Elementary-Particle Spectrum*

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The relativistic quantum theory of the symmetric top implies that the spin J of a particle state can be represented as the sum of three commuting spin operators of lengths given by the positive integers or half-integers n, n', $(l+l')$. If $(l+l')$ is identified with the isospin, $(l'-l)$ with I_3, $2(n'-n)$ with the strangeness, $(-1)^{2n'}$ with the parity (for baryons), and $(-1)^{n+n'}$ with the G parity (for mesons), the simplest choices of n, n', l, l' lead to a correct description of J, I, I_3, S, G, P for every known strongly interacting meson or baryon state.

FOR some years it has been known that the electric charge Q of a state of the spectrum of strongly interacting elementary particles may be represented by the general formula[1]

$$Q=\tfrac{1}{2}B+\tfrac{1}{2}S+I_3\,, \tag{1}$$

where $B=1, -1, 0$ for baryons, antibaryons, and mesons, respectively, S is an integer denoting the strangeness of the state, and the value of I_3 may range in integer steps from I to $-I$, where I is the isospin. In order that Q may be an integer it is therefore necessary to assume that, for the baryons and their antiparticles, S is odd if I is an integer and even if I is half-integral, whereas for the mesons S must be even if I is an integer and odd if I is half-integral. Since the strongly interacting mesons have integral spin and the baryons have half-integral spin, these conditions may be expressed by the requirement that, if J is the spin angular momentum of the state and we define

$$\mathfrak{N}\equiv J-I+\tfrac{1}{2}S\,, \tag{2}$$

then $\mathfrak{N}$ must always be an integer. Thus, Q becomes the sum of the three integers

$$Q=(\tfrac{1}{2}B-J)+(I+I_3)+\mathfrak{N}$$

and hence itself is an integer.

The baryons with $S=0$, -1 known at present are listed in Table I. Classified according to the values of $\mathfrak{N}$, or its equivalent $(J-I)$, the states appear to satisfy the following *Rule I: To every baryon state with strangeness S, parity P, spin J, isospin I and mass M there corresponds another baryon state with the same strangeness and parity, but with spin J+1, isospin I+1, mass M+ΔM*. For $S=-1$, ΔM is approximately 250 MeV; but for $S=0$, the data are too uncertain to decide. If this rule is generally valid, we should expect many more states of higher spin, isospin, and mass, the quantum numbers of which may be obtained by direct extrapolation of the data so far accumulated. There is some evidence for this in the case of the $N^*(1560)$. In other words, on a plot of rest energy against angular momentum we should expect almost parallel trajectories of constant S, P, $(J-I)$ with a slope of approximately 250 MeV. By the same token, states with $S=-2$, $I=\frac{1}{2}$ have been found at 1320 MeV ($J^P=\frac{1}{2}^+$), 1530 MeV ($J^P=\frac{3}{2}^+$), 1705?, 1810, and 1933 MeV, and if our rule is applicable to these states also, there should exist states with $S=-2$, $I=\frac{3}{2}$ corresponding to these and lying 250 MeV or so higher and with one more unit of spin. Similarly, corresponding to the $I=0$, $S=-3$, $J=\frac{3}{2}^+$ state $\Omega^-(1675)$, there should exist a state with $S=-3$, $J^P=\frac{5}{2}^+$, $I=1$, mass approximately 1925 MeV.

Trajectories of precisely this nature have indeed been found as solutions of one of the equations which describes the properties of a symmetrical top according to the principles of relativistic quantum mechanics. Nonrelativistically, the energy of such a top may be expressed thus:

$$E=p^2/2m+\tfrac{1}{2}\mathbf{J}\cdot\boldsymbol{\omega}\,, \tag{3}$$

where $\mathbf{J}$, $\boldsymbol{\omega}$ respectively denote the angular momentum

TABLE I. Pairs of baryon states with the same strangeness and parity and the same $(J-I)$.

