value of g_{33} from Eq. (23), the resonance occurs at approximately the correct position in the πN channel but with a branching ratio greater than $\frac{1}{2}$ (i.e., with phase $=\pi/2$). If we vary our couplings slightly so as to fit exactly the two parameters, $W^*=12.1\mu$ and $\Gamma_e/\Gamma=0.35$,[1] then we need the couplings: $g_3'=0.53g_3$ and $g_{33}'=1.01g_{33}$. However, for these new couplings the total width is only $\Gamma=40$ MeV as compared to the experimental value of $\Gamma=325$ MeV.

In conclusion we have seen that a simple model of inelastic coupling in a pole approximation gives an excellent prediction of the position of the $N^*(\frac{5}{2}-)$ resonance. Thus both the $N^*(\frac{3}{2}-)$[2,11] and $N^*(\frac{5}{2}-)$ resonances and their octet character are understandable from a dynamical model. The width of the $N^*(\frac{5}{2}-)$, as calculated in the three-channel case, is far too small as was that of the $N^*(\frac{7}{2}+)$, discussed in AB2. In that paper, it is argued that this is probably the result of having only one inelastic channel open for the decay. Other states are going to play a role and we would expect the resonance width to be increased by this Ball-Frazer[12] effect, but we know of no simple way to estimate it. A test of this model would be to observe that the dominant decay product of the $N^*(\frac{5}{2}-)$ is $\pi+N^*(\frac{3}{2}+)$.

[11] J. J. Brehm, Phys. Rev. **136**, B216 (1964).

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[12] J. S. Ball and W. R. Frazer, Phys. Rev. Letters **7**, 204 (1961).

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Electromagnetic Interaction in Static Strong-Coupling Theory

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Electromagnetic properties of isobars are studied using the group-theoretic formulation of the strong-coupling theory due to Cook, Goebel, and Sakita. Expressions for magnetic moment and electromagnetic mass shifts for isobars are obtained in the scalar and the pseudoscalar theories. The significance of these results is briefly discussed.

1. INTRODUCTION

RECENTLY, Cook, Goebel, and Sakita[1] have given a group-theoretic formulation of the static strong-coupling theory. As shown by these authors the strong-coupling approximation is closely related to the mathematical concept of group contraction and the "dynamical group" which emerges in the strong-coupling limit has the structure of a semidirect product of the "primitive group of invariance" with a suitable Abelian invariant subgroup. In this formulation the infinite number of isobar states that occur in the static scalar strong-coupling theory are put in a single unitary irreducible representation of the group $SU(2)\times T_3$. Similarly, the isobar states of pseudoscalar strong-coupling theory are put in a single unitary irreducible representation of the group $SU(2)\otimes SU(2)\times T_9$. The purpose of this paper is to study the electromagnetic properties of isobars in the static scalar and pseudoscalar strong-coupling theories using this group-theoretic formulation. In Sec. 2 we construct the explicit matrix representation of the infinitesimal generators of $SU(2)\times T_3$. These results are then utilized in Sec. 3 to obtain a magnetic-moment formula and an electromagnetic mass formula in static scalar theory. The pseudoscalar theory is studied in Sec. 4. The relevant representation of the group $SU(2)\otimes SU(2)\times T_9$ is first constructed and then these results are applied to obtain expressions for magnetic-moment and electromagnetic mass shifts of isobars. The significance of these results is then briefly discussed.

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[1] T. Cook, C. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965).

2. REPRESENTATIONS OF $SU(2)\times T_3$

Let us denote the six infinitesimal generators of this group by M_i and A_i. The commutation relations for these generators are

$$[M_i,M_j]=i\epsilon^{ijk}M_k, \quad (1)$$

$$[M_i,A_j]=i\epsilon^{ijk}A_k, \quad (2)$$

$$[A_i,A_j]=0. \quad (3)$$

We now define appropriate "ladder" generators:

$$M_\pm=M_1\pm iM_2;\quad A_\pm=A_1\pm iA_2. \quad (4)$$

The commutation relations of these follow from Eqs.

