look for the reaction

$$K^- + d \to 2K^+ + 3K^0 + \text{bi-}\Omega^-.$$

We now wish to show that it is not at all unreasonable to expect the bi-Ω^- to be stable. Indeed, it is well known that the stablest particles of triangular $SU(3)$ representations are those with lowest isospin. For instance, the Ω^- is the only stable particle of the baryon decuplet, while the deuteron is the only stable particle of the $B=2$ antidecuplet.[26] Both have zero isospin.

Likewise, the four-baryon system, which belongs (according to our criteria) to the **490*** representation[27] of $SU(6)$, has three stable states, namely the α particle, and the ${}_\Lambda H^4$ and ${}_\Lambda He^4$ hypernuclei.[28] They are, respectively, the isosinglet and isodoublet of the spin-0 anti-28-plet. The analogy of the α particle with the bi-Ω^- (which is the isosinglet of the spin-0 28-plet) is striking.

A further search for stable hyperstringe compounds can also be made among $B=3$ systems. The latter consist of nine quarks (always in s states) and belong to the **48620** representation of $SU(18)$. Again, $SU(3)'$ must be a singlet, and its Young diagram is (003), so that the Young diagram of $SU(6)$ is (00300), yielding the **980** representation.[14,27,29] When the latter is reduced with respect to $SU(2)\otimes SU(3)$, we find **15** distinct representations, the most interesting of which seem to be a spin-$\frac{1}{2}$ 35-plet and a spin-$\frac{1}{2}$ anti-35-plet. The anti-35-plet contains three known stable particles, namely H^3 and He^3 (an isodoublet) and the ${}_\Lambda H^3$ hypernucleus[28] (an isosinglet). Correspondingly, we may expect the 35-plet to contain a stable isodoublet with strangeness -6, and a stable isosinglet with strangeness -5. Their production, however, is hardly conceivable with current techniques—the simplest possibility would be to use a bubble chamber with liquid He^3.

Finally, we must mention that all these $SU(6)$ representations were, of course, known long ago.[14] The only novelty here is that we have found *a way of selecting those representations which are physically interesting*, thus showing us where to look for hyperstrange compounds.

ACKNOWLEDGMENTS

I am very much indebted to Professor Harry J. Lipkin, Professor Yuval Ne'eman, and Professor Paul Singer for many stimulating discussions.

[26] This antidecuplet belongs, of course, to the **490** representation of $SU(6)$.
[27] H. Bebie and S. Iwao, Berne (unpublished report).
[28] R. Levi Setti, Endeavour 24, 119 (1965).
[29] B. Ram, Phys. Rev. **141**, 1581 (1966).

PHYSICAL REVIEW VOLUME 145, NUMBER 4 27 MAY 1966

Nonexponential Decays in a Lee-Type Model

Hugh Osborn
Department of Physics, University College London, London, England
(Received 8 November 1965)

A generalization of the Lee model, due to Bell and Goebel, which allows the existence of double poles on the unphysical sheet of the S matrix, is considered. The time behavior of wave packets formed in the infinite past is investigated for two cases, (i) two distinct complex poles, and (ii) a single double pole, both being cases in weak coupling so that the unstable state formed is long-lived. It is claimed that the type of process considered represents a typical experimental situation in which the effects of double poles would be investigated.

I. INTRODUCTION

THE problem of consistently defining the physical properties, such as energy and lifetime, of an unstable particle or decaying state has been a matter of interest for some time.[1–3] The essential difficulty is that an unstable particle is not a well-defined state but must be considered as a somewhat arbitrary superposition of scattering states. However in scattering theory we are concerned with relating definite asymptotic states in the distant past to similar asymptotic states in the distant future whereas in a theory of unstable systems we are principally concerned with the behavior of the systems over finite periods of time. The general conclusion is that unstable states are associated with complex poles or zeros in the second Riemann sheet of the complex energy plane into which the S matrix, Jost function or propagator can be continued analytically. Except for very short or very long times the dominating term in the probability amplitude for finding the system in its initial state has a pure exponential form as a function of time. This gives rise to the usual exponential-decay law, provided only the poles or zeros are assumed simple.

Recently[4–7] attention has been focused on the possi-

[1] G. Källén and V. Glaser, Nucl. Phys. **2**, 706 (1956).
[2] M. Lévy, Nuovo Cimento **13**, 115 (1959); **14**, 274 (1960).
[3] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), Chap. 8, which contains further references.
[4] M. L. Goldberger and K. M. Watson, Phys. Rev. **136**, B1472 (1964), which is herein referred to as A.
[5] J. S. Bell and C. J. Goebel, Phys. Rev. **138**, B1198 (1965), which is herein referred to as B.
[6] R. J. Eden and P. V. Landshoff, Phys. Rev. **138**, B1817 (1964).
[7] C. J. Goebel and K. W. McVoy, Ann. Phys. (N. Y.) (to be published).

bility of other than pure exponential decays. There is no *a priori* reason why higher order poles or zeros should not occur and in A[4] the consequences for decay curves of higher order complex poles of any order in the S matrix were considered. In B[5] calculations based on a generalization of the Lee model showing the existence of double poles and their consequences for the time behavior of the system were studied. In this paper the model is reconsidered in a different way.

The main part of the paper consists of four sections, II and III are only of a preliminary nature, the main results are in Secs. IV and V. In Sec. II the model is set up and the time behavior for an arbitrary initial state considered in Sec. III. In Sec. IV the whole process is looked at more carefully in terms of relationship to experiment and transition to momentum rather than energy variables is made. In Sec. V a point source only is considered and the decay laws for two distinct poles in weak coupling and a single double pole found. Conclusions are drawn in Sec. VI and a few mathematical details presented in three appendices.

II. THE MODIFIED LEE MODEL

The model considered is an obvious generalization of the Lee model,[8] the generalization amounting to the introduction of an additional internal state, but it is also applicable to systems which to some approximation can be considered as a single state coupled to one other discrete state and also to continuum states. This case might arise in some atomic problems. Nonrelativistic kinematics is assumed and as is usual in the Lee model a form factor is introduced to supply convergence for the various integrals. Since there is no need here, details of renormalization are not considered.

The system consists of two particles in which the state vector space for the relative motion is spanned by free-particle states $|\mathbf{p}\rangle$ of momentum $\mathbf{p}$ and two discrete states $|\chi_1\rangle$ and $|\chi_2\rangle$.

The orthonormality and completeness relations are

$$\begin{aligned}&\langle\chi_i|\chi_j\rangle=\delta_{ij},\quad \langle\chi_i|\mathbf{p}\rangle=0,\quad i,j=1,2\\&\langle\mathbf{q}|\mathbf{p}\rangle=\delta(\mathbf{q}-\mathbf{p}),\\&1=|\chi_1\rangle\langle\chi_1|+|\chi_2\rangle\langle\chi_2|+\int|\mathbf{p}\rangle d\mathbf{p}\langle\mathbf{p}|.\end{aligned}\tag{2.1}$$

The free Hamiltonian H_0 is

$$H_0=|\chi_1\rangle E_1\langle\chi_1|+|\chi_2\rangle E_2\langle\chi_2|+\int|\mathbf{p}\rangle d\mathbf{p}\frac{p^2}{2m}\langle\mathbf{p}|.\tag{2.2}$$

The interaction Hamiltonian H_I introduces a coupling so that

$$|\chi_1\rangle\leftrightarrow|\chi_2\rangle\leftrightarrow|\mathbf{p}\rangle,$$

$$H_I=\frac{G}{(2\pi)^{3/2}}\int d\mathbf{p}\{U^*(p)|\mathbf{p}\rangle\langle\chi_2|+U(p)|\chi_2\rangle\langle\mathbf{p}|\}+g\{|\chi_1\rangle\langle\chi_2|+|\chi_2\rangle\langle\chi_1|\}.\tag{2.3}$$

[8] L. Fonda, G. C. Ghirardi, and A. Rimini, Phys. Rev. **133**, B196 (1964) consider the classical Lee model in a way similar to that here.

G, g are real coupling constants, and $U(p)$ is a form factor.

The scattering state for outgoing waves is as usual

$$|\mathbf{p}\rangle^+=|\mathbf{p}\rangle+\frac{1}{p^2/2m-H+i\epsilon}H_I|\mathbf{p}\rangle,\tag{2.4}$$

where ϵ is real, positive, and infinitesimal. For ingoing waves

$$|\mathbf{q}\rangle^-=|\mathbf{q}\rangle^++2\pi i\delta(q^2/2m-H)H_I|\mathbf{q}\rangle\tag{2.5}$$

The S-matrix element for plane waves, using (2.4) and (2.5), is

$$\langle\mathbf{q}|S|\mathbf{p}\rangle={}^-\langle\mathbf{q}|\mathbf{p}\rangle^+=\delta(\mathbf{q}-\mathbf{p})-2\pi i\delta(q^2/2m-p^2/2m)\langle\mathbf{q}|T|\mathbf{p}\rangle,\tag{2.6}$$

$$\begin{aligned}\langle\mathbf{q}|T|\mathbf{p}\rangle&=\langle\mathbf{q}|H_I|\mathbf{p}\rangle+\langle\mathbf{q}|H_I\frac{1}{p^2/2m-H+i\epsilon}H_I|\mathbf{p}\rangle\\&=\frac{G^2}{(2\pi)^3}|U(p)|^2\langle\chi_2|\frac{1}{p^2/2m-H+i\epsilon}|\chi_2\rangle\\&=\frac{G^2}{(2\pi)^3}|U(p)|^2S_2'\left(\frac{p^2}{2m}+i\epsilon\right),\end{aligned}\tag{2.7}$$

$$S_2'(z)=\langle\chi_2|\frac{1}{z-H}|\chi_2\rangle.\tag{2.8}$$

$S_2'(z)$ would be the propagator of the χ_2 particle in a field theory. It is easy to see from (2.7) that the scattered wave is spherically symmetrical so only s waves are present.