S	$J-I$	States	J^P	I	ΔM (MeV)
-1	$\frac{1}{2}$	$Y_1^*(1385)$	$\frac{3}{2}^+$	1	270
		$\Lambda(1115)$	$\frac{1}{2}^+$	0	
		$Y_1^*(1660)$	$\frac{3}{2}^-$?	1	255
		$Y_0^*(1405)$	$\frac{1}{2}^-$	0	
	$\frac{3}{2}$	$Y_1^*(1765)$	$\frac{5}{2}^-$	1	245
		$Y_0^*(1520)$	$\frac{3}{2}^-$	0	
	$\frac{5}{2}$	$Y_1^*(2050)$	$\frac{7}{2}^+$	1	235
		$Y_0^*(1815)$	$\frac{5}{2}^+$	0	
	$-\frac{1}{2}$	$\Sigma\pi(1415)$?	?	2	222?
		$\Sigma(1193)$	$\frac{1}{2}^+$	1	
0	0	$N^*(1560)$?	?	$\frac{5}{2}$	324
		$\Delta(1236)$	$\frac{3}{2}^+$	$\frac{3}{2}$	296
		$N(940)$	$\frac{1}{2}^+$	$\frac{1}{2}$	
	1	$\Delta(1710)$?	?	$\frac{3}{2}$	198?
		$N^*(1512)$	$\frac{3}{2}^-$	$\frac{1}{2}$	
	2	$\Delta(1920)$	$\frac{7}{2}^+$	$\frac{3}{2}$	232
		$N^*(1688)$	$\frac{5}{2}^+$	$\frac{1}{2}$	
	3	$\Delta(2360)$	$\frac{9}{2}^-$(?)	$\frac{3}{2}$	170?
		$N^*(2190)$	$\frac{7}{2}^-$(?)	$\frac{1}{2}$	
	4	$\Delta(2825)$	$11/2^+$(?)	$\frac{3}{2}$	175?
		$N^*(2650)$	$\frac{9}{2}^+$(?)	$\frac{1}{2}$	

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[1] T. Nakano and K. Nishijima, Progr. Theoret. Phys. (Kyoto) **10**, 581 (1953); K. Nishijima, Fortschr. Physik **4**, 519 (1956); M. Gell-Mann, Phys. Rev. **92**, 833 (1953); Nuovo Cimento **4**, Suppl. 2, 848 (1956).

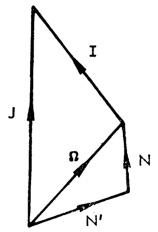

FIG. 1. Resolution of spin angular momentum according to Eq. (13).

and angular velocity. If I_A, I_B, I_C are the moments of inertia about the principal axes, and $I_C=I_B$, the angular momentum may be written in the form

$$\mathbf{J}=I_B\boldsymbol{\omega}+(I_A-I_B)\boldsymbol{\omega}\cdot\mathbf{a}\,\mathbf{a}\,, \tag{4}$$

where $\mathbf{a}$ is a unit vector along the A axis of symmetry.

Hence, if we define

$$\lambda=\mathbf{J}\cdot\mathbf{a}=I_A\boldsymbol{\omega}\cdot\mathbf{a}\,,$$

Eq. (3) becomes

$$E=\frac{p^2}{2m}+\frac{1}{2I_B}\mathbf{J}^2+\frac{1}{2}\left(\frac{1}{I_A}-\frac{1}{I_B}\right)\lambda^2, \tag{5}$$

the Hamiltonian which, taken over directly into quantum theory, describes the rotational energy levels of those molecules and atomic nuclei to which the model of a symmetric top is applicable.

In the same way that the rotational energy is added to the translational energy in Eq. (3) to yield the nonrelativistic expression for the total energy, we should expect that, in order to do the same thing relativistically we should add to the Dirac equation, for example,

$$[i\gamma_\mu p_\mu+mc]\psi=0\,,$$

a Lorentz-invariant term of the form

$$\tfrac{1}{4}J_{\mu\nu}\omega_{\mu\nu}\,, \tag{6}$$

where the axial components of $J_{\mu\nu}$, $\omega_{\mu\nu}$ become $\mathbf{J}$, $\boldsymbol{\omega}$ in the nonrelativistic limit.

If this extra term is absent, or is a constant, we know that

$$J_{\mu\nu}=-\tfrac{1}{4}i\hbar\gamma_{\mu\nu}\,, \tag{7}$$

where $\gamma_{\mu\nu}=(\gamma_\mu,\gamma_\nu)$. The particle is then allowed only the value $\frac{1}{2}$ for its spin. This case corresponds to a *spherical* top with fixed spin, for only in this special case, at least nonrelativistically, is there only one value of the energy available to a particle at rest. We therefore regard the relation between $\mathbf{J}$ and $\boldsymbol{\omega}$ for a spherical top,

$$J_{\mu\nu}=I_B\omega_{\mu\nu}\,, \tag{8}$$

as the classical analog of Eq. (7), so that

$$\omega_{\mu\nu}=-(i\hbar/4I_B)\gamma_{\mu\nu}\,.$$

The generalized Dirac equation now becomes

$$[i\gamma_\mu p_\mu+Mc-(i\hbar/16I_B)\gamma_{\mu\nu}J_{\mu\nu}]\psi=0\,, \tag{9}$$

reducing to the ordinary Dirac equation when Eq. (7) is satisfied, for in that case the extra term in Eq. (9) is a constant and may be absorbed into M.

