency after an initial increase, would return to zero again as the field is increased. This zero position has been observed but is not shown in the figure. Finally, a transition region with multiple beats is sometimes observed, which is similar to that seen with axial magnetic fields.[1] This is most predominant in the regions of zero beat frequency, and is not observed when the cavity tuning is far from line center. Figures 17(a) and 17(b) show typical beat spectra obtained in such regions. It is to be noted that at times the beats are not regular, or harmonically related. As the magnetic field is increased beyond this region, only one beat between the σ and π modes is finally observed, suggesting that both σ transitions are combined by the oscillation with this polarization, which combines with the π mode of

[10] The probable reason for this different behavior is that the oscillation is already in the σ direction on applying the magnetic field. For a $J=1\rightarrow 0$ transition no change in polarization would then be expected. This follows from Eq. (32), where now $\alpha_1>\alpha_0$.

oscillation to give a single beat frequency, instead of the two which would otherwise be observed.

6. CONCLUSIONS

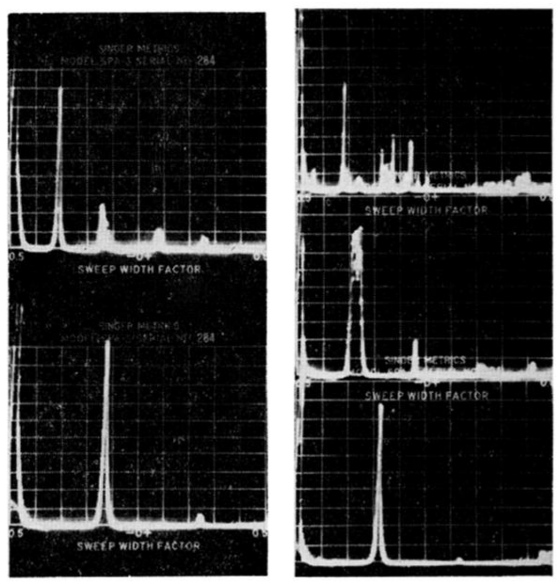
We have developed equations which govern the intensities and the frequencies of the π and σ modes when a transverse magnetic field is applied to a planar-type laser operating on the $J=1\rightarrow 0$ transition and in a single axial mode of the resonator. The results show that the coupling of the transitions, due to the common lower level, and determined by the coefficients θ_{01} and θ_{10}, plays an important role, particularly when the Zeeman separation is within the natural linewidth of the transition. Such a coupling may produce a quenching of the original π mode of oscillation, i.e., magnetic field applied along the direction of polarization in zero field, together with a more or less abrupt change in the polarization of the oscillation to the σ direction. The value of magnetic field at which this occurs is dependent on the operating conditions, and theory indicates that this value will be higher for lower levels of laser intensity, and for larger values of cavity detuning from the line center. As the magnetic field is increased the conditions for two frequency operation are satisfied, and since the two σ modes are combined in the theory, a single beat between the π and σ modes with a frequency dependent on the magnetic field, and on the operating conditions, should occur.

For a given detuning of the cavity from the line center, the beat frequency is zero in zero magnetic field, but it may also pass through zero again at higher values of magnetic field due to the relative balance between frequency pushing and pulling effects. In such regions of zero beat frequency there is a natural tendency for the oscillations to synchronize due to nonlinear effects, and the theory would need some modification in a way similar to that recently applied to consider a similar effect with an axial magnetic field.[1] The above coupling coefficients, and also those denoted by τ_{01} and τ_{10} in the equations for the frequencies of separate π and σ oscillations, depend on the linewidths and on the cavity tuning, but in general they become small at higher values of magnetic field, and may be neglected as a first approximation in such regions. When the cavity is tuned to the line center the beat frequency should remain zero for all values of transverse magnetic field, which may be useful as a means of centering the laser frequency on the Doppler distribution. Changes in the intensities of the π and σ modes at Zeeman separations corresponding to the cavity detuning are also indicated, as well as rapid variations in the beat frequency when the third order dispersion term ρ_1 passes through zero as shown in Figs. 6 and 7.

Similar results have been given for an axial magnetic field acting on the $J=1\rightarrow 0$ transition, and somewhat similar remarks apply. However, the conditions for two-frequency operation in right- and left-handed circular polarizations are more readily satisfied in this case, and the quenching phenomena does not occur. In zero-magnetic-field conditions are such that the oscillations should again become synchronized, and a rotation of the resultant polarization with magnetic field should be observed in this region, and in other regions of magnetic field where the beat frequency passes through zero. No such observations on this particular laser transition have, however, been made as yet. The same general remarks made above on the coupling coefficients apply, and in general their effects will be small at the higher values of magnetic field. Sharp resonances occur, particularly in the coefficients τ_{12} and τ_{21} in near zero magnetic fields, which apparently give rise to a sharp variation in the beat frequency, as shown in Figs. 11, 12, and 13. Such regions may be modified, however, when the very tight coupling or synchronization between the oscillations in such regions is considered.

The experimental results given on the effect of transverse magnetic fields on the $1.153\,\mu$, $J=1\rightarrow 2$, He-Ne laser transition represent the only such results available at present. Hence the theoretical results deduced here do not strictly apply in their entirety. However, the agreement as regards the general features of the observations is certainly suggestive that similar effects are occurring in these experiments. Thus we see a quenching of the initial π mode of oscillation, and a change to the σ mode of operation. As indicated by the theory the magnetic field at which this occurs increases with a decrease in laser intensity and with increased cavity detuning from the line center. Similar beat-frequency variations with magnetic field occur, and there is a region near the line center where the beat frequency remains constant with magnetic field. The theory indicates a zero beat frequency at line center, and the origin of the constant value of beat frequency in the experiments, see Fig. 16, is not clear, although there is some uncertainty in the exact experimental conditions at this time. One might expect from symmetry conditions that the beat frequency would be zero on line center even with this more complicated laser transition. However, any further comparison of the theory must await further definitive experimental observations on the particular laser transitions $J=1\rightarrow 0$ used in the present deductions, or an extension of the theory to the more complicated $J=1\rightarrow 2$ laser transition.[11]

[11] *Note added in proof.* Experiments on a dc excited $J=1\rightarrow 0$ laser transition at $2.65\,\mu$, using an enriched sample of Xe^{136}, show that quenching effects actually occur between axial modes acting on well resolved σ components in an axial magnetic field. There are also indications of similar quenching effects between the σ oscillations on a single mode in axial magnetic fields of a few gauss. Such results are not predicted in the present theory. One possible explanation is that relaxation effects on transitions between the Zeeman levels also occur due to collision effects within the discharge. We hope to discuss these later results in a further communication.



(a) (b)

FIG. 17. Beat spectra in the transition between the finite and zero beat frequency regions with transverse magnetic fields. (a) Top trace—Magnetic field 50 G. Bottom trace—Increased to 65 G. (b) Slightly different cavity setting. Magnetic field 39, 41, and 50 G from top to bottom. Spectrum analyzer sweep 20 kc/sec/cm. Zero frequency on left of all traces.