Writing in the usual way

$$\langle\mathbf{q}|S|\mathbf{p}\rangle=\sum_{lm}Y_{lm}(\hat{q})Y_{lm}^*(\hat{p})\frac{1}{p^2}S_l(p)\delta(q-p),$$

we have $S_l(p)=1$ for $l=1, 2, 3\cdots$ since there is no scattering, and

$$S_0(p)=1-4\pi 2\pi i(G^2/(2\pi)^3)\times mp|U(p)|^2S_2'(p^2/2m+i\epsilon).\tag{2.9}$$

Thus the propagator $S_2'(p^2/2m+i\epsilon)$ plays an essential role in the model, its poles determining those of the S matrix.

In Appendix I it is shown that

$$S_2'(z)=\left[z-E_2-\frac{g^2}{z-E_1}-\frac{G^2}{2\pi^2}\int_0^\infty dk\frac{k^2|U(k)|^2}{z-k^2/2m}\right]^{-1}=\frac{z-E_1}{h_0(z)},\tag{2.10}$$

$$h_0(z)=(z-E_1)\times\left[z-E_2-\frac{G^2}{2\pi^2}\int_0^\infty dk\frac{k^2|U(p)|^2}{z-k^2/2m}\right]-g^2.\tag{2.11}$$

The term in square brackets is just the V-particle propagator in the ordinary nonrelativistic Lee model. $h_0(z)$ is analytic in the finite z plane cut along the real axis from $z=0$ to $z=\infty$.

$$h_0(z)\sim z^2 \quad \text{as} \quad |z|\to\infty\,,$$

$$h_0\left(\frac{p^2}{2m}+i\epsilon\right)-h_0\left(\frac{p^2}{2m}-i\epsilon\right)=\left(\frac{p^2}{2m}-E_1\right)2\pi imp\frac{G^2}{2\pi^2}|U(p)|^2. \quad (2.12)$$

Using (2.12) in (2.9), we have

$$S_0(p)=h_0(p^2/2m-i\epsilon)/h_0(p^2/2m+i\epsilon)=e^{2i\delta_0}. \quad (2.13)$$

This exhibits the unitarity of the S matrix in this model and shows how S_0 can be continued analytically to the first sheet in the complex-energy plane. In Appendix II it is shown that S_0 can only have poles on the negative axis and that at most there can be two poles corresponding to two bound states which are strictly separate, i.e., this model cannot give a double pole describing bound states as is expected generally.

III. TIME BEHAVIOR FOR A GENERAL STATE

We now proceed to a discussion of the time evolution of the system from an arbitrary initial state $|\Psi(0)\rangle$ at $t=0$:

$$|\Psi(t)\rangle=e^{-iHt}|\Psi(0)\rangle. \quad (3.1)$$

By using the method of Laplace transforms[9] we can obtain

$$|\Psi(t)\rangle=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}dE\, e^{-iEt}\frac{1}{E-H+i\epsilon}|\Psi(0)\rangle. \quad (3.2)$$

Suppose the initial state is given by

$$|\Psi(0)\rangle=N_1|\chi_1\rangle+N_2|\chi_2\rangle+\int d\mathbf{k}a(\mathbf{k})|\mathbf{k}\rangle. \quad (3.3)$$

In Appendix I it is shown that

$$\begin{aligned}
\frac{1}{z-H}|\chi_2\rangle&=\left\{\frac{g}{z-E_1}|\chi_1\rangle+|\chi_2\rangle+\frac{G}{(2\pi)^{3/2}}\int d\mathbf{k}\frac{U^*(k)}{z-k^2/2m}|\mathbf{k}\rangle\right\}S_2'(z)\,,\\
\frac{1}{z-H}|\chi_1\rangle&=\frac{1}{z-E_1}\left\{|\chi_1\rangle+\frac{g}{z-H}|\chi_2\rangle\right\}\,,\\
\frac{1}{z-H}|\mathbf{k}\rangle&=\frac{1}{z-k^2/2m}\left\{|\mathbf{k}\rangle+\frac{G}{(2\pi)^{3/2}}U(k)\frac{1}{z-H}|\chi_2\rangle\right\}.
\end{aligned} \quad (3.4)$$

Using (3.2), (3.3) and (3.4), we obtain

$$|\Psi(t)\rangle=\mathrm{N}_1e^{-iE_1t}|\chi_1\rangle+\int d\mathbf{k}\, a(\mathbf{k})e^{-i(k^2/2m)t}|\mathbf{k}\rangle-\frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dz\, e^{-izt}S_2'(z)$$
$$\times\left\{\frac{gN_1}{z-E_1}+N_2+\frac{G}{(2\pi)^{3/2}}\int d\mathbf{k}\frac{U(k)}{z-k^2/2m}a(\mathbf{k})\right\}\left\{\frac{g}{z-E_1}|\chi_1\rangle+|\chi_2\rangle+\frac{G}{(2\pi)^{3/2}}\int d\mathbf{k}\frac{U^*(k)}{z-k^2/2m}|\mathbf{k}\rangle\right\}. \quad (3.5)$$

The probability amplitudes of finding the system in the various possible channels at time t is

$$\begin{aligned}
\chi_1(t)&=\langle\chi_1|\Psi(t)\rangle=N_{1e}{}^{-iE_1t}-\frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dz\, e^{-izt}S_2'(z)G_0(z)\frac{g}{z-E_1}\\
\chi_2(t)&=\langle\chi_2|\Psi(t)\rangle=-\frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dz\, e^{-izt}S_2'(z)G_0(z)\\
\phi(\mathbf{k},t)&=\langle\mathbf{k}|\Psi(t)\rangle=a(\mathbf{k})e^{-i(k^2/2m)t}-\frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dz\, e^{-izt}S_2'(z)G_0(z)\frac{G}{(2\pi)^{3/2}}\frac{U^*(k)}{z-k^2/2m}\,,
\end{aligned} \quad (3.6)$$

where

$$G_0(z)=\frac{gN_1}{z-E_1}+N_2+\frac{G}{(2\pi)^{3/2}}\int d\mathbf{k}\frac{U(k)}{z-k^2/2m}a(\mathbf{k}). \quad (3.7)$$

[9] Reference 3, p. 433.

By suitable manipulation Eqs. (3.6) can be cast into simpler form:

$$\chi_1(t)=-\frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dz\frac{e^{-izt}}{h_0(z)}\left[N_1\left\{z-E_2-\frac{G^2}{2\pi^2}\int_0^\infty dk\frac{k^2|U(k)|^2}{z-k^2/2m}\right\}+g\left\{N_2+\frac{G}{(2\pi)^{3/2}}\int dk\frac{U(k)}{z-k^2/2m}a(k)\right\}\right],$$

$$\chi_2(t)=-\frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dz\frac{e^{-izt}}{h_0(z)}\left[N_1g+(z-E_1)\left\{N_2+\frac{G}{(2\pi)^{3/2}}\int d\mathbf{k}\frac{U(k)}{z-k^2/2m}a(\mathbf{k})\right\}\right],\tag{3.8}$$

$$\phi(\mathbf{k},t)=a(\mathbf{k})e^{-i(k^2/2m)t}-\frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dz\frac{e^{-izt}}{h_0(z)}\frac{G}{(2\pi)^{3/2}}\frac{U^*(k)}{z-k^2/2m}\left[N_1g+(z-E_1)\left\{N_2+\frac{G}{(2\pi)^{3/2}}\int d\mathbf{p}\frac{U(p)}{z-p^2/2m}a(\mathbf{p})\right\}\right].$$

The probability amplitude of finding the system in its initial state at a time t is

$$\begin{aligned}A(t)&=\langle\Psi(0)|\Psi(t)\rangle=N_1^*\chi_1(t)+N_2^*\chi_2(t)+\int d\mathbf{k}a^*(\mathbf{k})\phi(\mathbf{k},t)\\&=\int d\mathbf{k}|a(\mathbf{k})|^2e^{-i(k^2/2m)t}-\frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dz\frac{e^{-izt}}{h_0(z)}\left[|N_1|^2\left\{z-E_2-\frac{G^2}{2\pi^2}\int_0^\infty dk\frac{k^2|U(k)|^2}{z-k^2/2m}\right\}+|N_2|^2g(z-E_1)\right.\\&\quad+(N_1^*N_2+N_1N_2^*)g+\frac{G^2}{(2\pi)^3}\int d\mathbf{k}\frac{U^*(k)}{z-k^2/2m}a^*(\mathbf{k})\int d\mathbf{p}\frac{U(p)}{z-p^2/2m}a(\mathbf{p})+\frac{G}{(2\pi)^{3/2}}g\left\{N_1^*\int d\mathbf{k}\frac{U(k)}{z-k^2/2m}a(\mathbf{k})\right.\\&\quad\left.\left.+N_1\int d\mathbf{k}\frac{U^*(k)}{z-k^2/2m}a^*(\mathbf{k})\right\}+(z-E_1)\frac{G}{(2\pi)^{3/2}}\left\{N_2^*\int d\mathbf{k}\frac{U(k)}{z-k^2/2m}a(\mathbf{k})+N_2\int d\mathbf{k}\frac{U^*(k)}{z-k^2/2m}a^*(\mathbf{k})\right\}\right].\end{aligned}\tag{3.9}$$

The above expressions are rather complicated but the essential behavior at different times is brought out by manipulating the contour in different ways. It is easily seen that the only factor which can produce poles in the integrand is $[h_0(z)]^{-1}$. This has, as shown in Appendix II only real bound-state poles on the first sheet, but in the usual way the unstable states are defined by the poles of $[h_0(z)]^{-1}$ on the second sheet. Assuming the analytical continuation of $[h_0(z)]^{-1}$ and the other terms in (3.8) and (3.9) from $\text{Im}z>0$ across the real axis cut into the second sheet is possible, then the poles of $[h_0(z)]^{-1}$ may be made to occur explicitly when the integration contour is transformed as shown in Fig. 1. Poles in this region give rise to exponentially decaying factors for $t>0$ when the above integrals are evaluated along the contour C_0'. The integral along the contour C_1' gives a contribution which depends on the nature of the initial state. This behaves after a long time like some inverse power of t and so eventually will always dominate the exponentially decreasing terms but this is proportional to some power of G^2 and so for weak coupling is quite small. The nature of the exponentially decreasing term thus depends on the nature of the zeros of $h_0(z)$ in the lower half of the second Riemann sheet.