In general, however, $\mathbf{J}$ is not required to be a constant times $\boldsymbol{\omega}$, and if we write

$$J_{\mu\nu}=-\tfrac{1}{4}i\hbar\gamma_{\mu\nu}-i\hbar\Gamma_{\mu\nu}\,, \tag{10}$$

we find that angular momentum is conserved by Eq. (9) provided that the new operators $\Gamma_{\mu\nu}$ commute with the $\gamma_{\mu\nu}$ and satisfy among themselves the standard commutation rules for angular momenta. These rules may be most easily expressed by writing the axial components of $-i\Gamma_{\mu\nu}$ as the sum of two vectors $\mathbf{L}+\mathbf{L}'$, and the polar components as $\mathbf{L}-\mathbf{L}'$. Then $\mathbf{L}$, $\mathbf{L}'$ commute and are simple angular momentum operators of lengths given by $\mathbf{L}^2=l(l+1)$, $\mathbf{L}'^2=l'(l'+1)$, where l, $l'=0$, $\frac{1}{2}$, 1, $\cdots$. The intrinsic spin of a particle satisfying Eq. (9) is now given by the vector sum

$$\mathbf{J}=\hbar[\tfrac{1}{2}\boldsymbol{\sigma}+\mathbf{L}+\mathbf{L}']\,. \tag{11}$$

If in Eq. (9) we write

$$M+\frac{3\hbar^2}{4I_B}=m\,,\quad \frac{\hbar^2}{16I_B}=m_0=am\,,$$

the equation becomes

$$[i\gamma_\mu p_\mu+mc[1-a\gamma_{\mu\nu}\Gamma_{\mu\nu}]]\psi=0\,,$$

and to first order in a there are two solutions for the rest energy[2]:

$$\left(\frac{E}{mc}\right)^2=1-a+2aJ \tag{12}$$

or

$$=1-3a-2aJ\,,$$

the latter leading to imaginary rest energies for $J>(1/2a)-\frac{3}{2}$. For the former solution, however, it is found that $\mathbf{L}$ and $\mathbf{L}'$ are parallel, and if we identify the length $(l+l')$ of the vector $\mathbf{L}+\mathbf{L}'=\hbar^{-1}\mathbf{I}$ with the isospin I we are lead to trajectories along which $(J-I)$ is constant and which, again to first order in a, have a constant slope m_0c^2. These are the basic characteristics of the states listed in Table I. We now write $I_3=l'-l$, since this may then assume the values I, $I-1$, $\cdots -I$ as required.

Equation (9) has a more general meaning if we do not require that the γ_μ and $\gamma_{\mu\nu}$ that appear in it should be Dirac operators, but allow them to be general spin operators, like the $\Gamma_{\mu\nu}$ of Eq. (10). In this case, we write the axial components of $-\frac{1}{4}i\gamma_{\mu\nu}$ as the sum of two vectors $\mathbf{N}+\mathbf{N}'$ and the polar components as $\mathbf{N}-\mathbf{N}'$, where, as before, $\mathbf{N}^2=n(n+1)$, $\mathbf{N}'^2=n'(n'+1)$, $(n, n'=0, \frac{1}{2}, 1\cdots)$. Thus,

$$\mathbf{J}=\hbar[\mathbf{N}+\mathbf{N}'+\mathbf{L}+\mathbf{L}']\,, \tag{13}$$

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TABLE II. Lowest quantum numbers of states allowed by Fig. 1.

