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Longitudinal Critical Current in Type-II Superconductors

R. G. BOYD
Inorganic Materials Research Division, Lawrence Radiation Laboratory, University of California, Berkeley, California

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The longitudinal critical current of a bulk type-II superconductor near the transition field H_{c2} has been calculated from the Ginzburg-Landau equations via a simple extension of Abrikosov's treatment of the mixed state, valid when the self-field of the current can be considered negligible compared with the external field. The critical current is given by

$$(4\pi J_c/c)=(2/3)^{3/2}(H_{cb}/\lambda)(1-H/H_{c2})^{3/2}/\beta(1-1/2\kappa^2).$$

This expression should be obeyed very near H_{c2} by small wires with strong pinning and with surfaces properly treated to inhibit surface currents.

1. INTRODUCTION AND DISCUSSION

We have calculated the longitudinal critical current of a bulk type-II superconductor very near the transition field H_{c2}, neglecting the self-field of the current (which goes to zero at H_{c2}). The calculation is a simple extension of the Abrikosov[1] treatment of the mixed state (based on the Ginzburg-Landau[2] equations). The result is

$$(4\pi J_c/c)=(\tfrac{2}{3})^{3/2}(H_{cb}/\lambda)(1-H/H_{c2})^{3/2}/\beta(1-1/2\kappa^2).$$

This differs from the result obtained by a simple free-energy argument[3] only by a numerical factor of order unity. It is of the same form and order of magnitude as the critical surface current density,[4,5] except for the substitution of H_{c2} for H_{c3}. Although the result is derived subject to the approximation $H_{c2}-H\ll H_{c2}$, it is interesting to note that for zero field the result is identical to the Ginzburg-Landau critical current of a thin film in zero field[6] except for the numerical factor $\beta(1-1/2\kappa^2)$, which is of order unity.

The self-field of a current is most likely to be negligible compared to the external field for wires of small radius in a longitudinal field H very near H_{c2}. This is the limit in which surface currents are particularly important.[4,5,7,8] In order to observe the bulk critical current it may be necessary to treat the surface appropriately to inhibit surface currents. It is known[7] that a normal metal such as copper will inhibit surface currents; perhaps a magnetic metal[9] would be even more effective.

A nominally longitudinal critical current in a uniform external field gives rise to helical fluxoids which approach straight lines as $H\rightarrow H_{c2}$. The current may have a small component transverse to the local magnetic field, in which case the flux lines must be pinned to prevent flux flow if we are to expect our expression to apply. Such scanty data as are available on longitudinal critical currents indicate that our predicted values are approached for materials with strong pinning[10] but not for materials with weak pinning.[3,11] Although our calculation is based on a model in which pinning is absent, it seems necessary to have some pinning in order to make it applicable. We are not troubled by the apparent logical inconsistency, since physicists generally tolerate the same sort of logical inconsistency when they calculate the thermodynamic equilibrium of a noninteracting gas (which needs to interact if it is to achieve thermodynamic equilibrium).

2. DERIVATION OF LONGITUDINAL CRITICAL CURRENT

We shall follow Abrikosov[1] closely. We assume an infinite superconductor. The external field $\mathbf{H}_0$ and vector potential $\mathbf{A}$ are assumed to be directed along the z and y axes, respectively. The units of field and length are the thermodynamic critical field $\sqrt{2}H_{cb}$ and penetration depth λ, respectively. In these units the Ginzburg-Landau[2] equations can be written

$$\left(\frac{i\nabla}{\kappa}+\mathbf{A}\right)^2\Psi=\Psi(1-|\Psi|^2), \tag{1}$$

$$-\nabla\times(\nabla\times\mathbf{A})=\mathbf{A}|\Psi|^2+\frac{i}{2\kappa}(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*). \tag{2}$$