IV. CONSIDERATION OF A WAVE PACKET

The results of Sec. III give the time evolution from an arbitrary initial state and it would now be possible to consider different special cases leading to different decay laws. This is done in B [the first two of Eqs. (3.8) are the parallel of the first two of Eqs. (9) in B] but this procedure is rather artificial. To obtain greater realism and to reduce the complexity of the integrals somewhat it is necessary to inquire more carefully about what is actually measured in any practical investigation of such unstable states as occur here. Essentially the preparation apparatus is used in the distant past to form a wave packet of continuum states with the two interacting parts spatially separated and moving toward each other with a fairly well-defined energy and momentum. Thus it is possible to define an approximation time of collision when the interaction occurs. After interaction the probability that the system decays to a continuum state with momentum and energy in some predetermined range, depending on the "bandwidth" of the detection instruments, is measured as a function of time.

The initial wave packet is assumed to be formed at a time t_0 with the relative coordinate of the two interacting particles fairly well defined at a value $\mathbf{b}$. The amplitude in momentum space is $a(\mathbf{p})e^{-i(p^2/2m)t_0}$ [the phase of the amplitude is brought out explicitly so $a(\mathbf{p})$ is essentially real and is peaked around some value $\mathbf{p}_0$]. Initially at $t=t_0$ the state vector is

$$|t_0\rangle=\int d\mathbf{p}\,a(\mathbf{p})e^{-i(p^2/2m)t_0}|\mathbf{p}\rangle.\tag{4.1}$$

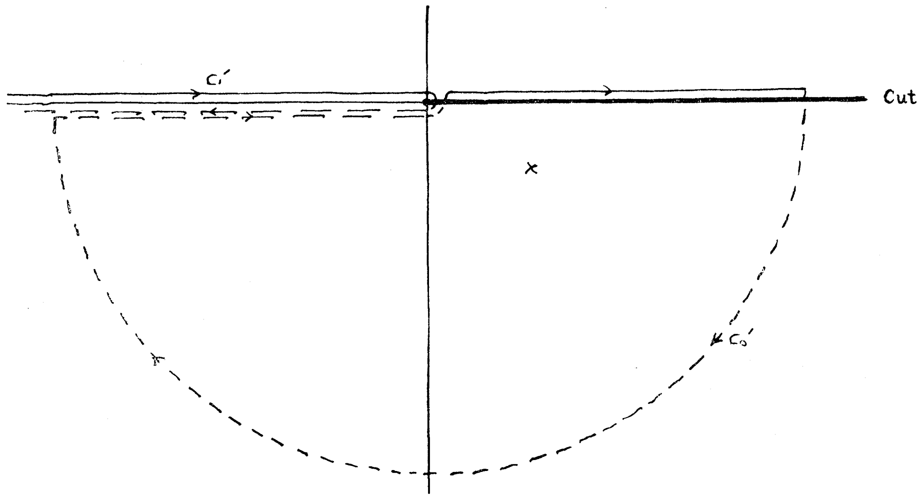

FIG. 1. Complex z plane. Path of integration for (3.8) and (3.9) is transformed to sum of C_0' and C_1'. Dashed lines represent parts of contours on second Riemann sheet. The $\times$ indicates a resonance pole on the second sheet.

For normalization

$$\int d\mathbf{p}\,|a(\mathbf{p})|^2=1. \tag{4.2}$$

In the absence of any interaction the state at some subsequent time t is given by

$$|t\rangle=\int d\mathbf{p}\; a(\mathbf{p})e^{-i(p^2/2m)t}|\mathbf{p}\rangle. \tag{4.3}$$

The probability amplitude for the relative position vector of the two particles is then

$$\langle \mathbf{r}|t\rangle=\frac{1}{(2\pi)^{3/2}}\int d\mathbf{p}\; e^{i[\mathbf{p}\cdot\mathbf{r}-(p^2/2m)t]}a(\mathbf{p}). \tag{4.4}$$

The equation of motion of the average relative position vector of the two particles found from (4.4) by the condition for stationary phase is

$$\mathbf{r}_{\mathrm{av}}=(\mathbf{p}_0/m)t. \tag{4.5}$$

Thus $\mathbf{r}_{\mathrm{av}}=0$ at time $t=0$ so that the two particles reach the zone of interaction near the origin $\mathbf{r}=0$ at time $t=0$. Also we have

$$\mathbf{b}=(\mathbf{p}_0/m)t_0. \tag{4.6}$$

From the last of Eqs. (3.8) the probability amplitude for finding the system in a continuum state with momentum in the range at a time t is then

$$\phi_0(\mathbf{k},t)=a(\mathbf{k})e^{-i(k^2/2m)t}-\frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dz\frac{e^{-iz(t-t_0)}}{h_0(z)}(z-E_1)\frac{G^2}{(2\pi)^3}\frac{U^*(k)}{z-k^2/2m}\times\int d\mathbf{p}\frac{U(p)}{z-p^2/2m}a(\mathbf{p})e^{-i(p^2/2m)t}. \tag{4.7}$$

It can be assumed that the detecting instruments are placed outside the direction of the incident wave packet, corresponding to a direction where $a(\mathbf{k})$ is negligible, so that the first term can be neglected. Putting $\omega=p^2/2m$, suppose

$$a_1(\omega)d\omega=\int d\hat{p}\; p^2a(\mathbf{p})dp.$$

Remembering the restriction now placed on $\mathbf{k}$, we have

$$\phi_0(\mathbf{k},t)=-\frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dz\frac{e^{-izt}}{h_0(z)}(z-E_1)\frac{G^2}{(2\pi)^3}\frac{U^*(k)}{z-k^2/2m}\times\int_0^\infty d\omega\frac{U(\sqrt{(2m\omega)})}{z-\omega}a_1(\omega)e^{i(z-\omega)t_0}. \tag{4.8}$$

This is independent of the direction of $\mathbf{k}$ as is to be expected from previous results. The exact time of formation of the wave packet is not of any consequence here except that it is in the distant past. In the usual way the limit $t_0\to-\infty$ is taken. Since in the integral in (4.8) $z=x+i\epsilon$ where x is real, we have

$$\int_0^\infty d\omega\frac{U(\sqrt{(2m\omega)})}{x-\omega+i\epsilon}\times e^{i(x-\omega)t_0}a_1(\omega)\underset{t_0\to-\infty}{\sim}-2\pi iU(\sqrt{(2mx)})a_1(x). \tag{4.9}$$

Then

$$\phi_0(\mathbf{k},t)=\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dz\frac{e^{-izt}}{h_0(z)}(z-E_1)\times\frac{G^2}{(2\pi)^3}\frac{U^*(k)}{z-k^2/2m}U(\sqrt{(2mz)})a_1(z). \tag{4.10}$$

To evaluate (4.10) and bring out explicit dependence of $\phi_0(\mathbf{k},t)$ on time it is necessary at various steps to make strong assumptions about the analyticity of various functions. We do not intend to try to justify any of these since we are dealing only with a special model.

Suppose $U(p)$ is analytic in $\mathrm{Im}p>0$ or at least its singularities are farther from the real z axis than those of $a_1(z)$ and $a_1(z)$ has some definite analytic properties in the complex z plane, in particular no cuts near the real axis and no essential singularity at infinity. For a particular case we might suppose $a_1(z)$ to be the result of a set of high-order complex poles[10] which contrive to make $a_1(\omega)$ as ω varies along the real axis peaked around $\omega=p_0^2/2m$. If $a_1(\omega)$ is sharply peaked the poles will be close to the real axis, if it is fairly smooth they will be some distance away.

For $t<0$ the contour in (4.10) can be completed in the upper half plane by a semicircle at infinity on which the integrand vanishes. The bottom part of the contour can be shifted upwards in the direction of increasing $\mathrm{Im}z$ until it encounters the first singularity of $a_1(z)$. If this is an ordinary pole the integral will give the residue in the usual way and the factor e^{-izt} will supply a damping factor such that $\phi_0(\mathbf{k},t)$ vanishes at $t\rightarrow-\infty$. The broader the wave packet and hence the farther any pole in $a_1(z)$ from the real axis the more well defined is the moment of collision. For the case when $a_1(z)$ has no singularities in $\mathrm{Im}z>0$, $\phi_0(\mathbf{k},t)$ is rigorously zero for $t<0$ and the moment of collision can be exactly defined.

For $t>0$ it is assumed that $h_0(z)$ is analytically continuable into the second Riemann sheet of the complex z plane through the cut along the real axis from $\mathrm{Im}z>0$ and then the contour can be modified as in Fig. 1.

$$\phi_0(\mathbf{k},t)=\left\{\int_{c_0'}+\int_{c_1'}\right\}dz\frac{e^{-izt}}{h_0(z)}(z-E_1)\times\frac{G^2}{(2\pi)^3}\frac{U^*(k)}{z-k^2/2m}U(\sqrt{(2mz)})a_1(z). \quad (4.11)$$

We can also go one stage further in our assumptions and require that $h_0(p^2/2m)$ is an analytic function of p in the uncut complex p plane. This is valid for a wide range of cutoff functions $U(p)$ and corresponds to two Riemann sheets in the complex z plane being sufficient to contain the function $h_0(z)$.