(1)–(3). A given representation of $SU(2)\times T_3$ will automatically realize a representation of the subgroup $SU(2)$. The irreducible representations of the latter are characterized by an integral or half-integral nonnegative number k. Let $\mathfrak{M}^k$ denote the corresponding (invariant) vector space and let f_ν^k, $\nu=-k, -k+1,\cdots+k$ be a canonical basis for $\mathfrak{M}^k$. On this basis operators $M_\pm$, M_3 are given by the usual formula

$$\begin{aligned} M_+f_\nu^k&=[(k-\nu)(k+\nu+1)]^{1/2}f_{\nu+1}^k\,,\\ M_-f_\nu^k&=[(k+\nu)(k-\nu+1)]^{1/2}f_{\nu-1}^k\,,\\ M_3f_\nu^k&=\nu f_\nu^k\,. \end{aligned} \tag{5}$$

We consider the representation space R of given representation of $SU(2)\times T_3$ to be a closed direct sum of the subspaces $\mathfrak{M}^k$. The general form of the operators $A_\pm$, A_3 in R can be obtained from commutation relation (2) and Eq. (4). This derivation is given in Naimark's book[2] and the result is

$$\begin{aligned} A_+f_\nu^k=&[(k-\nu)(k-\nu-1)]^{1/2}c_kf_{\nu+1}^{k-1}\\ &-[(k-\nu)(k+\nu+1)]^{1/2}A_kf_{\nu+1}^k\\ &+[(k+\nu+1)(k+\nu+2)]^{1/2}c_{k+1}f_{\nu+1}^k\,, \end{aligned} \tag{6}$$

$$\begin{aligned} A_-f_\nu^k=&-[(k+\nu)(k+\nu-1)]^{1/2}c_kf_{\nu-1}^{k-1}\\ &-[(k+\nu)(k-\nu+1)]^{1/2}A_kf_{\nu-1}^k\\ &-[(k-\nu+1)(k-\nu+2)]^{1/2}c_{k+1}f_{\nu-1}^{k+1}\,, \end{aligned} \tag{7}$$

$$\begin{aligned} A_3f_\nu^k=&[k^2-\nu^2]^{1/2}c_kf_\nu^{k-1}-\nu A_kf_\nu^k\\ &-[(k+1)^2-\nu^2]^{1/2}c_{k+1}f_\nu^{k+1}\,. \end{aligned} \tag{8}$$

In the above $k\geqslant k_0$, i.e., $c_{k_0}=0$, when k_0 is a nonnegative integral or half-integral number. It remains now to determine the coefficients c_k and A_k. For this purpose we use the remaining commutation relations (3). We apply relations (3) to a vector f_ν^k and use Eqs. (6)–(8) and then compare coefficients of the same vectors f_ν^p. In this manner we obtain

$$[(k+1)A_k-(k-1)A_{k-1}]c_k=0\,, \tag{9}$$

$$[(k+2)A_{k+1}-kA_k]c_{k+1}=0\,, \tag{10}$$

$$(2k-1)c_k^2-(2k+3)c_{k+1}^2-A_k^2=0\,. \tag{11}$$

Equations (9)–(11) together with condition $c_{k_0}=0$ determine the parameters c_k and A_k and thus lead to the classification of the irreducible representations of $SU(2)\times T_3$. We first notice a particularly simple solution of the above equations, viz.,

$$A_k=0=c_k\,. \tag{12}$$

In this case $A_\pm$ and A_3 are zero; and the irreducible representations of $SU(2)\times T_3$ coincide with those of the $SU(2)$. We call this the O_0 type of representation. Actually except for this case, the group $SU(2)\times T_3$ does not have any finite-dimensional irreducible representation.

We now consider the case of irreducible representations in which the generators A_i are not zero. The parameters c_k and A_k can be found following the method given in Naimark.[2] This is done as follows. First we suppose that c_k does not vanish for any k

$$c_k\neq0;\quad k=k_0+n\,,\quad n=1,2,3\cdots\,. \tag{13}$$

Now relations (9) and (10) mean the same thing. Multiplying both sides of (9) by k and introducing the notation

$$k(k+1)A_k=\rho_k\,, \tag{14}$$

we obtain

$$\rho_k-\rho_{k-1}=0\,. \tag{15}$$

Equation (15) shows that ρ_k is independent of k, i.e., is a constant. We denote this constant by ik_0C. Thus

$$A_k=ik_0C/k(k+1)\,. \tag{16}$$

Now consider Eq. (11). Introducing the notation

$$\sigma_k=(2k-1)(2k+1)c_k^2\,, \tag{17}$$

we obtain from (11) and (16)

$$\sigma_k-\sigma_{k+1}=-(2k+1)k_0^2C^2/k^2(k+1)^2\,. \tag{18}$$

Therefore,

$$\begin{aligned} \sigma_{k_0}-\sigma_k&=\sum_{\nu=k_0}^{k-1}(\sigma_\nu-\sigma_{\nu+1})\\ &=-C^2(k^2-k_0^2)/k^2\,. \end{aligned} \tag{19}$$