J^P	n'	n	$S=2(n'-n)$	ω	I	State	Unitary symmetry assignment
$\frac{1}{2}^+$	0	0	0	0	$\frac{1}{2}$	$N(939)$	
		$\frac{1}{2}$	−1	$\frac{1}{2}$	0	$\Lambda(1115)$	α octet (3)
		$\frac{1}{2}$	−1	$\frac{1}{2}$	1	$\Sigma(1193)$	
		1	−2	1	$\frac{1}{2}$	$\Xi(1317)$	
		1	−2	1	$\frac{3}{2}$	$\tilde{\Xi}(?)$	
		$\frac{3}{2}$	−3	$\frac{3}{2}$	1, 2	$\tilde{\Omega}(?)$	
$\frac{1}{2}^-$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$N^*(1500)$?	
		$\frac{1}{2}$	0	1	$\frac{1}{2}$	$N^*(1700)$?	
		1	−1	$\frac{1}{2}$	0	$Y_0^*(1405)$	Singlet (3), ninth baryon (5)
		$\frac{1}{2}$	0	1	$\frac{3}{2}$	$\Delta(1692)$?	
		1	−1	$\frac{1}{2}$	1	$\Sigma^*(?)$	70^- decimet without Ω^- (4)
		$\frac{3}{2}$	−2	1	$\frac{1}{2}$	$\Xi^*(?)$	
		$\frac{3}{2}$	−2	1	$\frac{3}{2}$	$\tilde{\Xi}(?)$	
		$\frac{3}{2}$	−2	2	$\frac{3}{2}, \frac{5}{2}$	$\tilde{\tilde{\Xi}}(?)$	
		2	−3	$\frac{3}{2}, \frac{5}{2}$	1, 2, 3	$\tilde{\Omega}(?)$	
$\frac{1}{2}^+$	1	1	0	0, 1	$\frac{1}{2}, \frac{3}{2}$	$N^*(1450)$	
		$\frac{3}{2}$	−1	$\frac{1}{2}$	0	$Y_0^*(1663)$	70^- η octet (4)
		$\frac{3}{2}$	−1	$\frac{1}{2}$	1	$\Sigma^*(?)$	
		2	−2	1	$\frac{1}{2}$	$\Xi^*(?)$	
		$\frac{3}{2}$	−1	$\frac{3}{2}, \frac{5}{2}$	1, 2, 3	$\tilde{\Sigma}(?)$	
		2	−2	1, 2, 3	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$	$\tilde{\Xi}(?)$	
$\frac{3}{2}^+$	0	0	0	0	$\frac{3}{2}$	$\Delta(1236)$	
		$\frac{1}{2}$	−1	$\frac{1}{2}$	1	$Y_1^*(1385)$	δ decimet (3)
		1	−2	1	$\frac{1}{2}$	$\Xi^*(1530)$	
		$\frac{3}{2}$	−3	$\frac{3}{2}$	0	$\Omega^-(1675)$	
		$\frac{1}{2}$	−1	$\frac{1}{2}$	2	$\Sigma\pi(1415)$	
		1	−2	1	$\frac{3}{2}, \frac{5}{2}$	$\Xi^*(?)$	
		$\frac{3}{2}$	−3	$\frac{3}{2}$	0, 1, 2, 3	$\tilde{\Omega}(?)$	
$\frac{3}{2}^-$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$N^*(1512)$	Decimet (7); γ octet (3) (4)
		1	−1	$\frac{3}{2}$	0	$Y_0^*(1520)$	Part of γ octet (3)
		$\frac{1}{2}$	0	0, 1	$\frac{3}{2}$	$\Delta(1640)$?	
		1	−1	$\frac{1}{2}, \frac{3}{2}$	1	$Y_1^*(1660)$	Decimet with $S=1, 0, -1, -2$ (7); rest of γ octet (3)
		$\frac{3}{2}$	−2	1, 2	$\frac{1}{2}$	$\Xi^*(1705)$?	Completed by $\Xi^*(1810)\Lambda^*(1661)$ (4)
		2	−3	$\frac{3}{2}, \frac{5}{2}$	0	$\Omega^*(?)$	
$\frac{5}{2}^+$	0	0	0	0	$\frac{5}{2}$	$N^*(1560)$	
		$\frac{1}{2}$	−1	$\frac{1}{2}$	2	$\tilde{\Sigma}(?)$	
		1	−2	1	$\frac{3}{2}$	$\tilde{\Xi}(?)$	
		$\frac{3}{2}$	−3	$\frac{3}{2}$	1	$\tilde{\Omega}(?)$	
		2	−4	2	$\frac{1}{2}$	$\psi(?)$	35-plet (8) with $S=1, I=2$ replaced by $S=-5, I=0$
		$\frac{5}{2}$	−5	$\frac{5}{2}$	0	$\Upsilon(?)$	
$\frac{5}{2}^-$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{3}{2}$	$\Delta(1710)$	
		1	−1	$\frac{3}{2}$	1	$Y_1^*(1765)$	
		$\frac{3}{2}$	−2	2	$\frac{1}{2}$	$\Xi^*(1810)$?	
		2	−3	$\frac{5}{2}$	0	$\Omega^*(?)$	
$\frac{5}{2}^+$	1	1	0	2	$\frac{1}{2}$	$N^*(1688)$	
		$\frac{3}{2}$	−1	$\frac{5}{2}$	0	$Y_0^*(1815)$	Part of α octet (3)
		2	−2	2	$\frac{1}{2}$	$\Xi^*(1933)$	
$\frac{7}{2}^+$	1	1	0	2	$\frac{3}{2}$	$\Delta(1920)$	
		$\frac{3}{2}$	−1	$\frac{5}{2}$	1	$Y_1^*(2050)$	Decimet (9)
		2	−2	3	$\frac{1}{2}$	$\Xi(?)$	
		$\frac{5}{2}$	−3	$\frac{7}{2}$	0	$\Omega(?)$	
$\frac{7}{2}^-$	$\frac{3}{2}$	$\frac{3}{2}$	0	3	$\frac{1}{2}$	$N^*(2190)$	
		2	−1	$\frac{7}{2}$	0	$Y_0^*(?)$	
$\frac{9}{2}^-$	$\frac{3}{2}$	$\frac{3}{2}$	0	3	$\frac{3}{2}$	$\Delta(2360)$	
$\frac{9}{2}^+$	2	2	0	4	$\frac{1}{2}$	$N^*(2650)$	
$11/2^+$	2	2	0	4	$\frac{3}{2}$	$\Delta(2825)$	