The integration contours then become as shown in Fig. 2, and

$$\phi_0(\mathbf{k},t)=\left\{\int_{c_0}+\int_{c_1}\right\}dp\frac{p}{m}\frac{e^{-i(p^2/2m)t}}{h_0(p^2/2m)}\left(\frac{p^2}{2m}-E_1\right)\times\frac{G^2}{(2\pi)^3}\frac{U^*(k)}{p^2/2m-k^2/2m}U(p)a_1(p^2/2m) \quad (4.12)$$

[10] Perhaps the simplest form would be the Lorentzian shape
$$a_1(z)=(\Delta/\pi)[(z-p_0^2/2m)^2+\Delta^2]^{-1},$$
which has poles at equal distances Δ on either side of $w=p_0^2/2m$. This form (in momentum variables) is considered by Goebel and McVoy (Ref. 7).

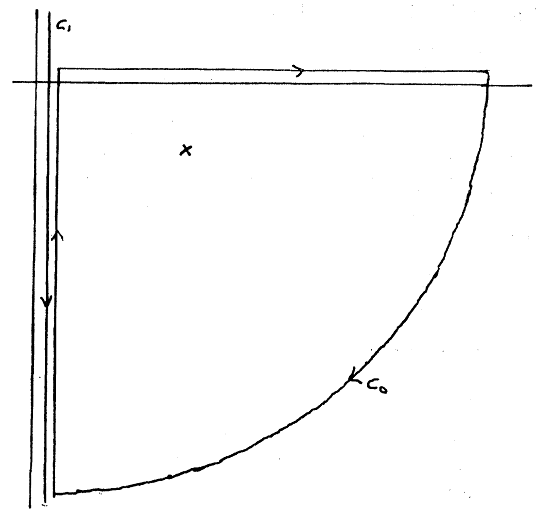

FIG. 2. Integration contours transferred from Fig. 1 to complex p plane. The upper half represents the first sheet in the complex z plane. The $\times$ indicates a resonance pole.

V. DECAY LAWS FOR TWO DISTINCT POLES AND ONE DOUBLE POLE

We now specialize the model even farther to the limit of a point source, $U(p)=1$ for all p, so that an explicit form for $h_0(p^2/2m)$ can be obtained for use in (4.12). The limiting case $U(p)=1$ introduces a linear divergence but this can be removed by one renormalization. By manipulation of (2.11), we obtain

$$h_0(z)=(z-E_1)\times\left\{z-E_2^R+z2m^2\frac{G^2}{\pi}\int_0^\infty dk\frac{|U(k)|^2}{k^2-2mz}\right\}-g^2, \quad (5.1)$$

$$E_2^R=E_2-m\frac{G^2}{\pi^2}\int_0^\infty dk|U(k)|^2. \quad (5.2)$$

If the limit $U(p)=1$ is taken E_2^R diverges linearly,

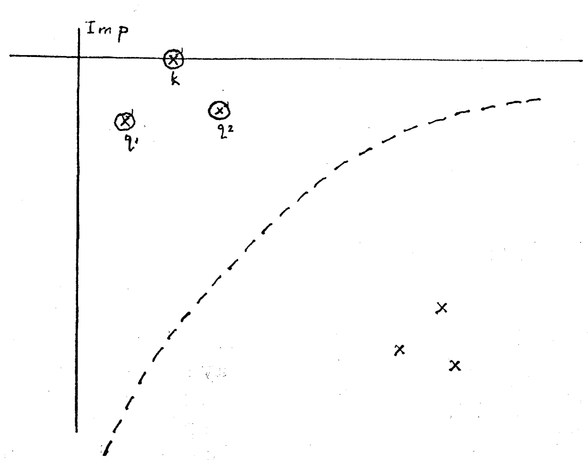

FIG. 3. Complex p plane. Modification of contour from Fig. 2 for evaluation of I_2. The uncircled $\times$'s represent singularities of $a_1(p^2/2m)$. The dashed line represents $2\,\mathrm{Re}p\,\mathrm{Im}p=\mathrm{Im}p^2=\mathrm{const}$.

but assuming it to remain finite then

$$\tilde{h}_0(p^2/2m)=(p^2/2m-E_1)\times\{p^2/2m-E_2{}^R+ipmG^2/2\pi\}-g^2, \quad (5.3)$$

where $\mathrm{Im}p>0$ on the first sheet of the complex z plane. The expression (5.3) corresponds exactly to $\Delta(\omega)$ in B but here the explicit expression for energy in terms of momentum has been introduced. It is convenient to multiply (5.3) by $4m^2$ and define a new expression

$$f(p)=(p^2-2mE_1)\times(p^2-2mE_2{}^R+ipG^2m^2/\pi)-4m^2g^2. \quad (5.4)$$

As required, this is analytic throughout the complex p plane. The same properties as are proved about $h_0(z)$ in Appendix II can be proved about $f(p)$; the principal property that $f(p)$ can only have zeros in the upper half plane on the imaginary axis, so these zeros correspond to bound states, is proved in Appendix III. Since $f(p)$ is a quartic there are only four roots, and since $f(p)^*=f(-p^*)$ if $p=q_1$ is a root then so is $p=-q_1{}^*$. Consequently complex roots occur in pairs either side of the imaginary axis, and if two corresponding to unstable states are assumed to lie in the bottom right-hand segment of the complex p plane we do not in (4.12) have to worry about poles on the imaginary axis. (4.12) now becomes

$$\phi_0(\mathbf{k},t)=\int_{i\infty}^{-i\infty}dp\;2pe^{-i(p^2/2m)t}\frac{1}{f(p)}(p^2-2mE_1)\frac{G^2}{(2\pi)^3}2m\frac{1}{p^2-k^2}a_1(p^2/2m)$$
$$+\int_{c_0}dp\;2pe^{-i(p^2/2m)t}\frac{1}{f(p)}(p^2-2mE_1)\frac{G^2}{(2\pi)^3}2m\frac{1}{p^2-k^2}a_1(p^2/2m)$$
$$=I_1+I_2. \quad (5.5)$$

We initially investigate the first term I_1 in (5.5). Putting $p=iq$ this becomes

$$I_1=\frac{G^2}{(2\pi)^3}4m\int_{-\infty}^{\infty}dq\;e^{i(q^2/2m)t}q\frac{1}{f(-iq)}\frac{q^2+2mE_1}{q^2+k^2}a_1(-q^2/2m), \quad (5.6)$$

$$f(-iq)=(q^2+2mE_1)(q^2+2mE_2{}^R-qG^2m^2/\pi)-4m^2g^2. \quad (5.7)$$

By further manipulation, since a_1 is a function of q^2,

$$I_1=\frac{G^2}{(2\pi)^3}4m\frac{2G^2m^2}{\pi}\int_0^{\infty}dq\;e^{i(q^2/2m)t}q^2\frac{1}{f(-iq)f(iq)}\frac{(q^2+2mE_1)^2}{(q^2+k^2)}a_1(-q^2/2m). \quad (5.8)$$

It is at once apparent that a factor G^2 has been gained in (5.8) and to this extent for weak coupling I_1 is smaller than the second term, I_2, in (5.5). The asymptotic form of (5.8) for large t is found by integrating by parts assuming $a_1(0)$ is not zero:

$$I_1\sim\frac{G^4}{(2\pi)^3}m^2(2m)^{1/2}\frac{1}{\sqrt{\pi}}e^{i(\pi/4)}\frac{E_1{}^2}{(E_1E_2{}^R-g^2)^2}a_1(0)\frac{1}{k^2}\frac{1}{t^{3/2}}+O(1/t^{5/2}). \quad (5.9)$$

This term will eventually dominate any exponentially damped terms but for weak coupling, G^2 small, this will only occur after a long time.

The second term I_2 in (5.5) can now be dealt with. The contributions to this are from the pole at $p=k$, the poles due to the zeros of $f(p)$ in the bottom right-hand quadrant, and the singularities of $a_1(p^2/2m)$ in this region. If $f(p)$ has zeros at q_1, q_2 where both $\mathrm{Re}q_1$, $\mathrm{Re}q_2>0$ and $\mathrm{Im}q_1$, $\mathrm{Im}q_2<0$, then it has the form

$$f(p)=(p-q_1)(p-q_2)(p+q_1{}^*)(p+q_2{}^*). \quad (5.10)$$

If $f(p)$ has a second order zero at q_0, $\mathrm{Re}q_0>0$ and $\mathrm{Im}q_0<0$, then it has the form

$$f(p)=(p-q_0)^2(p+q_0{}^*)^2. \quad (5.11)$$

Assuming form (5.10), the contour C_0 in (5.5) can be modified as shown in Fig. 3. If the singularities of $a_1(z)$ in $\mathrm{Im}z<0$ are the same distance away from the real axis as those in $\mathrm{Im}z>0$ then they produce terms which cut off as rapidly for increasing positive t, $t>0$, as the terms resulting from (4.10) cut off for increasing negative t, $t<0$. In the limit when $a_1(z)$ has no singularities in $\mathrm{Im}z<0$ there are no terms except those due to the pole at $p=k$ and the zeros of $f(p)$. If Δ is the distance of the closest singularity of $a_1(z)$ to the real axis then for $t<-8/\Delta$, $\phi_0(\mathbf{k},t)$ is essentially zero ($e^{-8}\simeq 1/3000$) and for $t>8/\Delta$ I_2 is found by evaluating the residues at the poles k, q_1, and q_2 in the usual way, assuming of course these are closer to the real axis in the complex energy plane.

Remembering the restrictions on t, we have

$$I_2=-\frac{G^2}{(2\pi)^3}4\pi mi\Big[e^{-i(k^2/2m)t}\frac{1}{f(k)}(k^2-2mE_1)a_1(k^2/2m)+e^{-i(q_1^2/2m)t}\frac{2q_1(q_1^2-2mE_1)}{(q_1-q_2)(q_1+q_2^*)(q_1+q_1^*)(q_1^2-k^2)}a_1(q_1^2/2m)$$
$$+e^{-i(q_2^2/2m)t}\frac{2q_2(q_2^2-2mE_1)}{(q_2-q_1)(q_2+q_1^*)(q_2+q_2^*)(q_2^2-k^2)}a_1(q_2^2/2m)\Big]. \quad (5.12)$$

Assuming from (5.11), we get

$$I_2=-\frac{G^2}{(2\pi)^3}4\pi mi\Big[e^{-i(k^2/2m)t}\frac{1}{f(k)}(k^2-2mE_1)a_1(k^2/2m)+\frac{d}{dq}e^{-i(q^2/2m)t}\frac{2q(q^2-2mE_1)}{(q+q_0^*)^2(q^2-k^2)}a_1(q^2/2m)\Big|_{q=q_0}\Big]. \quad (5.13)$$

We now consider two distinct cases, when there is a double pole and for contrast, when there are two simple poles and G^2 small, dealing with the latter first. For this case we work to first order in G^2 throughout. The equation $f(p)=0$ is solved to first order in G^2 in Appendix III.