But $\sigma_{k_0}=0$ since $c_{k_0}=0$. Thus we obtain from (19)

$$c_k=\frac{[C^2]^{1/2}}{k}\left[\frac{k^2-k_0^2}{4k^2-1}\right]^{1/2}. \tag{20}$$

Before proceeding further we must emphasize that the condition (13) is not a real restriction on the representations. To see this let us suppose the contrary case, viz., that c_k vanishes for some of the values $k=k_0+n$; $n=1, 2, 3,\cdots$. Let k_1 be one such value. Thus $c_{k_1}=0$ so that also $\sigma_{k_1}=0$. Repeating the same argument as before we now obtain

$$\sigma_{k_0}-\sigma_{k_1}=-c^2\frac{k_1^2-k_0^2}{(k_1+1)^2}=0\,,$$

i.e.,

$$k_1^2=k_0^2;\quad k_1=k_0+n\,,\quad n=1,2,3,\cdots\,. \tag{21}$$

Relation (21) cannot, of course, be satisfied, showing thus that the possibility of vanishing c_k for some $k=k_0+n$ can never actually occur. This proves the earlier assertion regarding the absence of finite-dimensional irreducible representations other than O_0. We now go back to Eq. (20). This equation shows that C has to be either purely real or purely imaginary. For the physically interesting case of the unitary representations C is purely imaginary (see below), so that we write $C=iz$. We now notice a curious thing. From Eqs. (6)–(8), (16), and (20) it is clear that the representations

[2] M. A. Naimark, *Linear Representations of the Lorentz Group* (Permagon Press, Inc., New York, 1964).

are actually independent of z which occurs as an overall multiplicative factor. One can simply redefine[3] the operators A_i as A_i/z and thus eliminate z from all the expressions. We thus obtain finally

$$A_k = -\frac{k_0}{k(k+1)}, \quad c_k = \frac{i}{k}\left[\frac{k^2-k_0^2}{4k^2-1}\right]^{1/2}. \tag{22}$$

Equations (5)–(8) and (22) give the desired irreducible representations. These equations show that if a weight k is contained in an irreducible representation, so is the weight $k+1$, and hence all weights $k=k_0+n$, $n=1, 2, 3, \cdots$. These representations are clearly infinite-dimensional. These equations also show that the matrices M_3 and A_3 are Hermitian and $M_+(A_+)$ is Hermitian adjoint of $M_-(A_-)$. Thus these representations are unitary. We thus conclude that the unitary irreducible representations of $SU(2)\times T_3$ are characterized by a single non-negative integral or half-integral number k_0 and these contain an infinite number of irreducible representations of the subgroup $SU(2)$, namely, k_0, k_0+1, $k_0+2, \cdots$. Finally, we may note the connection of k_0 with the Casimir operators. From Eqs. (1)–(3) we see that $SU(2)\times T_3$ has two Casimir operators, namely, A^2 and $\mathbf{M}\cdot\mathbf{A}$. By direct computation one may verify that

$$\mathbf{M}\cdot\mathbf{A} f_\nu{}^k = k_0 f_\nu{}^k,$$
$$A^2 f_\nu{}^k = f_\nu{}^k.$$

Thus although $SU(2)\times T_3$ has two Casimir operators, its irreducible representation is characterized by only a single number. This sort of situation arises from the fact that group $SU(2)\times T_3$ is not semisimple.

Apart from the irreducible representations studied above, $SU(2)\times T_3$ could also have finite-dimensional representations which are not completely reducible. The regular representation is one such. These are not considered in this paper.

3. ELECTROMAGNETIC PROPERTIES IN SCALAR THEORY

In this section we will utilize the results of the previous section in studying the electromagnetic properties of isobars in static scalar strong-coupling theory. We have to first specify the transformation property of the electromagnetic current. From physical considerations we require the current to transform as a component of a finite-dimensional tensor. If we further require the tensor to be irreducible then the only possibility is a O_0 type of tensor.[4] It follows thus that we have to specify the behavior of the current under isotopic rotations alone. But now we already know from the relation $Q=\frac{1}{2}+T_z$ that the electromagnetic current transforms as a superposition of an isotopic scalar and the third component of an isotopic vector. From Eq. (2) we see that A_3 transform as the third component of an isotopic vector and A^2 as a scalar. But in every irreducible representation of our group A^2 has eigenvalue unity. Thus we finally conclude that the desired transformation property of the current is a linear combination[5] of unit operator and A_3.