TABLE II (*continued*)

J^P	n'	n	$S=2(n'-n)$	ω	I	$G=(-1)^{n+n'}$	State	Unitary symmetry assignment
0	0	0	0	0	0	+	$\eta(549)$	
	0	$\frac{1}{2}$	−1	$\frac{1}{2}$	$\frac{1}{2}$		$\bar{K}(496)$	β octet (3) (6)
	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$\frac{1}{2}$		$K(496)$	
	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	−	$\pi(137)$	
	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	−	?	
	$\frac{1}{2}$	1	−1	$\frac{1}{2}$	$\frac{1}{2}$		$\kappa(725)$?	
	1	1	0	0	0	+	$X_0(960)$	Singlet (3); ninth meson (5)
	1	1	0	1	1	+	?	
	1	0	2	1	1		$K^+K^+(1055)$?	
	$\frac{1}{2}$	1	−1	$\frac{3}{2}$	$\frac{3}{2}$		?	
1	0	0	0	0	1	+	$\rho(763)$	
	0	$\frac{1}{2}$	−1	$\frac{1}{2}$	$\frac{1}{2}$		$\bar{K}^*(890)$	γ octet (3)
	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$\frac{1}{2}$		$K^*(890)$	
	$\frac{1}{2}$	$\frac{1}{2}$	0	1	0	−	$\omega(783)$	
	$\frac{1}{2}$	$\frac{1}{2}$	0	0, 1	1	−	$A_1(1080)$	
	0	$\frac{1}{2}$	−1	$\frac{1}{2}$	$\frac{3}{2}$		$\bar{K}^*(1270)$	
	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$\frac{3}{2}$		$K^*(1270)$	
	1	1	0	0	1	+	$B(1215)$?	
	$\frac{1}{2}$	1	−1	$\frac{1}{2}$	$\frac{1}{2}$		$C(1215)$?	
	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$		$\bar{C}(1215)$?	
	$\frac{3}{2}$	$\frac{3}{2}$	0	1	0	−	$\phi(1020)$	Singlet (3); ninth meson (5)
2	0	$\frac{1}{2}$	−1	$\frac{1}{2}$	$\frac{3}{2}$		?	
	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	−	$A_2(1320)$	
	$\frac{1}{2}$	1	−1	$\frac{3}{2}$	$\frac{1}{2}$		$\bar{K}^{**}(1430)$	
	1	$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{1}{2}$		$K^{**}(1430)$	Nonet (5)
	1	1	0	2	0	+	$f^0(1250)$	
	2	2	0	0	0	+	$f'(1530)$	
	1	0	2	1	1		$K^+K^+(1280)$?	
	$\frac{1}{2}$	1	−1	$\frac{1}{2}, \frac{3}{2}$	$\frac{3}{2}$		?	
	1	1	0	0, 1	1, 2	+	?	
	$\frac{3}{2}$	$\frac{3}{2}$	0	0, 1	1, 2, 3	−	?	

the case of the generalized Dirac equation corresponding to $\mathbf{N}=\frac{1}{4}(1-\gamma_5)\boldsymbol{\sigma}$, $\mathbf{N}'=\frac{1}{4}(1+\gamma_5)\boldsymbol{\sigma}$. We write

$$\hbar(\mathbf{N}+\mathbf{N}')=\boldsymbol{\Omega}\,,\quad \hbar(\mathbf{L}+\mathbf{L}')=\mathbf{I}\,,$$

so that, for $\mathbf{L}$, $\mathbf{L}'$ parallel, the simple vector diagram of Fig. 1 describes the composition of the spin $\mathbf{J}$.