Suppose

$$\mathcal{E}=k^2/2m,\quad \mathcal{E}_1-\tfrac{1}{2}i\Gamma_1=g_1^2/2m,\quad \mathcal{E}_2-\tfrac{1}{2}i\Gamma_2=q_2^2/2m,$$

then

$$\mathcal{E}_{1,2}\approx\tfrac{1}{2}(E_1+E_2^R)\pm\tfrac{1}{2}[(E_2^R-E_1)^2+4g^2]^{1/2}=(1/2m)p_{1,2}^2,$$
$$\Gamma_{1,2}\approx\pm\frac{G^2m}{2\pi}p_2'\frac{(E_2^R-E_1)\pm[(E_2^R-E_1)^2+4g^2]^{1/2}}{[(E_2^R-E_1)^2+4g^2]^{1/2}}, \quad (5.14)$$

where the first subscript corresponds to the upper sign of the $\pm$ and the second subscript to the lower sign.

The probability amplitude for finding the system in a continuum state with energy in the range $\mathcal{E}$ to $\mathcal{E}+d\mathcal{E}$ at time t is then, neglecting I_1 since this is of order G^4,

$$\phi(\mathcal{E},t)=-\frac{G^2}{(2\pi)^3}i(4\pi km)^{1/2}\exp[(-i\mathcal{E}t)]\Big[\frac{\mathcal{E}-E_1}{(\mathcal{E}-\mathcal{E}_1+\frac{1}{2}i\Gamma_1)(\mathcal{E}-\mathcal{E}_2+\frac{1}{2}i\Gamma_2)}\frac{k+q_1}{k+q_1^*}\frac{k+q_2}{k+q_2^*}a_1(\mathcal{E})$$
$$+\frac{E_1-\mathcal{E}_1+\frac{1}{2}i\Gamma_1}{\mathcal{E}-\mathcal{E}_1+\frac{1}{2}i\Gamma_1}\exp[i(\mathcal{E}-\mathcal{E}_1)t-\tfrac{1}{2}\Gamma_1t]\frac{4mq_1}{(q_1-q_2)(q_1+q_2^*)(q_1+q_1^*)}a_1(\mathcal{E}_1)$$
$$-\frac{E_1-\mathcal{E}_2+\frac{1}{2}i\Gamma_2}{\mathcal{E}-\mathcal{E}_2+\frac{1}{2}i\Gamma_2}\exp[i(\mathcal{E}-\mathcal{E}_2)t-\tfrac{1}{2}\Gamma_1t]\frac{4mq_2}{(q_1-q_2)(q_2+q_1^*)(q_2+q_2^*)}a_1(\mathcal{E}_2)\Big]. \quad (5.15)$$

With further approximation, one obtains

$$\phi(\mathcal{E},t)=-\frac{G^2}{(2\pi)^3}i(4\pi km)^{1/2}\exp(-i\mathcal{E}t)\Big[\frac{\mathcal{E}-E_1}{(\mathcal{E}-\mathcal{E}_1+\frac{1}{2}i\Gamma_1)(\mathcal{E}-\mathcal{E}_2+\frac{1}{2}i\Gamma_2)}a_1(\mathcal{E})+\frac{E_1-\mathcal{E}_1+\frac{1}{2}i\Gamma_1}{(\mathcal{E}-\mathcal{E}_1+\frac{1}{2}i\Gamma_1)(\mathcal{E}_1-\mathcal{E}_2-\frac{1}{2}i\{\Gamma_1-\Gamma_2\})}$$
$$\times\exp[i(\mathcal{E}-\mathcal{E}_1)t-\tfrac{1}{2}\Gamma_1t]a_1(\mathcal{E}_1)-\frac{E_1-\mathcal{E}_2+\frac{1}{2}i\Gamma_2}{(\mathcal{E}-\mathcal{E}_2+\frac{1}{2}i\Gamma_2)(\mathcal{E}_1-\mathcal{E}_2-\frac{1}{2}i\{\Gamma_1-\Gamma_2\})]}\exp[i(\mathcal{E}-\mathcal{E}_2)t-\tfrac{1}{2}\Gamma_2t]a_1(\mathcal{E}_2)\Big]. \quad (5.16)$$

If $E_1-\mathcal{E}_1$, $E_1-\mathcal{E}_2$ and $\mathcal{E}_1-\mathcal{E}_2$ are not small compared with Γ_1, Γ_2 which are of order G^2, it is possible to go even further to the same degree of approximation as in (5.16):

$$\phi(\mathcal{E},t)=-\frac{G^2}{(2\pi)^3}i(4\pi km)^{1/2}e^{-i\mathcal{E}t}\Big[\frac{\mathcal{E}-E_1}{(\mathcal{E}-\mathcal{E}_1+\frac{1}{2}i\Gamma_1)(\mathcal{E}-\mathcal{E}_2+\frac{1}{2}i\Gamma_2)}a_1(\mathcal{E})$$
$$-\frac{\mathcal{E}_1-E_1}{(\mathcal{E}-\mathcal{E}_1+\frac{1}{2}i\Gamma_1)(\mathcal{E}_1-\mathcal{E}_2)}e^{i(\mathcal{E}-\mathcal{E}_1)t-\frac{1}{2}\Gamma_1t}a_1(\mathcal{E}_1)-\frac{\mathcal{E}_2-E_1}{(\mathcal{E}-\mathcal{E}_2+\frac{1}{2}i\Gamma_2)(\mathcal{E}_2-\mathcal{E}_1)}e^{i(\mathcal{E}-\mathcal{E}_2)t-\frac{1}{2}\Gamma_2t}a_1(\mathcal{E}_1)\Big]. \quad (5.17)$$

If $|\mathcal{E}_1-\mathcal{E}_2|\gg\Delta$, where though Δ has been defined previously it can now be regarded as a measure of the width of $a_1(\mathcal{E})$, and also Γ_1, $\Gamma_2\ll\Delta$, then the two unstable states are quite distinct and can only be observed separately. Suppose we are only looking at the unstable state at energy $\mathcal{E}_1$; then, assuming $a_1(\mathcal{E})$ is centered on the energy

$\mathcal{E}_1$ and is substantially constant over a width Γ_1 about $\mathcal{E}_1$, we obtain

$$\phi(\mathcal{E},t)\approx-\frac{G^2}{(2\pi)^3}i(4\pi k_1 m)^{1/2}\exp(-i\mathcal{E}t)\frac{\mathcal{E}_1-E_1}{\mathcal{E}_1-\mathcal{E}_2}a_1(\mathcal{E}_1)\frac{1}{\mathcal{E}-\mathcal{E}_1+\frac{1}{2}i\Gamma_1}\{1-\exp[i(\mathcal{E}-\mathcal{E}_1)t-\tfrac{1}{2}\Gamma_1 t]\}. \tag{5.18}$$

This is the usual decay law for a single unstable particle.[11] In particular after an infinite time the energy distribution is governed by a factor $[(\mathcal{E}-\mathcal{E}_1)^2+\frac{1}{4}\Gamma_1^2]^{-1}$ and the total probability of finding the system in a continuum state varies with time according to a factor $[1-e^{-\Gamma_1 t}]$. This shows the usual exponential decay law with lifetime $1/\Gamma_1$. For $\Delta\gg|\mathcal{E}_1-\mathcal{E}_2|$ if we are only interested in energies in the range $\mathcal{E}_1$ to $\mathcal{E}_2$ and approximately equal amounts $|\mathcal{E}_1-\mathcal{E}_2|$ either side then if $a_1(\mathcal{E})$ is substantially constant over this range, $a_1(\mathcal{E})\approx a_1(\mathcal{E}_1)\approx a_1(\mathcal{E}_2)$ (5.17) becomes

$$\phi(\mathcal{E},t)=-\frac{G^2}{(2\pi)^3}i(4\pi km)^{1/2}e^{-i\mathcal{E}t}a_1(\mathcal{E}_{\rm av})\Bigg[\frac{(\mathcal{E}-E_1)}{(\mathcal{E}-\mathcal{E}_1+\frac{1}{2}i\Gamma_1)(\mathcal{E}-\mathcal{E}_2+\frac{1}{2}i\Gamma_2)}$$
$$-\frac{(\mathcal{E}_1-E_1)}{(\mathcal{E}-\mathcal{E}_1+\frac{1}{2}i\Gamma_1)(\mathcal{E}_1-\mathcal{E}_2)}\exp[i(\mathcal{E}-\mathcal{E}_1)t-\tfrac{1}{2}\Gamma_1 t]-\frac{(\mathcal{E}_2-E_1)}{(\mathcal{E}-\mathcal{E}_2+\frac{1}{2}i\Gamma_2)(\mathcal{E}_2-\mathcal{E}_1)}\exp[i(\mathcal{E}-\mathcal{E}_2)-\tfrac{1}{2}\Gamma_2 t]\Bigg]. \tag{5.19}$$

$\mathcal{E}_{\rm av}$ is some average value of the energy between $\mathcal{E}_1$ and $\mathcal{E}_2$. After an infinite time the energy distribution is given by

$$|\phi(\mathcal{E},\infty)|^2=\frac{G^4}{(2\pi)^3}2km|a_1(\mathcal{E}_{\rm av})|^2\frac{(\mathcal{E}-E_1)^2}{[(\mathcal{E}-\mathcal{E}_1)^2+\frac{1}{4}\Gamma_1^2][(\mathcal{E}-\mathcal{E}_2)^2+\frac{1}{4}\Gamma_2^2]} \tag{5.20}$$

