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Isobar Model for Resonance Scattering*

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An explicit representation of the Rarita-Schwinger wave function is constructed, and with its help it is shown that many calculations of resonance scattering within an isobar model can easily be performed without explicitly calculating the propagator function.

I. INTRODUCTION

IN calculations involving resonances of high spin, the resonant particle is frequently described by a field and perturbation theory used to calculate the decay width of that particle. The expression for the width in this "isobar model" differs from the frequently used $\Gamma=g^2p^{2l+1}$ (l the relative orbital angular momentum of the two decay products in the center-of-momentum system, p the three-momentum, and g the phenomenological coupling constant) and is more appropriate when p is not very small.

Isobar-model calculations have been performed for spin-$\frac{3}{2}$[1] and spin-$\frac{5}{2}$[2] resonances. The difficulty of this method is that for higher spin the expression for the positive-energy projection operator becomes extremely lengthy and unwieldy. For spin-$\frac{3}{2}$ this is

$$X_{\mu\nu}(p)=\frac{m-p}{2m}\left\{\delta_{\mu\nu}-\tfrac{1}{3}\gamma_\mu\gamma_\nu+\frac{i}{3m}(\gamma_\mu p_\nu-\gamma_\nu p_\mu)+\frac{2}{3m^2}p_\mu p_\nu\right\}$$

which is of moderate length; but for spin $\frac{5}{2}$ it is very long,[2] and even more so for spin $\frac{7}{2}$, etc.[3]

It is the purpose of this paper to show that if one use an explicit representation of the wave function in the Rarita-Schwinger formalism[4] many calculations can be performed quite easily and quickly. We will demonstrate these techniques with two examples: (1) calculating the decay width of an arbitrary spin fermion decaying into spin-$\frac{1}{2}$ and spin-0 particles,[5] and the associated amplitude for creation and decay of such a resonance, and (2) calculating the multipole transition amplitudes for photoproduction of these resonances.

II. GENERAL THEORY

In the Rarita-Schwinger formalism,[4] the wave function representing a particle of spin J, where $J=n+\frac{1}{2}$, n an integer, is an n-component tensor-spinor,

$$u_{\alpha_1\cdots\alpha_n}{}^{(m)}(p),$$

i.e., under a Lorentz transformation, it transforms like the product of n vectors and one four-component spinor. The superscript (m) denotes the z component of the angular momentum. In addition, this function must satisfy three conditions[3]:

(i) $(p+m)u_{\alpha_1\cdots\alpha_n}{}^{(m)}(p)=0$ for all $(\alpha_1,\cdots,\alpha_n)$

(ii) $u_{\alpha_1\cdots\alpha_i\cdots\alpha_j\cdots\alpha_n}{}^{(m)}(p)=u_{\alpha_1\cdots\alpha_j\cdots\alpha_i\cdots\alpha_n}{}^{(m)}(p)$ for all i and j

(iii) $\gamma_{\alpha_1}u_{\alpha_1\cdots\alpha_n}{}^{(m)}(p)=0$ for all $(\alpha_2,\cdots,\alpha_n)$,

γ_α being the 4×4 Dirac matrices. As a result of these, the following conditions are also satisfied:

(iv) $p_{\alpha_1}u_{\alpha_1\cdots\alpha_n}{}^{(m)}(p)=0$ for all $(\alpha_2,\cdots,\alpha_n)$ (Lorentz condition),

(v) $u_{\alpha_1\cdots\alpha_1\cdots\alpha_n}{}^{(m)}(p)=0$ for all $(\alpha_2,\cdots,\alpha_{i-1},\alpha_{i+1},\cdots,\alpha_n)$.