In order to assess the possibility that this more general form of the wave equation may be able to describe the elementary-particle spectrum, we examine the ways in which a state of given spin may be formed according to this figure. We assume as before that

$$I=l+l'\,,\quad I_3=l'-l$$

and that similar quantum numbers in the $\mathbf{N}$, $\mathbf{N}'$ space define the strangeness:

$$\tfrac{1}{2}S=n'-n\,. \tag{14}$$

In terms of these quantum numbers Eq. (1) becomes

$$Q=\tfrac{1}{2}B+(n'-n)+(l'-l) \tag{15}$$

showing a similarity between $\frac{1}{2}S$ and I_3. The quantity S defined by Eq. (14) is automatically an integer. Further, since the vectors of length n, n', l, l' must combine to form $\mathbf{J}$ according to Eq. (13), it follows that, for fermions, if I is half-integral S must be even and if I is integral S must be odd. For bosons, on the other hand, it follows that if I is half-integral S must be odd, and if I is integral S must be even. This result is just that quoted earlier, i.e., it automatically rules out the wrong combinations of S and I, and therefore always makes Q as defined by (15) equal to an integer.

The lowest values of n, n', $(l+l')$ which can combine to form a given spin are listed in Table II, together with various unitary symmetry assignments that have been proposed for the resulting states.[3–9] Since the space of the vectors $\mathbf{L}$, $\mathbf{L}'$ is not coupled directly to the momenta in Eq. (9), we suppose that keeping n, n' fixed and changing l, l' does not change the parity, and assume that, for the baryons,

$$P=(-1)^{2n'}\ \text{(baryons)}\,. \tag{16}$$

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For antibaryons, n and n' are interchanged, as also are l and l'. Since the π meson appears in the table with $n=n'=\frac{1}{2}$, we suppose that, for the mesons, the G parity is given by

$$G=(-1)^{n+n'} \text{ (mesons)}. \tag{17}$$

Since $\mathbf{J}$ is the vector sum of $\mathbf{I}$ and $\mathbf{\Omega}$, we write $\mathbf{\Omega}^2=\omega(\omega+1)$ and note that, from Fig. 1,

$$\omega=\tfrac{1}{2}|S|,\ \tfrac{1}{2}|S|+1,\ \cdots(n+n'), \tag{18}$$
$$I=|J-\omega|,\ |J-\omega|+1,\ \cdots(J+\omega).$$

The states allowed by Fig. 1 are similar to those allowed by some of the representations of SU_3 and SU_6. The appearance of decimets is common to both, but here we cannot construct a decimet of spin 0, $\frac{1}{2}$, or 1 but we are led very naturally to a decimet at spin $\frac{3}{2}$. Thereafter, decimets occur for higher spins, which also allow 21-plets for $J\geqslant\frac{5}{2}$, with $S=0,\ -1,\ \cdots -5$. The baryon octet appears here as the simplest way of putting together two spins of length 0, $\frac{1}{2}$, or 1 to form spin $\frac{1}{2}$. However, for spin 0 we obtain a basic sextet until we admit the antiparticle to the $\bar{K}$. In addition, there are other states of spin 0 allowed, including an $S=2$, $I=1$ state. From Eq. (18) it is impossible to form $I=0$ states with $J<\frac{1}{2}|S|$, so that for $J=0$, $I=0$ there is no state with $S=\pm 2$, and for $J=\frac{1}{2}$, $I=0$ there is no state with $S=\pm 3$, etc. Were such states to be found, this analysis would be proved to be invalid.

For the spin-$\frac{1}{2}^+$ octet and spin-$\frac{3}{2}^+$ decimet we have $n'=0$, so that for these particle states

$$\mathbf{J}=\mathbf{\Omega}+\mathbf{I},$$

with $\omega=\frac{1}{2}|S|=\frac{1}{2}(1-Y)$. However, we may also resolve $\mathbf{J}$ into the vector sum

$$\mathbf{J}=\mathbf{X}+\mathbf{U},$$

where $\mathbf{X}^2=x(x+1)$ and $x=\frac{1}{2}(1+Q)$. Thus, x is obtained by replacing Y in ω by $-Q$. We may then regard the spin-$\frac{1}{2}$ octet as the sum of two vectors of lengths $(x,U)=(0,\frac{1}{2})$, $(\frac{1}{2},0)$, $(\frac{1}{2},1)(1,\frac{1}{2})$, i.e., $(Q,U)=(-1,\frac{1}{2})$, $(0,0)$, $(0,1)$, $(1,\frac{1}{2})$. Thus, U may be regarded as the U spin, although there is no reason here for taking linear combinations of the Λ and Σ_0 wave functions to give the $U=0$, $U=1$ components. Similarly, for the decimet, in order to make spin $\frac{3}{2}$ we may take $(x,U)=(\frac{3}{2},0)$, $(1,\frac{1}{2})$, $(\frac{1}{2},1)$, $(0,\frac{3}{2})$, i.e., a singlet with $Q=2(N^{*++})$ a doublet with $Q=1(N^{*+},\Sigma^+)$ a triplet with $Q=0(N^{*0},\Sigma^0,\Xi^0)$ and a quartet with $Q=-1(N^{*-},\Sigma^-,\Xi^-,\Omega^-)$.