The total probability of finding the system in a continuum state varies with time, once more considering only the leading terms in G^2, according to

$$P(t)=\frac{G^4}{(2\pi)^2}2k_{\rm av}m|a_1(\mathcal{E}_{\rm av})|^2\left[\left(\frac{\mathcal{E}_1-E_1}{\mathcal{E}_1-\mathcal{E}_2}\right)^2\frac{1}{\Gamma_1}(1-e^{-\Gamma_1 t})+\left(\frac{\mathcal{E}_2-E_1}{\mathcal{E}_1-\mathcal{E}_2}\right)^2\frac{1}{\Gamma_2}(1-e^{-\Gamma_2 t})\right]. \tag{5.21}$$

We have neglected the variation in k and put it equal to some average value, $k_{\rm av}$. There are some oscillating terms in the result but these are smaller by a factor G^2 (it must be remembered that Γ_1, Γ_2 are of order G^2) and in any case would average to zero over a very short interval. The result is clearly the superposition of two simple exponential decays. The factor $\mathcal{E}_1-\mathcal{E}_2$ which often occurs in the above equations is given in terms of E_1 and E_2^R by

$$\mathcal{E}_1-\mathcal{E}_2=[(E_2^R-E_1)^2+4g^2]^{1/2}.$$

It now remains to consider the case of a double pole and see how it differs from the above. The conditions required to ensure that $f(p)=0$ has a double root are found in Appendix III. From (C11) and (C12) it can be seen that we can again assume weak coupling, G^2 small, and this gives poles close to the real axis in the complex energy plane. This latter condition is necessary for the decaying state to give rise to observable effects which are not masked by effects due to the poles of $a_1(\mathcal{E})$. To first order in G^2, when the conditions for a double pole are satisfied,

$$q_0\approx(2mE_1)^{1/2}-iG^2m^2/4\pi,$$
$$q_0^2/2m=\mathcal{E}_0-\tfrac{1}{2}i\Gamma_0\approx E_1-\tfrac{1}{2}i(2mE_1)^{1/2}G^2m/8\pi. \tag{5.22}$$

We are here implicitly assuming $E_1\gg G^4m^3$. Since G^2 is again small we can neglect as before the term I_1. The probability amplitude for the energy distribution of the decay products is given by

$$\phi(\mathcal{E},t)=-\frac{G^2}{(2\pi)^2}i(4\pi km)^{1/2}e^{-i\mathcal{E}t}\Bigg[\frac{\mathcal{E}-E_1}{(\mathcal{E}-\mathcal{E}_0+\frac{1}{2}i\Gamma_0)^2}\left(\frac{k+g_0}{k+g_0^*}\right)^2a_1(\mathcal{E})$$
$$-e^{i\mathcal{E}t}\frac{d}{d\alpha}e^{-i\alpha t}\frac{\alpha-E_1}{\mathcal{E}-\alpha}\frac{4qq_0}{(q+q_0^*)^2}a_1(\alpha)\Big|_{\alpha=q^2/2m=\mathcal{E}_0-\frac{1}{2}i\Gamma_0}\Bigg] \tag{5.23}$$

Working to first order in G^2, we have

$$\left(\frac{k+q_0}{k+q_0^*}\right)^2=1,\quad \frac{d}{dq}\frac{4qq_0}{(q+q_0^*)^2}\Bigg|_{q=q_0}=0.$$

[11] Reference 3, p. 451.

Remembering that $\mathcal{E}_0=E_1$ in this order of approximation, we obtain

$$\phi(\mathcal{E},t)=-\frac{G^2}{(2\pi)^2}i(4\pi km)^{1/2}\exp(-i\mathcal{E}t)\left[\frac{\mathcal{E}-E_1}{(\mathcal{E}-E_1+\frac{1}{2}i\Gamma_0)^2}a_1(\mathcal{E})-\frac{1}{(\mathcal{E}-E_1+\frac{1}{2}i\Gamma_0)}\exp[i(\mathcal{E}-E_1)t-\tfrac{1}{2}\Gamma_0t]a_1(E_1)\right.$$
$$\left.\times\left\{1-\tfrac{1}{2}\Gamma_0t-\frac{\frac{1}{2}i\Gamma_0}{(\mathcal{E}-E_1+\frac{1}{2}i\Gamma_0)}-\tfrac{1}{2}i\Gamma_0\frac{1}{a_1(E_1)}\frac{d}{d\alpha}a_1(\alpha)|_{\alpha=E_1}\right\}\right].\quad(5.24)$$

We can assume that the width of the wave packet $\Delta\gg\Gamma_0$, that $a_1(\mathcal{E})$ is substantially constant over a wide range centered on the resonance energy E_1, and that its derivative at this point can be neglected. The derivative term is in any case multiplied by a factor Γ_0 which is of order G^2. Within the range of energies in which $a_1(\mathcal{E})$ is nearly constant,

$$\phi(\mathcal{E},t)\approx-\frac{G^2}{(2\pi)^2}i(4\pi km)^{1/2}e^{-i\mathcal{E}t}a_1(E_1)\frac{\mathcal{E}-E_1}{(\mathcal{E}-E_1+\frac{1}{2}i\Gamma_0)^2}\left[1-\left\{1-\tfrac{1}{2}\Gamma_0t\frac{\mathcal{E}-E_1+\frac{1}{2}i\Gamma_0}{\mathcal{E}-E_1}\right\}\exp[i(\mathcal{E}-E_1)t-\tfrac{1}{2}\Gamma_0t]\right].\quad(5.25)$$

There are two special cases which are of some interest. For energies such that $|\mathcal{E}-E_1|$ is greater than a few times Γ_0

$$\phi(\mathcal{E},t)\approx-(G^2/(2\pi)^2)i(4\pi km)^{1/2}e^{-i\mathcal{E}t}a_1(E_1)(\mathcal{E}-E_1+\tfrac{1}{2}i\Gamma_0)^{-1}[1-(1-\tfrac{1}{2}\Gamma_0t)\exp[i(\mathcal{E}-E_1)t-\tfrac{1}{2}\Gamma_0t]].\quad(5.26)$$

The energy distribution after an infinite time is the same as that for a simple resonance but the decay law is that given generally in B for excitation of the unstable state by a wave packet of continuum states.

The other special case is when $\mathcal{E}=E_1$. All the terms except one proportional to t vanish.

$$\phi(E_1,t)=-(G^2/(2\pi)^2)(4\pi k_1m)^{1/2}\times e^{-iE_1t}a_1(E_1)te^{-\frac{1}{2}\Gamma_0t}.\quad(5.27)$$

This is quite distinctive [it is similar to the decay laws (14) and (15) in B]. This energy does not occur in the spectrum of the final decay products after an infinite time. This is shown in the asymptotic energy distribution

$$|\phi(\mathcal{E},\infty)|^2=\frac{G^4}{(2\pi)^3}2km|a_1(E_1)|^2\times\frac{(\mathcal{E}-E_1)^2}{[(\mathcal{E}-E_1)^2+\frac{1}{4}\Gamma_0^2]^2}.\quad(5.28)$$

The total probability of finding the system in a continuum state varies with time, once more considering only the leading terms in G^2, according to

$$p(t)=(G^4/(2\pi)^2)k_1m|a_1(E_1)|^2(1/\Gamma_0)\times[1-(1-\Gamma_0t+\tfrac{1}{2}\Gamma_0^2t^2)e^{-\Gamma_0t}].\quad(5.29)$$

We have here neglected the variation in k and put it equal to $k_1=(2mE_1)^{1/2}$. The dissimilarity between (5.29) and (5.21) is obvious.

VI. CONCLUSIONS

We now appear to have added another decay law to the list of those associated with double poles. What relationship does this bear to those given in A or B? In B the conclusion drawn is that double poles give rise to a continuously variable rather than a unique decay law, in practice the amplitude is some arbitrary linear combination of $e^{-\Gamma t}$ and $\Gamma te^{-\Gamma t}$. This is true and to obtain any greater degree of definiteness it is necessary to make some kind of assumption about the initial state. In A a definite decay law is obtained by requiring that the initial state is strongly localized spatially. Here we assume that the unstable state is formed, as referred to in B, by excitation from a wave packet of states in one of the decay channels coupled to it. At first sight our decay law is slightly different from that in B since in the corresponding case the decay law given there by Eq. (16) is $|\psi_2(t)|=(1-\frac{1}{2}\Gamma t)e^{-\frac{1}{2}\Gamma t}$, where $\psi_2(t)$ is the probability amplitude for finding the system in the χ_2 particle state. However, these results can be reconciled since we have calculated the probability that the system has actually decayed into the final continuum states. The corresponding quantity is then

$$\int_0^t|\psi_2(t')|^2dt'=\frac{1}{2\Gamma}[1-(1-\Gamma t+\tfrac{1}{2}\Gamma^2t^2)e^{-\Gamma t}]$$

which agrees with our results.

If physical systems with double poles are ever investigated it is very unlikely that the unstable states will be particularly long-lived, and the unstable states will have to be formed in the course of the experiment. There are only two possible ways of forming an unstable state, by excitation from a decay channel which is essentially considered here, and by decay of another unstable state. The model considered here obviously does not extend to the latter case, but in such circumstances peculiar energy dependences of the various parameters might in any case be expected.

There is one other difficulty about the formulas given here. As they are given, the probability amplitude for finding the systems in a final continuum state of definite energy at a definite time violating the uncertainty principles and also the usual result that the energy distribution at a resonance can only be observed if the time behavior is not observed and vice versa. The reason is that we have only considered half the experimental situation, neglecting the final detection instruments. If these only detect a small band of energies they will not be able to define the time behavior. The case when all energies are detected gives rise to the decay laws (5.21) and (5.29) given here.

ACKNOWLEDGMENTS

I should like to thank Dr. S. Zienau for his warm help and encouragement and for suggesting this problem to me. I should also like to acknowledge a Research Studentship grant from the University of London.