An explicit representation[6] of the Rarita-Schwinger wave function may be constructed in the following manner: Couple a spin-$\frac{1}{2}$ spinor to a spin-1 vector, with appropriate Clebsch-Gordan coefficients, to give a spin-$\frac{3}{2}$ function; couple this spin-$\frac{3}{2}$ function to a spin-1 vector, again with appropriate Clebsch-Gordan coefficients, to give a spin-$\frac{5}{2}$

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[1] See, for example, M. Gourdin and Ph. Salin, Nuovo Cimento **27**, 193 (1963); **27**, 309 (1963).
[2] D. M. Brudnoy, Phys. Rev. Letters **14**, 273 (1965).
[3] C. Fronsdal, Nuovo Cimento Suppl. **9**, 416 (1958). A method for obtaining an expression for the positive-energy projection operator for arbitrary spin is contained in this article.
[4] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).
[5] Many of these calculations were performed by J. G. Rushbrooke, Phys. Rev. **143**, 1345 (1966) using the expressions of C. Fronsdal, Ref. 3.
[6] This construction has been given for spin-$\frac{3}{2}$ functions by S. Kusaka, Phys. Rev. **60**, 61 (1941).

function; and so on, n times. The resulting expression can be written

$$u_{\alpha_1\cdots\alpha_n}{}^{(m)}(p)=\sum_{m_1,\cdots,m_n}\langle n-\tfrac{1}{2}\ 1(m-m_1)m_1|n+\tfrac{1}{2}\ m\rangle\langle n-\tfrac{3}{2}\ 1(m-m_1-m_2)m_2|n-\tfrac{1}{2}\ (m-m_1)\rangle$$
$$\cdots\langle\tfrac{1}{2}1(m-m_1-\cdots-m_n)m_n|\tfrac{3}{2}(m-m_1-\cdots-m_{n-1})\rangle u^{(m-m_1-\cdots-m_n)}(p)(\epsilon_{m_1}(p))_{\alpha_1}(\epsilon_{m_2}(p))_{\alpha_2}\cdots(\epsilon_{m_n}(p))_{m_n}, \quad (1)$$

where the $\langle j1m_sm_v|j+1M\rangle$ are Clebsch-Gordan coefficients, $u^{(m_s)}(p)$ is the four-component Dirac spinor,[7] and $\epsilon_{m_v}(p)$ the four-component spherical spin-1 vector.[7] Note that any term for which $m-m_1-\cdots-m_n\neq-\frac{1}{2}$ or $+\frac{1}{2}$ in Eq. (1) is zero.

(i) It is trivial to see that this function obeys the Dirac equation, for any vector component, by construction.
(ii) Because of the property of the Clebsch-Gordan coefficients,[8]

$$\langle j1m_sm_v|j+1\ M\rangle=\left(\frac{2!(2j)!}{(2(j+1))!}\frac{(j+1+M)!(j+1-M)!}{(j+m_s)!(j-m_s)!(1+m_v)!(1-m_v)!}\right)^{1/2},$$

Eq. (1) can be written as

$$u_{\alpha_1\cdots\alpha_n}{}^{(m)}(p)=\left(\frac{2^n(n+\frac{1}{2}+m)!(n+\frac{1}{2}-m)!}{(2n+1)!}\right)^{1/2}\sum_{m_1,\cdots,m_n}\frac{1}{[(1+m_1)!(1-m_1)!\cdots(1+m_n)!(1-m_n)!]^{1/2}}$$
$$\times u^{(m-m_1-\cdots-m_n)}(p)(\epsilon_{m_1}(p))_{\alpha_1}\cdots(\epsilon_{m_n}(p))_{\alpha_n}. \quad (1')$$

Since all the m_k's are summation indices from -1 to $+1$, the function is manifestly symmetric under the exchange $\alpha_i\leftrightarrow\alpha_j$, for any i and j.

(iii) Similarly it is easy to show that $\gamma_{\alpha_1}u_{\alpha_1\cdots\alpha_n}{}^{(m)}(p)=0$. Because $u_{\alpha_1\cdots\alpha_n}{}^{(m)}(p)$ is totally symmetric, one need prove this only for the case $n=1$. Also $\gamma_{\alpha_1}u_{\alpha_1}{}^{(m)}(p)$ transforms like a four-spinor, i.e., $\gamma_{\alpha_1}u_{\alpha_1}{}^{(m)}(p)=S\gamma_{\alpha_1}u_{\alpha_1}{}^{(m)}(0)$ where S is the "boost" transformation for the spin-$\frac{1}{2}$ spinor, $u^{(m_s)}(p)=Su^{(m_s)}(0)$; hence, one need consider only $\gamma_{\alpha_1}u_{\alpha_1}{}^{(m)}(0)$. The proof that this is zero is then straightforward.