We also note that, for these states,

$$-\mathbf{J}\cdot\mathbf{\Omega}=\tfrac{1}{2}[I(I+1)-\tfrac{1}{4}Y^2+Y-\tfrac{3}{4}-J(J+1)],$$
$$-\mathbf{J}\cdot\mathbf{X}=\tfrac{1}{2}[U(U+1)-\tfrac{1}{4}Q^2-Q-\tfrac{3}{4}-J(J+1)],$$

which resemble the Gell-Mann–Okubo mass formula

$$M=a+bY+c[I(I+1)-\tfrac{1}{4}Y^2] \tag{19}$$

(with $a=m_\Lambda$, $b=-189$ MeV, $c=-2b/9$) and the SU_6 formula for the magnetic moments:

$$\mu=\mu_n[U(U+1)-\tfrac{1}{4}Q^2-Q-1]. \tag{20}$$

The assumption that

$$\boldsymbol{\mu}=\tfrac{1}{2}\mu_n(\mathbf{U}-2\mathbf{X}) \tag{21}$$

[analogous to the Landé g-factor rule $\boldsymbol{\mu}\sim(\mathbf{L}+2\mathbf{S})$] leads directly to the SU_6 result (20) for the octet, but of course there is no *a priori* reason for the choice of the coefficients in Eq. (21). Similarly, the masses of the $J=\frac{1}{2}^+$ octet may be expressed by the two-parameter formula

$$M=m_p[1+\tfrac{1}{2}\omega]-m_0\mathbf{J}\cdot\mathbf{\Omega},$$

where m_p is the proton mass and $m_0=2c=m_p/11$.

In summary, the spin of a symmetrical top in relativistic quantum theory is the sum of four commuting

TABLE III. Regge recurrences for which $\mathbf{N}$, $\mathbf{N}'$, $\mathbf{I}$ are parallel. Higher member of pair is obtained from lower member by increasing both n and n' by unity and J by 2.

n'	n	I	$P=(-1)^{2n'}$	$S=2(n'-n)$	$J=n+n'+I$	States	ΔM (MeV)
1 0	1 0	$\frac{1}{2}$	+	0	$\frac{5}{2}$ $\frac{1}{2}$	$N^*(1688)$ $N(940)$	748
1 0	$\frac{3}{2}$ $\frac{1}{2}$	0	+	−1	$\frac{5}{2}$ $\frac{1}{2}$	$Y_0^*(1815)$ $\Lambda(1115)$	700
1 0	1 0	$\frac{3}{2}$	+	0	$\frac{7}{2}$ $\frac{3}{2}$	$\Delta(1920)$ $\Delta(1236)$	684
1 0	$\frac{3}{2}$ $\frac{1}{2}$	1	+	−1	$\frac{7}{2}$ $\frac{3}{2}$	$Y_1^*(2050)$ $Y_1^*(1385)$	665
$\frac{3}{2}$ $\frac{1}{2}$	$\frac{3}{2}$ $\frac{1}{2}$	$\frac{1}{2}$	−	0	$\frac{7}{2}$ $\frac{3}{2}$	$N^*(2190)$ $N^*(1512)$	678
$\frac{3}{2}$ $\frac{1}{2}$	$\frac{3}{2}$ $\frac{1}{2}$	$\frac{3}{2}$	−	0	$\frac{9}{2}$? $\frac{5}{2}$?	$\Delta(2360)$ $\Delta(1710)$	650

TABLE IV. Some baryon states predicted by this analysis.