APPENDIX I

We are here initially concerned with the evaluation of the propagator $S_2'(z)$ for the χ_2 particle:

$$S_2'(z)=\langle\chi_2|\frac{1}{z-H}|\chi_2\rangle. \tag{A1}$$

This is achieved by using the expansion[12]

$$\frac{1}{z-H}=\frac{1}{z-H_0}+\frac{1}{z-H_0}H_I\frac{1}{z-H} \tag{A2a}$$

$$=\frac{1}{z-H_0}+\frac{1}{z-H_0}H_I\frac{1}{z-H_0}+\frac{1}{z-H_0}H_I\frac{1}{z-H_0}H_I\frac{1}{z-H}. \tag{A2b}$$

Inserting (A2b) in (A1) and using $\langle\chi_2|H_I|\chi_2\rangle=0$, we obtain

$$S_2'(z)=\frac{1}{z-E_2}\left[1+\langle\chi_2|H_I\frac{1}{z-H_0}H_I\frac{1}{z-H}|\chi_2\rangle\right]$$

$$=\frac{1}{z-E_2}\left[1+\langle\chi_2|H_I\frac{1}{z-H_0}H_I|\chi_1\rangle\langle\chi_1|\frac{1}{z-H}|\chi_2\rangle\right.$$

$$\left.+\langle\chi_2|H_I\frac{1}{z-H_0}H_I|\chi_2\rangle\langle\chi_2|\frac{1}{z-H}|\chi_2\rangle+\int\langle\chi_2|H_I\frac{1}{z-H_0}H_I|\mathbf{k}\rangle d\mathbf{k}\langle\mathbf{k}|\frac{1}{z-H}|\chi_2\rangle\right], \tag{A3}$$

where the completeness relation (2.1) has been used. Since

$$\langle\chi_2|H_I\frac{1}{z-H_0}H_I|\chi_1\rangle=\langle\chi_2|H_I\frac{1}{z-H_0}H_I|\mathbf{k}\rangle=0,$$

we get

$$S_2'(z)\left[z-E_2-\langle\chi_2|H_I\frac{1}{z-H_0}H_I|\chi_2\rangle\right]=1. \tag{A4}$$

By further use of the completeness relation, we obtain

$$\langle\chi_2|H_I\frac{1}{z-H_0}H_I|\chi_2\rangle=\frac{g^2}{z-E_1}+\frac{G^2}{2\pi^2}\int_0^\infty dk\,k^2|U(k)|^2\frac{1}{z-k^2/2m}. \tag{A5}$$

So finally, using (A5) in (A4), we have

$$S_2'(z)=\left[z-E_2-\frac{g^2}{z-E_1}-\frac{G^2}{2\pi^2}\int_0^\infty dk\frac{k^2|U(k)|^2}{z-k^2/2m}\right]^{-1}. \tag{A6}$$

[12] These expansions have been used in similar contexts in A. Messiah, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1961), Vol. II, p. 995.

The other matrix elements of the resolvent $(z-H)^{-1}$ can be evaluated in terms of $S_2'(z)$. Using (A2a), we obtain

$$\langle\chi_1|\frac{1}{z-H}|\chi_2\rangle=\frac{1}{z-E_1}\langle\chi_1|H_I\frac{1}{z-H}|\chi_2\rangle=\frac{g}{z-E_1}S_2'(z)\,, \tag{A7}$$

$$\langle\mathbf{k}|\frac{1}{z-H}|\chi_2\rangle=\frac{G}{(2\pi)^{3/2}}\frac{U^*(k)}{z-k^2/2m}S_2'(z)\,. \tag{A8}$$

Using the completeness relation and (A1), (A7), and (A8), we find

$$\frac{1}{z-H}|\chi_2\rangle=\left[\frac{g}{z-E_1}|\chi_1\rangle+|\chi_2\rangle+\frac{G}{(2\pi)^{3/2}}\int d\mathbf{k}\frac{U^*(k)}{z-k^2/2m}|\mathbf{k}\rangle\right]S_2'(z)\,. \tag{A9}$$

Again using (A2a), we get

$$\begin{aligned}\frac{1}{z-H}|\chi_1\rangle&=\frac{1}{z-E_1}\left[|\chi_1\rangle+\frac{1}{z-H}H_I|\chi_1\rangle\right]\\&=\frac{1}{z-E_1}\left[|\chi_1\rangle+\frac{g}{z-H}|\chi_2\rangle\right],\end{aligned} \tag{A10}$$

$$\frac{1}{z-H}|\mathbf{k}\rangle=\frac{1}{z-k^2/2m}\left[|\mathbf{k}\rangle+\frac{G}{(2\pi)^{3/2}}U(k)\frac{1}{z-H}|\chi_2\rangle\right]. \tag{A11}$$

APPENDIX II

We here study the properties of $h_0(z)$ on the first sheet of the complex z plane

$$h_0(z)=(z-E_1)\left[z-E_2-\frac{G^2}{2\pi^2}\int_0^\infty dk\frac{k^2|U(k)|^2}{z-k^2/2m}\right]-g^2. \tag{B1}$$

As noted before, this is analytic in the finite z plane cut along the real axis from $z=0$ to $z=\infty$. By taking the real and imaginary parts,

$$\mathrm{Re}h_0(z)=(\mathrm{Re}z-E_1)\left[\mathrm{Re}z-E_2-\frac{G^2}{2\pi^2}\mathrm{Re}\int_0^\infty dk\frac{k^2|U(k)|^2}{z-k^2/2m}\right]-g^2-\mathrm{Im}z\left[\mathrm{Im}z+\mathrm{Im}z\frac{G^2}{2\pi^2}\int_0^\infty dk\frac{k^2|U(k)|^2}{|z-k^2/2m|^2}\right] \tag{B2}$$

$$\mathrm{Im}h_0(z)=\mathrm{Im}z\left[\mathrm{Re}z-E_2-\frac{G^2}{2\pi^2}\mathrm{Re}\int_0^\infty dk\frac{k^2|U(k)|^2}{z-k^2/2m}\right]+(\mathrm{Re}z-E_1)\left[\mathrm{Im}z+\mathrm{Im}z\frac{G^2}{2\pi^2}\int_0^\infty dk\frac{k^2|U(k)|^2}{|z-k^2/2m|^2}\right]. \tag{B3}$$

If $h_0(z)$ has a zero the real and imaginary parts must be separately zero. By putting the right-hand sides of (B2) and (B3) equal to zero and substituting from (B2) and (B3), then

$$0=\mathrm{Im}z\left[g^2+|z-E_1|^2\left\{1+\frac{G^2}{2\pi^2}\int_0^\infty dk\frac{k^2|U(k)|^2}{|z-k^2/2m|^2}\right\}\right]. \tag{B4}$$

From (B4) it is clear that necessarily $\mathrm{Im}z=0$ for $h_0(z)$ to have a zero. Further, for $p^2/2m$ real,

$$\mathrm{Im}h_0(p^2/2m\pm i\epsilon)=\pm(p^2/2m-E_1)(G^2/2\pi)mp|U(p)|^2. \tag{B5}$$

If $U(p)$ does not vanish then $h_0(p^2/2m\pm i\epsilon)$ has no zeros. Hence the S matrix S_0 only has poles on the first sheet of the complex energy plane on the negative real axis. These correspond to bound states in the usual way.

Suppose x, where x is negative and real, is a zero of $h_0(z)$; then

$$E_2-x-\frac{G^2}{2\pi^2}\int_0^\infty dk\frac{k^2|U(k)|^2}{k^2/2m-x}=\frac{g^2}{E_1-x}. \tag{B6}$$

The left-hand side of (B6) is a monotonic decreasing function of x starting from $+\infty$ at $x=-\infty$ and continuing at $x=0$ to $E_2{}^R$ where

$$E_2{}^R=E_2-\frac{G^2}{\pi^2}m\int_0^\infty dk\,|U(k)|^2. \tag{B7}$$

The right-hand side of (B6) is a monotonic increasing function of x increasing from zero at $x=-\infty$ to g^2/E_1 at $x=0$ if $E_1>0$ or if $E_1<0$ to ∞ at $x=E_1$ and then from $-\infty$ to g^2/E_1 at $x=0$. Thus if $E_1>0$ the two curves can intersect and produce a zero of $h_0(z)$ only if $E_2{}^R<g^2/E_1$. If $E_1<0$ the two curves always cross at least once and if $E_2{}^R<g^2/E_1$ they cross twice. The conditions for the number of zeros of $h_0(z)$ on the negative real axis are thus

	$E_1>0$		$E_1<0$	
	$E_2{}^R>g^2/E_1$	$E_2{}^R<g^2/E_1$	$E_2{}^R>g^2/E_1$	$E_2{}^R<g^2/E_1$
No. of bound states	0	1	1	2

or alternatively

	$E_2{}^R>0$		$E_2{}^R<0$	
	$E_1>g^2/E_2{}^R$	$E_1<g^2/E_2{}^R$	$E_1>g^2/E_2{}^R$	$E_1<g^2/E_2{}^R$
No. of bound states	0	1	1	2

It is clear that there can be at most two bound states and that these must always be distinct since the corresponding zeros of $h_0(z)$ must occur either side of E_1, i.e., where there are two bound states one has an energy greater than and the other less than E_1. This conforms to the usual rule that double S matrix poles describing bound states are forbidden.