Using Eq. (1) we see that the wave function is normalized according to

$$\bar{u}_{\alpha_1\cdots\alpha_n}{}^{(m')}(p)u_{\alpha_1\cdots\alpha_n}{}^{(m)}(p)=\delta_{m'm}.$$

Thus, Eq. (1) [or (1′)] may be used to represent a particle of spin $(n+\frac{1}{2})$ in the Rarita-Schwinger formalism. The positive-energy projection operator is defined as

$$X_{\alpha_1\cdots\alpha_n;\beta_1\cdots\beta_n}(p)\equiv\sum_{m=-J}^{J}u_{\alpha_1\cdots\alpha_n}{}^{(m)}(p)\bar{u}_{\beta_1\cdots\beta_n}{}^{(m)}(p), \quad (2)$$

and because of the normalization of the wave function obeys

$$X_{\alpha_1\cdots\alpha_n;\gamma_1\cdots\gamma_n}(p)X_{\gamma_1\cdots\gamma_n;\beta_1\cdots\beta_n}(p)=X_{\alpha_1\cdots\alpha_n;\beta_1\cdots\beta_n}(p).$$

III. DECAY OF RESONANCE INTO SPIN-$\frac{1}{2}$ FERMION AND SPIN-0 BOSON

In order to calculate the decay

$$J\to\tfrac{1}{2}+0, \quad (3)$$

we consider the process described by the graph of Fig. 1. We use the following Lagrangian density to describe the coupling

$$\mathcal{L}(x)=\frac{g}{m_\pi{}^n}\bar{\psi}(x)[\gamma_5]\psi_{\alpha_1\cdots\alpha_n}(x)\partial_{\alpha_1}\cdots\partial_{\alpha_n}\varphi(x)+\text{H.c.}, \quad (4)$$

[7] The following conventions are used:

$$u^{(m_s)}(p)=\frac{1}{[2m(m+E)]^{1/2}}\begin{pmatrix}(m+E)\chi_{m_s}\\ \boldsymbol{\sigma}\cdot\mathbf{p}\chi_{m_s}\end{pmatrix},\ \boldsymbol{\sigma}\text{ the Pauli matrices,}$$

and χ_{m_s} the Pauli two-component spinors, $\chi_{\frac{1}{2}}=\begin{pmatrix}1\\0\end{pmatrix}$, $\chi_{-\frac{1}{2}}=\begin{pmatrix}0\\1\end{pmatrix}$. For p in the z direction, say,

$$\epsilon_{+1}(p)=-\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\\0\\0\end{bmatrix},\quad\epsilon_{-1}(p)=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\\0\\0\end{bmatrix},\quad\epsilon_0(p)=\begin{bmatrix}0\\0\\E/m\\ip/m\end{bmatrix}.$$

[8] See, e.g., A. Messiah, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1962), Vol. II, Appendix C, p. 1059, Eq. (C.25)

where $1/m_\pi{}^n$ is inserted (m_π being the mass of the pion) to make g dimensionless. For the lowest order perturbation theory, any different combination of couplings which are both Lorentz-invariant and parity-conserving, will be equivalent to the above because of the conditions (i)-(v) of Sec. II. The brackets around γ_5 are to indicate that γ_5 is included in the coupling whenever $J=l-\frac{1}{2}$ and is not included whenever $J=l+\frac{1}{2}$.

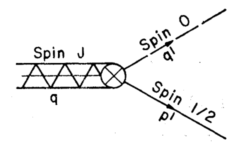


FIG. 1. Decay of resonance of spin-J into spin-$\frac{1}{2}$ fermion and spin-0 meson.

Using the Feynman rules, we get for the transition rate of (3)

$$\Gamma_D=\frac{g^2}{m_\pi{}^{2n}}\frac{p}{2\pi}\frac{m_N}{m_R}\frac{1}{2J+1}\sum_{m'}\bar{u}(p')^{(m')}q_{\alpha_1}'\cdots q_{\alpha_n}'q_{\beta_1}'\cdots q_{\beta_n}'[\gamma_5]X_{\alpha_1\cdots\alpha_n;\beta_1\cdots\beta_n}(q)[-\gamma_5]u^{(m')}(p'),\tag{5}$$

where p is the three-momentum in the center-of-momentum system, m_N the spin-$\frac{1}{2}$ fermion mass, m_R the resonance mass.