Near the transition field $|\Psi|^2\ll1$, so that we may approximate the field with

$$|\mathbf{A}|=A_y=H_0x. \tag{3}$$

[1] A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)].
[2] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).
[3] G. D. Cody and G. W. Cullen, RCA Technical Report No. AFML-TR-65-169, p. 48, 1965 (unpublished); Bull. Am. Phys. Soc. **10**, 360 (1965).
[4] A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **47**, 720 (1964) [English transl.: Soviet Phys.—JETP **20**, 480 (1965)].
[5] J. G. Park, Phys. Rev. Letters **15**, 352 (1965).
[6] G. Rickayzen, *Theory of Superconductivity* (Interscience Publishers, Inc., New York, 1965), p. 317.
[7] P. S. Swartz and H. R. Hart, Jr., Phys. Rev. **137**, A818 (1965).
[8] H. J. Fink and L. J. Barnes, Phys. Rev. Letters **15**, 792 (1965).
[9] M. A. Woolf and F. Reif, Phys. Rev. **137**, A557 (1965). *Note added in proof.* In experimental work done at Atomic International, L. J. Barnes and H. J. Fink have shown that plating with magnetic metals eliminates hysteresis near H_{c2} due to surface currents. Herman Fink, private communication.
[10] S. T. Sekula, R. W. Boom, and C. J. Bergeron, Appl. Phys. Letters **2**, 102 (1963).
[11] J. W. Heaton and A. C. Rose-Innes, Phys. Letters **9**, 112 (1964).

We wish to introduce a longitudinal transport current but neglect the magnetic field due to this current, which amounts to neglecting A_z. We need a longitudinal phase gradient to yield a longitudinal current. The longitudinal phase gradient must be uniform to preserve the longitudinal translational invariance of the transverse currents. Accordingly, we must include a phase factor $\exp(ijz)$ in Ψ which gives rise to a longitudinal current proportional to $|\Psi|^2$; the current therefore avoids the vortex cores.

We assume that Ψ is otherwise a function of x only:

$$\Psi=\exp(ijz)f(x). \tag{4}$$

Then, neglecting the term $\Psi|\Psi|^2$, we obtain for $f(x)$ the harmonic-oscillator equation

$$d^2f/dx^2+(\kappa^2-j^2)[1-\kappa^2H_0^2x^2/(\kappa^2-j^2)]f=0, \tag{5}$$

which has solutions when $H_0=(\kappa^2-j^2)/\kappa(2n+1)$. The nucleation field is given by the largest eigenvalue

$$H_0=(\kappa^2-j^2)/\kappa, \tag{6}$$

and corresponds to the solution

$$\Psi=\exp(ijz)\exp[-(\kappa^2-j^2)x^2/2]. \tag{7}$$

Equation (1) is also satisfied by the functions

$$\Psi=\exp\{ijz+iky-\tfrac{1}{2}(\kappa^2-j^2)[x-k/(\kappa^2-j^2)]^2\}. \tag{8}$$

We choose a general solution of the form

$$\Psi=e^{ijz}\sum_n C_ne^{ikny}\psi_n(x),$$
$$\psi_n=\exp\{-\tfrac{1}{2}(\kappa^2-j^2)[x-kn/(\kappa^2-j^2)]^2\}, \tag{9}$$

where k, C_n are arbitrary constants.

Let $\mathbf{A}_0$ be the vector potential for the nucleation field,

$$A_0=(\kappa^2-j^2)x/\kappa. \tag{10}$$

We note the useful identity

$$\partial\Psi/\partial x=-\kappa[(i/\kappa)\partial/\partial y+A_0]\Psi, \tag{11}$$

which may be easily verified by differentiation of Eq. (9).

We substitute Ψ and A_0 into the equation for the current, Eq. (2), to obtain the first-order correction to A. On making use of the identity Eq. (11), we get

$$\partial^2A/\partial x^2=-(1/2\kappa)(\partial/\partial x)|\Psi|^2, \tag{12}$$

$$H=\partial A/\partial x=H_0-(1/2\kappa)|\Psi|^2, \tag{13}$$

$$A=H_0x-\frac{1}{2\kappa}\int^x|\Psi|^2\,dx' \tag{14}$$

which are identical to Abrikosov's results, since the phase j is contained only in $|\Psi|^2$. The constant H_0 was shown by Abrikosov to be the external field strength.