The above conditions on the number of bound states can also be summarized that if $E_1E_2{}^R<g^2$ there is one bound state, if $E_1E_2{}^R>g^2$ and both E_1, $E_2{}^R$ are positive there are none, and if $E_1E_2{}^R>g^2$ and both E_1, $E_2{}^R$ are negative then there are two. The significance of these conditions becomes more transparent in the limit $G^2\to 0$ when $E_2{}^R\to E_2$. In the absence of any coupling to the continuous states the matrix representative of the Schrödinger equation is

$$\begin{pmatrix} E_1 & g \\ g & E_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=E\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{B8}$$

where the physical state is

$$|\chi\rangle=x_1|\chi_1\rangle+x_2|\chi_2\rangle. \tag{B9}$$

The eigenvalues of the energy are given by

$$E=\tfrac{1}{2}(E_1+E_2)\pm[\tfrac{1}{4}(E_1-E_2)^2+g^2]^{1/2}. \tag{B10}$$

As g^2 is increased from zero the separation of the energy levels is increased. This "repulsion" of energy levels is of the same nature as that which occurs in ϕ-ω mixing. If $g^2>E_1E_2$ then the square root term is greater than $\frac{1}{2}(E_1+E_2)$, so one energy level is positive and the other negative. On coupling to the continuum state the positive energy level immediately becomes unstable and is no longer an exact energy eigenstate. For $g^2<E_1E_2$ then either both the energy levels are positive or both negative depending on whether E_1, E_2 are positive or negative. The positive energy eigenstates naturally disappear on coupling to continuum channels.

APPENDIX III

We study here the properties of the function $f(p)$ analytic everywhere in the finite complex p plane

$$f(p)=(p^2-2mE_1)(p^2-2mE_2{}^R+ipG^2m^2/\pi)-4m^2g^2. \tag{C1}$$

As in Appendix II we take the real and imaginary parts

$$\begin{aligned}\mathrm{Re}f(p)=\{(\mathrm{Re}p)^2-(\mathrm{Im}p)^2-2mE_1\}\{(\mathrm{Re}p)^2-(\mathrm{Im}p)^2-2mE_2{}^R-\mathrm{Im}p(G^2m^2/\pi)\}\\ -4m^2g^2-2\,\mathrm{Im}p\,\mathrm{Re}p\{2\,\mathrm{Im}p\,\mathrm{Re}p+\mathrm{Re}pG^2m^2/\pi)\}\end{aligned} \tag{C2}$$

$$\begin{aligned}\mathrm{Im}f(p)=\{(\mathrm{Re}p)^2-(\mathrm{Im}p)^2-2mE_1\}\{2\,\mathrm{Im}p\,\mathrm{Re}p\,\mathrm{Re}p+\mathrm{Re}p(G^2m^2/\pi)\}\\ +2\,\mathrm{Im}p\,\mathrm{Re}p\{(\mathrm{Re}p)^2-(\mathrm{Im}p)^2-2mE_2{}^R-\mathrm{Im}p(G^2m^2/\pi)\}.\end{aligned} \tag{C3}$$

If $f(p)$ has a zero (C2) and (C3) are separately zero. By substituting one such equation into the other and after some calculation we obtain

$$0=\mathrm{Re}p[8m^2g^2\,\mathrm{Im}p+|p^2-2mE_1|^2\{2\,\mathrm{Im}p+G^2m^2/\pi\}]. \tag{C4}$$

For $\mathrm{Im}p\geqslant 0$ the only zeros of $f(p)$ occur for $\mathrm{Re}p=0$, i.e., on the positive imaginary axis. These correspond to bound states. It is easily shown that there are at most two such zeros and that they are strictly separate, being on either side of $p=(2mE_1)^{1/2}$. The same conditions on the number of bound states as in Appendix II still apply in this special case.

We now solve the equation $f(p)=0$ for small G^2 to see what kind of unstable states occur for weak coupling. For $G^2=0$ the roots are essentially given by (B10):

$$p_{1,2}^2=m(E_1+E_2^R)\pm m[(E_2^R-E_1)^2+4g^2]^{1/2}. \tag{C5}$$

It is assumed here that $g^2<E_1E_2^R$ and both E_1 and E_2^R are positive so that on coupling to the continuum states there are no stable but two unstable states. We are here only interested in the two roots with positive real part since the other two can be found by reflection in imaginary axis. For $G^2\neq 0$ but small the correction to the roots can be given by Newton's method. If q_1 and q_2 are the exact roots in $\mathrm{Re}p>0$, then, working to first order in G^2,

$$\begin{aligned} q_{1,2}&\approx p_{1,2}-\frac{f(p_{1,2})}{f'(p_{1,2})}\approx p_{1,2}-ip_{1,2}\frac{G^2m^2}{\pi}\,\frac{m(E_2^R-E_1)\pm m[(E_2^R-E_1)^2+4g^2]^{1/2}}{f'(p_{1,2})_{G^2=0}} \\ &\approx p_{1,2}\mp i\frac{G^2m^2}{4\pi}\,\frac{(E_2^R-E_1)\pm[(E_2^R-E_1)^2+4g^2]^{1/2}}{[(E_2^R-E_1)^2+4g^2]^{1/2}}. \end{aligned} \tag{C6}$$

The result is to give the roots a negative imaginary part, and it is easily seen from (C5) and (C6) that there can be no double root in this approximation.

Finally we consider the conditions required for $f(p)$ to have a double complex root. Since $f(p)$ is a polynomial, in fact a quartic, this is a straightforward algebraic problem and results in an equation of constraint on the coefficients of the powers of p in $f(p)$. For a quartic equation

$$g(x)=a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0,\quad a_0\neq 0$$

the necessary and sufficient conditions for a double root to exist is that the discriminant D shall be zero:

$$D=a_0^6(\theta_1-\theta_2)^2(\theta_1-\theta_3)^2(\theta_1-\theta_4)^2(\theta_2-\theta_3)^2(\theta_2-\theta_4)^2(\theta_3-\theta_4)^2,$$

where θ_1, θ_2, θ_3, θ_4 are the roots of the above equation. If

$$I=a_0a_4-4a_1a_3+3a_2^2$$
$$J=a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_4a_1^2-a_2^2$$

then $D=256(I^3-27J^2)$. In this case, putting $\alpha=G^2m^2/\pi$, $\beta=mE_1$, $\gamma=mE_2^R$, $\delta=m^2g^2$, we obtain

$$f(p)=p^4+\alpha ip^3-2(\beta+\gamma)p^2-2\alpha\beta ip+4(\beta\gamma-\delta). \tag{C7}$$

Thus here we have

$$I=4(\beta\gamma-\delta)-\tfrac{1}{2}\alpha^2\beta+\tfrac{1}{3}(\beta+\gamma)^2,$$
$$J=-\tfrac{4}{3}(\beta+\gamma)(\beta\gamma-\delta)+\tfrac{1}{6}(\beta+\gamma)\alpha^2\beta-\tfrac{1}{4}\alpha^2\delta+(1/27)(\beta+\gamma)^3.$$

Putting $I^3=27J^2$ for a double root and then simplifying, we have the tediously long equation

$$\begin{aligned} &2\beta^3\alpha^6+\{8(\beta-\gamma)^2\beta^2-16\beta^3\gamma+12(\beta-3\gamma)\beta\delta+27\delta^2\}\alpha^4+8\{(\beta-\gamma)^4\beta-8(\beta-\gamma)^2\beta^2\gamma-(\beta+\gamma)(\beta-\gamma)^2\delta \\ &\qquad+16(\beta-\gamma)^2\beta\delta+32\beta\gamma^2\delta+12(\beta-3\gamma)\delta^2\}\alpha^2-64(\beta\gamma-\delta)\{(\beta-\gamma)^2+4\delta\}^2=0. \end{aligned} \tag{C8}$$

This does not appear to factorize except in artificial cases such as $\alpha=0$ or $\delta=0$ which cannot be used here since we require complex roots. (C8) can be regarded as a cubic in α^2 and hence in G^4. Since G must be real it is necessary to restrict the roots in α^2 to real positive values. Since the equation is a cubic there is always one real root. This root is positive if the value of the cubic at $\alpha^2=0$ is negative. This requires

$$\beta\gamma>\delta\quad\text{or}\quad g^2<E_1E_2^R \tag{C9}$$

which if E_1, E_2^R are positive is the condition for the existence of complex roots in weak coupling. Thus if (C9) is satisfied, for some value of G^2, there is a double root. By suitable manipulation of the coefficients in (C8) it would be possible to find the exact conditions under which there are one, two, or three values of G^2 which for given values of the other parameters produce a double root. However, this is too complicated and to obtain more information about double complex roots we adopt another approach.

If $p=q_0=a-ib$ (a, b positive) is a double root of $f(p)$, then all the coefficients and hence α, β, γ, and δ are determined in terms of a and b, and thus the required inter-relationships are found:

$$f(p)=(p-a+ib)^2(p+a+ib)^2$$
$$=p^4+4ibp^3-2(3b^2+a^2)p^2-4ib(b^2+a^2)p+(a^2+b^2)^2. \tag{C10}$$

By comparing coefficients, we have

$$\alpha=4b,\quad \beta=\tfrac{1}{2}(b^2+a^2),\quad \gamma=\tfrac{1}{2}(a^2+5b^2),\quad \delta=\tfrac{1}{4}(b^2+a^2)b^2.$$

Choosing G^2 and E_1 as independent parameters, the values of a and b and of $E_2{}^R$ and g^2 for a double root are given by

$$b=G^2m^2/4\pi,\qquad a^2=2mE_1-G^4m^4/16\pi^2,$$
$$E_2{}^R=\tfrac{1}{2}(E_1+G^4m^3/8\pi^2),\quad g^2=(G^4m^3/32\pi^2)E_1=\tfrac{1}{2}E_1(E_2{}^R-\tfrac{1}{2}E_1). \tag{C11}$$

The position of the double pole in the complex energy plane is given by

$$g_0{}^2/2m=\mathcal{E}_0-\tfrac{1}{2}i\Gamma_0,\quad \mathcal{E}_0=E_1-\frac{G^4m^3}{16\pi^2}=\tfrac{3}{2}E_1-E_2{}^R,$$
$$\Gamma_1=[2mE_1-G^4m^3/16\pi^2]^{1/2}G^2m/8\pi. \tag{C12}$$

The only condition on the parameters for the above solution to be valid is

$$E_1>G^4m^3/32\pi^2$$

but for the double pole actually to give rise to a resonance it is necessary that

$$E_1>G^4m^3/16\pi^2. \tag{C13}$$