In the center-of-momentum system, choosing $\mathbf{q}'$ along the z axis, $(q'\cdot\epsilon_{m_v}(q))=p\delta_{m_v0}$. Thus substituting Eq. (1′) into Eq. (2) we obtain

$$q_{\alpha_1}'\cdots q_{\alpha_n}'q_{\beta_1}'\cdots q_{\beta_n}'X_{\alpha_1\cdots\alpha_n;\beta_1\cdots\beta_n}(q)=\frac{2J+1}{2}\frac{(n!)^2 2^n}{(2n+1)!}p^{2n}\left(\frac{m_R-q}{2m_R}\right).\tag{6}$$

Combining Eqs. (5) and (6), we immediately obtain[5]

$$\Gamma_D=\frac{g^2}{4\pi}\frac{(n!)^2 2^n}{(2n+1)!}\frac{E_N\pm m_N}{m_R}\left(\frac{p}{m_\pi}\right)^{2n+1}m_\pi,\quad J=l\pm\tfrac{1}{2},\tag{7}$$

where E_N is the energy of the spin-$\frac{1}{2}$ fermion. For very small p, $\Gamma_D\sim g^2p^{2l+1}$, as required.

The scattering amplitude for meson-nucleon scattering is given by[9]

$$f_{m'm}(W,\cos\theta)=\chi_{m'}{}^\dagger[f_1(W,\cos\theta)+\boldsymbol{\sigma}\cdot\hat{p}'\boldsymbol{\sigma}\cdot\hat{p}f_2(W,\cos\theta)]\chi_m,\tag{8}$$

where W is the energy in the center-of-momentum system, θ the c.m. scattering angle, and χ_m the Pauli two-component spinors. The partial-wave decomposition of the functions f_1 and f_2 is[9]

$$f_1(W,\cos\theta)=\sum_{l=0}^{\infty}[f_{l+}(W)P_{l+1}'(\cos\theta)-f_{l-}(W)P_{l-1}'(\cos\theta)]\tag{9a}$$

$$f_2(W,\cos\theta)=\sum_{l=0}^{\infty}[f_{l-}(W)-f_{l+}(W)]P_l'(\cos\theta).\tag{9b}$$

Here l represents the relative orbital angular momentum of the final-state particles, and $l\pm$ means $J=l\pm\frac{1}{2}$; the energy-dependent functions $f_{l\pm}(W)$ are the partial-wave amplitudes, and $P_l(\cos\theta)$ the Legendre polynomials.

For scattering in only one total angular momentum state, the partial-wave amplitude may be calculated most easily by considering the scattering at $\theta=0$. By Eqs. (8) and (9):

$$f_{l\pm}(W)=(2J+1)^{-1}\textstyle\sum_m\chi_m{}^\dagger[f_1(\boldsymbol{\sigma}_1(W,\theta=0)+f_2(W,\theta=0)]\chi_m.\tag{10}$$

In order to calculate the process $\frac{1}{2}+0\rightarrow J\rightarrow\frac{1}{2}'+0'$ (see Fig. 2) we use the coupling, Eq. (4). Applying Eq. (6), we immediately obtain that at resonance

$$f_{l\pm}(m_R)=4\pi i\left(\frac{g_i}{4\pi}\right)\left(\frac{g_f}{4\pi}\right)\frac{[(E_{N_i}\pm m_{N_i})(E_{N_f}\pm m_{N_f})]^{1/2}}{m_R\Gamma}\frac{(n!)^2 2^n}{(2n+1)!}\left(\frac{|\mathbf{q}'|}{m_\pi}\right)^n\left(\frac{|\mathbf{k}|}{m_\pi}\right)^n.\tag{11}$$

This may also be written $f_{l\pm}(m_R)=(i/\Gamma)(\Gamma_D{}^i\Gamma_D{}^f/|\mathbf{k}||\mathbf{q}'|)^{1/2}$, where $\Gamma_D{}^{i(f)}$ are the partial decay widths into the initial (final) states [Eq. (7)] and Γ is the total width.

FIG. 2. Meson-nucleon scattering via a resonance of spin J.