We now add to the factors $C_n\psi_n$ in Eq. (9) small terms $\psi_n^{(1)}$:

$$\Psi=\Psi^{(0)}+\Psi^{(1)}=\sum e^{ikny}(C_n\psi_n+\psi_n^{(1)}). \tag{15}$$

We substitute Ψ together with A given by Eq. (14) into Eq. (1). Keeping terms of first order in small quantities, and subtracting the linear equation satisfied by $\Psi^{(0)}$, we obtain

$$\left[\left(\frac{i\nabla}{\kappa}+\mathbf{A}_0\right)^2-1\right]\Psi^{(1)}=\left[\left(\frac{i\nabla}{\kappa}+\mathbf{A}_0\right)^2-\left(\frac{i\nabla}{\kappa}+\mathbf{A}\right)^2-|\Psi^{(0)}|^2\right]\Psi^{(0)}. \tag{16}$$

On multiplication by $\exp(-ikny)$ and integration over y, we get an inhomogeneous equation for $\psi_n^{(1)}$. For a solution to exist, the inhomogeneous part must be orthogonal to the solution of the corresponding homogeneous equation. But this is simply ψ_n. We multiply the inhomogeneous part by ψ_n, integrate over x, and set the result equal to zero:

$$\int\int dx\,dy\,e^{-ikny}\psi_n\left[\left(\frac{i\nabla}{\kappa}+\mathbf{A}_0\right)^2-\left(\frac{i\nabla}{\kappa}+\mathbf{A}\right)^2-|\Psi|^2\right]\Psi=0. \tag{17}$$

We discard the superscript on Ψ hereafter.

Setting $\mathbf{A}_0+\mathbf{A}\approx 2\mathbf{A}_0$ and discarding $\nabla\cdot\mathbf{A}$, we get from Eq. (17) after applying Eq. (11)

$$\int\int e^{-ikny}\psi_n\left[-\frac{2}{\kappa}(A_0-A)\frac{d}{dx}-|\Psi|^2\right]=0. \tag{18}$$

At this point some caution is necessary. We wish to obtain a polynomial in $|\Psi|^2$ under the integral after we multiply by C_n^* and sum over n. For this purpose we must transfer the derivative to the term (A_0-A) by partial integration, which eliminates x as a coefficient. However, we must do this before performing the sum if the evaluated term is to be zero. The order of operations is as follows: We partially integrate half the derivative term, then multiply by C_n^* and sum over n. After adding the complex conjugate equation, we finally get

$$\int\int|\Psi|^2\left[\frac{1}{\kappa}\frac{d}{dx}(A_0-A)-|\Psi|^2\right]=0. \tag{19}$$

The phase j appears explicitly only in the derivative term. On evaluation we obtain Abrikosov's relation

$$\frac{\kappa^2-j^2-\kappa H_0}{\kappa^2}\langle|\Psi|^2\rangle_{\text{av}}+\left(\frac{1}{2\kappa^2}-1\right)\langle|\Psi|^4\rangle_{\text{av}}=0, \tag{20}$$

in which the quantity $(\kappa^2-j^2)/\kappa$ has taken the place of κ as the bulk nucleation field. From this we obtain

$$\langle|\Psi|^2\rangle_{\text{av}}/2\kappa=[(\kappa^2-j^2)/\kappa-H_0]/(2\kappa^2-1)\beta, \tag{21}$$

where

$$\beta=\langle|\Psi|^4\rangle_{\text{av}}/\langle|\Psi|^2\rangle_{\text{av}}^2 \tag{22}$$

and β is independent of H_0.