n'	n	S	I	J^P	Mass (MeV)
0	$\frac{1}{2}$	-1	2	$\frac{5}{2}^+$	1650
0	1	-2	$\frac{3}{2}$	$\frac{3}{2}^+$	1570
$\frac{1}{2}$	1	-1	2	$\frac{5}{2}^-$	1910
$\frac{1}{2}$	1	-1	2	$\frac{7}{2}^-$	2010
1	$\frac{3}{2}$	-1	2	$\frac{9}{2}^+$	2290
0	1	-2	$\frac{3}{2}$	$\frac{5}{2}^+$	1760
$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$	$\frac{5}{2}^-$	1710
$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{5}{2}$	$\frac{7}{2}^-$	1900
1	1	0	$\frac{5}{2}$	$\frac{9}{2}^+$	2125
$\frac{3}{2}$	$\frac{3}{2}$	0	$\frac{5}{2}$	$11/2^-$	2525
2	2	0	$\frac{5}{2}$	$13/2^+$	3000
0	1	-2	$\frac{3}{2}$	$\frac{1}{2}^+$	?
0	$\frac{3}{2}$	-3	1	$\frac{1}{2}^+$	?
$\frac{1}{2}$	2	-3	0	$\frac{3}{2}^-$	1950
0	2	-4	$\frac{1}{2}$	$\frac{5}{2}^+$	2100
0	$\frac{5}{2}$	-5	0	$\frac{5}{2}^+$	2230
$\frac{1}{2}$	2	-3	0	$\frac{5}{2}^-$	2060
0	$\frac{3}{2}$	-3	1	$\frac{5}{2}^+$	1925
1	2	-2	$\frac{3}{2}$	$\frac{7}{2}^+$	2200
$\frac{3}{2}$	2	-1	0	$\frac{7}{2}^-$	2200

spin operators, two of which, $\mathbf{N}$ and $\mathbf{N}'$, are described in the space of the γ_μ which are coupled to the momentum in the equation of motion, while the other two, $\mathbf{L}$ and $\mathbf{L}'$, are not so coupled. We have found that if the lengths of these vectors are n, n', l, l', we are faced with a limited choice of them to form a particle with given spin, and have examined the lowest values of these quantum numbers for various values of J. In the special case $\mathbf{N}+\mathbf{N}'=\frac{1}{2}\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is the Dirac spin operator, the multiplicity of the highest state is $2(l+l')+1$, corresponding to an isospin $I=(l+l')$. We have assumed that this is generally true, and in addition that the strangeness quantum number for particles is the integer $2(n'-n)$ the parities of the baryons are $(-1)^{2n'}$ and the G parities of the mesons $(-1)^{n+n'}$. Thereafter, the formation of the tables which correctly describe the observed spectrum of baryons and mesons simply consists of taking the lowest values of these quantum numbers consistent with the value of the spin of the particle.

The model illustrated in Fig. 1 gives rise in a very natural way to four distinct types of intersecting trajectories in the rest-energy angular-momentum plane such that the parity is constant along each one of them. The most familiar set of these trajectories is generated by keeping I fixed and allowing both n' and n to increase by one unit, with ω and J correspondingly increasing by two units. Since $\Delta n'=\Delta n$, the strangeness is constant along such a curve, and since $P=(-1)^{2n'}$ the parity remains constant. These are the Regge trajectories the evidence for which, in the case of the baryons, is summarized in Table III, and leads us to *Rule II: To every baryon state with strangeness S, parity P, spin J, isospin I, and mass M there corresponds another baryon state with the same strangeness, isospin, and parity but with spin J+2, mass M+ΔM.* For all pairs of states known at present, $\Delta M \doteqdot 700$ MeV, although ΔM appears to decrease slowly as M increases.

Alternatively, we may keep n', n fixed, so that S and P are fixed, and allow J and I to increase by integers. These are the trajectories of constant S, P, $(J-I)$ listed in Table I, described by Eq. (12) or Rule I and studied also by bootstrap techniques.[3] Another set of trajectories is generated by keeping n' and hence P constant, and allowing n and J to increase in integer steps while I remains constant. Along each of these trajectories the strangeness decreases by two units for each increase in J by one unit. This leads us to expect *Rule III: To every baryon state with strangeness S, parity P, spin J, isospin I, and mass M there corresponds another baryon state with the same parity and isospin, but with spin (J+1), strangeness (S−2), mass M+ΔM.* Two examples of such trajectories are known at present, one connecting the Λ and the Ω^-, the other connecting the neutron-proton with the $\Xi^*(1530)$. The slopes of these trajectories are approximately 575 MeV per unit of J, and they lead to the expectation that states of higher $|S|$ will be discovered at higher values of J.

Finally, of course, we may keep J, n' and $(n+1)$ fixed so that P and $I-\frac{1}{2}S$ remain constant. This implies *Rule IV: To every baryon state with strangeness S, parity P, spin J, isospin I, and mass M there corresponds another baryon state with the same parity and spin, but with strangeness S−1, isospin $I-\frac{1}{2}>0$, mass M+ΔM.* Prime example of this is the $J=\frac{3}{2}^+$ decimet, with $\Delta M = 147$ MeV.

This set of interlocking trajectories allows an estimate to be made of the masses of the states to be expected as these trajectories are extended to higher energies and spins. Table IV lists some of the lower lying baryon states predicted by this analysis.