We denote the isobar states as $|k_0k\nu\rangle$; k is the isotopic spin, ν is its third component and k_0 is the label of the irreducible representation. The magnetic moment of this state is

$$u = a + b\langle k_0k\nu|A_3|k_0k\nu\rangle.$$

Using Eqs. (8) and (22) we obtain

$$u = a + [\nu/k(k+1)]b; \tag{23}$$

a and b are constants independent of k and ν. For fixed k Eq. (23) reduces to that of Marshak, Okubo, and Sudarshan[6] as it must. Similar formulas can be derived for the transition moments also. We can now discuss electromagnetic mass splittings. The relevant transformation property is now a linear combination of an isotopic scalar, third component of a $k=1$ vector and third component of a $k=2$ tensor. Thus we obtain the electromagnetic mass formula[7]

$$M = A + \frac{\nu}{k(k+1)}B + \left[\frac{k^2-\nu^2}{k^2}\frac{k^2-k_0^2}{4k^2-1} + \frac{(k+1)^2-\nu^2}{(k+1)^2}\frac{(k+1)^2-k_0^2}{4(k+1)^2-1} + \nu^2\frac{k_0^2}{k^2(k+1)^2}\right]C; \tag{24}$$

A, B, and C are constants. Since we do not know how much trust to put in the scalar theory, we refrain from discussing the significance of these relations any further.

4. PSEUDOSCALAR STRONG-COUPLING THEORY

The relevant group structure of this theory is $SU(2)\otimes SU(2)\times T_9$. The two commuting $SU(2)$ refer to isotopic spin and ordinary spin, respectively, and the nine translation generators correspond to the nine states of a p-wave charged meson. The isobar states that occur in the pseudoscalar strong-coupling

[3] This procedure is allowed for $C\neq 0$. However if $C=0$ then also $A_i=0$ so that we merely have the O_0 representation discussed earlier in the text.

[4] It is worthwhile to recall in this connection that similar situations occur in other contexts too. For instance, the transformation property of electromagnetic current 4-vector $\mathcal{J}_\mu$ under the Poincaré group is also of the O_0 type, as the translation operators do not act on the tensor index μ.

[5] Notice that this is in conformity with the statement that the photon transforms as a superposition of η^0 and π^0, as from the theory of Ref. 1 we know that A_3 is indeed the π^0 source function.

[6] R. E. Marshak, S. Okubo, and E. C. G. Sudarshan, Phys. Rev. **106**, 599 (1957).

[7] Compare, however, with R. Ramachandran, Phys. Rev. **139**, B121 (1965).

[8] G. Wentzel, Rev. Mod. Phys. **19**, 1 (1947); further references are quoted there.

theory are characterized by the property that for each isobar the magnitude of isospin is equal to that of spin. We have thus to construct a representation of $SU(2)\otimes SU(2)\times T_9$ having this property. This representation is already found in Ref. 1, where it is identified with the $(\infty,0,0)$ representation of $SU(4)$. But here we want to construct explicitly the matrix representation of the generators for our later use. This is done below.

The infinitesimal generators and their commutation rules are as follows:

$$[M_i,M_j]=i\epsilon^{ijk}M_k\,,\quad [N_\alpha,N_\beta]=i\epsilon^{\alpha\beta\gamma}N_\gamma\,,\quad [M_i,N_\alpha]=0\,, \tag{25}$$

$$[M_i,T_{j\alpha}]=i\epsilon^{ijk}T_{k\alpha}\,,\quad [N_\alpha,T_{i\beta}]=i\epsilon^{\alpha\beta\gamma}T_{i\gamma}\,, \tag{26}$$

$$[T_{i\alpha},T_{j\beta}]=0\,. \tag{27}$$

We denote by $f_{\nu m}{}^{kj}$ a set of vectors which provide a unitary, irreducible representation of $SU(2)\otimes SU(2)$. k and j refer to isospin and spin, respectively, ν and m their third components. M_i and N_α are represented on this basis in the usual way. To obtain the desired representation we have to construct an invariant representation space for the operators $T_{i\alpha}$ using only those vectors for which $k=j$. Let us first consider T_{33}. Thus we seek a representation in the following form:

$$T_{33}f_{\nu m}{}^{kk}=[(k^2-\nu^2)(k^2-m^2)]^{1/2}\times d_k f_{\nu m}{}^{k-1,k-1}-\nu m B_k f_{\nu m}{}^{kk}-[((k+1)^2-\nu^2)((k+1)^2-m^2)]^{1/2}d_{k+1}f_{\nu m}{}^{k+1,k+1}\,. \tag{28}$$

The coefficients d_k and B_k are functions of k alone and do not depend on ν and m. The validity of this statement follows from Eq. (26) which says that $T_{i\alpha}$ is a regular tensor operator of $SU(2)\otimes SU(2)$ and hence according to the Wigner-Eckart theorem the dependence of matrix elements of T_{33} on the projection quantum numbers m and ν is given by the relevant Clebsch-Gordan coefficients. The functions of ν and m occurring in (28) are precisely these Clebsch-Gordan coefficients. From (28) we can generate every other $T_{i\alpha}$ using Eq. (26). We first obtain $T_{3+}(=T_{31}+iT_{32})$ and then use Eq. (27). In this way we obtain two equations

$$[(k-1)B_{k-1}-(k+1)B_k]d_k=0\,, \tag{29}$$

$$(k^2-m^2)(2k-1)d_k{}^2-[(k+1)^2-m^2]d_{k+1}{}^2-m^2B_k{}^2=0\,. \tag{30}$$

Solving Eqs. (29) and (30), we get

$$B_k=\frac{1}{k(k+1)}\,,\quad d_k=\frac{i}{k}\left[\frac{1}{4k^2-1}\right]^{1/2}. \tag{31}$$

In obtaining (31) we have also dropped an over-all multiplicative constant exactly in the same way as discussed in Sec. 2. All commutation relations (27) are now satisfied as can be verified by direct calculation. Equations (28) and (31) also show that the least value of k in a representation is either 0 or $\frac{1}{2}$. Thus for the half-integral case we have the desired representation with the spin-isospin content $(\frac{1}{2},\frac{1}{2})$, $(\frac{3}{2},\frac{3}{2})$, $(\frac{5}{2},\frac{5}{2})$, $\cdots$ etc.

Electromagnetic properties are now easy to discuss. The transformation property of electromagnetic current was considered in Sec. 3. From analogous reasoning we conclude that the relevant transformation property is now that of T_{i3}. Unlike the case with scalar theory, we cannot have a linear combination of the unit operator and T_{i3} without violating invariance under space rotation. We are able to discuss the isovector part of the electromagnetic current only. Using Eqs. (28) and (31) we thus obtain for the isovector magnetic moment

$$u=\frac{\nu}{k+1}b'; \tag{32}$$

b' is a constant, and we have adopted the (obvious) convention of defining the magnetic moment with respect to the state of maximum spin projection, i.e., with $m=k$. Similarly electromagnetic mass shift transforms like $T_{i3}T_{i3}(=T_{13}{}^2+T_{23}{}^2+T_{33}{}^2)$. Hence, the electromagnetic mass formula is

$$m=\langle k\nu m|T_{i3}T_{i3}|k\nu m\rangle_{m=k}C'\,, \tag{33}$$

where C' is a constant. Using Eqs. (26), (28), and (31) we evaluate the above matix element and find that

$$\langle k\nu m|T_{i3}T_{i3}|k\nu m\rangle_{m=k}=1\,. \tag{34}$$

Thus no electromagnetic mass splitting obtains in this theory.[9] This situation is due to the peculiar interrelation of spin and isospin in the pseudoscalar theory. These variables occur in a completely symmetric fashion, as is evident, for instance, from the isobar spectrum. Because of this situation, it is impossible to introduce anisotropy in the isotopic-spin space (i.e., obtain electromagnetic mass splitting) without necessarily introducing the same in ordinary space. By the same reasoning it is impossible to introduce, in a nontrivial way, the isoscalar part of the electromagnetic current without violating invariance under ordinary space rotations. This feature seems to be a serious drawback of the pseudoscalar strong-coupling theory.

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[9] A similar situation has been noted previously in Ref. 7.