The natural unit of current is

$$(c/4\pi)(\sqrt{2}H_{cb}/\lambda),$$

which ensures that $\mathbf{J}=\nabla\times\mathbf{H}$ in this system of units. Then the mean longitudinal current is given by

$$J=(j/\kappa)\langle|\Psi|^2\rangle_{\mathrm{av}}=2j[(\kappa^2-j^2)/\kappa-H_0]/(2\kappa^2-1)\beta. \quad (23)$$

We differentiate J with respect to j and set the derivative equal to zero to find the value of j for which J is a maximum:

$$\partial J/\partial j\propto\kappa^2-\kappa H_0-3j^2=0 \quad (24)$$

or

$$j^2=\tfrac{1}{3}(\kappa^2-\kappa H_0).$$

The maximum occurs at a value of j only $1/\sqrt{3}$ of the maximum possible value of j.

In ordinary cgs units, the critical current is then given by

$$\frac{4\pi J_c}{c}=(\tfrac{2}{3})^{3/2}\frac{H_{cb}}{\lambda}\frac{(1-H/H_{c2})^{3/2}}{\beta(1-1/2\kappa^2)}. \quad (25)$$

The magnetization at the transition falls to two-thirds of its equilibrium value in the absence of current. The smallness of the change in the magnetization accounts for the fact that a simple free-energy argument gives very nearly the same result for the critical current.

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Effects of Transverse and Axial Magnetic Fields on Gaseous Lasers*

W. Culshaw and J. Kannelaud
Lockheed Palo Alto Research Laboratory, Palo Alto, California
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The effects of transverse magnetic fields on a $J=1\rightarrow 0$ laser transition are considered, and expressions for the macroscopic-atomic-polarization terms are derived. The contributions from the σ modes are combined, and equations for the intensities and frequencies of the π and σ oscillations are deduced. Various coupling terms occur because of the common lower level of the transition, and a quenching of the initial π-mode oscillation with a change to the σ mode is indicated as the magnetic field increases from zero. With increased field, both modes oscillate simultaneously, and a beat occurs whose frequency depends on the operating conditions, and which may become zero again at higher magnetic fields. At line center the beat frequency should remain zero with increasing magnetic field, representing a method of laser tuning. Similar equations for the circularly polarized σ oscillations in axial magnetic fields are deduced. Here the conditions for stable two-frequency operation are more readily satisfied, and no such quenching is indicated. Again there are zero-beat-frequency regions of magnetic field in which a mutual synchronization of the oscillations should occur. Some experimental results with transverse magnetic fields on the 1.153-μ He-Ne laser are given. These display the general features indicated by the theory. Thus a quenching of the initial π-mode oscillation occurs, with more or less abrupt changes to the σ mode, depending on conditions. Similar single-beat-frequency variations with magnetic field occur, together with a region near the line center where the beat frequency, although finite, remains constant with increasing magnetic field.

1. INTRODUCTION

THE effects of relatively small magnetic fields on the operating characteristics of gaseous lasers are quite pronounced, and provide a fertile field for the investigation of the dispersive and nonlinear properties of the laser medium. Thus axial magnetic fields of a few tenths of a gauss can produce orthogonal circularly polarized oscillations and hence beat frequencies in a planar-type laser, which in zero magnetic field oscillated on a single frequency and was linearly polarized.[1] The beat frequency observed as the magnetic field increases depends on the detailed shape of the first- and third-order dispersion functions involved in the laser transition, or on the balance between frequency pulling and pushing effects. It is thus dependent on the lifetimes of the states involved, on the cavity tuning and Q value, on the applied magnetic field, and on the dispersive properties of the laser medium. When the magnetic field is zero the beat frequency is zero and the polarization is linear with a direction determined by small anisotropies, chiefly in the reflectors, of the laser cavity. The beat frequency may also pass through zero in one or more regions as the axial magnetic field increases.[1,2] In such regions the oscillations have a natural tendency to coalesce because of nonlinear effects and this again, for small anisotropy, gives rise to a linear

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[1] W. Culshaw and J. Kannelaud, Phys. Rev. **136**, A1209 (1964). See also Phys. Rev. **141**, 228 (1966); **141**, 237 (1966).
[2] R. L. Fork and M. Sargent, III, Phys. Rev. **139**, A617 (1965).