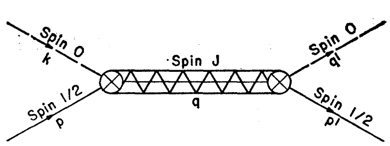
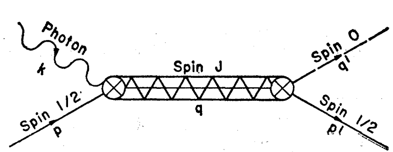


FIG. 3. Photo-meson production via a resonance of spin J.



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[9] See, e.g., S. Gasiorowicz, Fortschr. Physik 8, 665 (1960), especially Appendix A, p. 721.

IV. PHOTOPRODUCTION OF SPIN-0 MESON VIA SPIN-J RESONANCE

In discussing the photomeson production process (see Fig. 3) we use the kinematic description and partial-wave analysis given by Chew, Goldberger, Low, and Nambu.[10] There the scattering amplitude in the center-of-momentum system can be written[10]

$$f_{m'm}{}^{m_\gamma}(W,\theta)=\chi_{m'}{}^\dagger[i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_{m_\gamma}f_1(W,\theta)+\boldsymbol{\sigma}\cdot\hat{q}'\boldsymbol{\sigma}\cdot(\hat{k}\times\boldsymbol{\varepsilon}_{m_\gamma})f_2(W,\theta)+i\boldsymbol{\sigma}\hat{k}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_{m_\gamma}f_3(W,\theta)+i\boldsymbol{\sigma}\cdot\hat{q}'\hat{q}'\cdot\boldsymbol{\varepsilon}_{m_\gamma}f_4(W,\theta)]\chi_m, \tag{12}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices, $\boldsymbol{\varepsilon}_{m_\gamma}$ the photon helicity functions ($m_\gamma=+1$ or -1), $\hat{k}$ the direction of the photon, which here is taken in the z direction, and $\hat{q}'$ the direction of the meson. Specificially

$$f_{\frac{1}{2}\frac{1}{2}}{}^1(W,\theta)=f_{-\frac{1}{2}-\frac{1}{2}}{}^{-1}(W,\theta)=-\frac{i}{\sqrt{2}}\sin\theta[f_3(W,\theta)+\cos\theta f_4(W,\theta)], \tag{13a}$$

$$f_{\frac{1}{2}-\frac{1}{2}}{}^1(W,\theta)=-f_{-\frac{1}{2}\frac{1}{2}}{}^{-1}(W,\theta)=-\frac{i}{\sqrt{2}}[2(f_1(W,\theta)-\cos\theta f_2(W,\theta))+\sin^2\theta f_4(W,\theta)], \tag{13b}$$

$$f_{-\frac{1}{2}\frac{1}{2}}{}^1(W,\theta)=-f_{\frac{1}{2}-\frac{1}{2}}{}^{-1}(W,\theta)=-\frac{i}{\sqrt{2}}\sin^2\theta f_4(W,\theta), \tag{13c}$$

$$f_{-\frac{1}{2}-\frac{1}{2}}{}^1(W,\theta)=f_{\frac{1}{2}\frac{1}{2}}{}^{-1}(W,\theta)=\frac{i}{\sqrt{2}}\sin\theta[2f_2(W,\theta)+f_3(W,\theta)+\cos\theta f_4(W,\theta)]. \tag{13d}$$

The functions $f_r(W,\theta)$ can be decomposed[11]:

$$f_1(W,\theta)=\sum_{l=0}^{\infty}[lM_{l+}(W)+E_{l+}(W)]P_{l+1}'(\cos\theta)+[(l+1)M_{l-}(W)+E_{l-}(W)]P_{l-1}'(\cos\theta), \tag{14a}$$

$$f_2(W,\theta)=\sum_{l=0}^{\infty}[(l+1)M_{l+}(W)+lM_{l-}(W)]P_l'(\cos\theta), \tag{14b}$$

$$f_3(W,\theta)=\sum_{l=0}^{\infty}[E_{l+}(W)-M_{l+}(W)]P_{l+1}''(\cos\theta)+[M_{l-}(W)+E_{l-}(W)]P_{l-1}''(\cos\theta), \tag{14c}$$

$$f_4(W,\theta)=\sum_{l=0}^{\infty}[M_{l+}(W)-E_{l+}(W)-E_{l-}(W)-M_{l-}(W)]P_l''(\cos\theta), \tag{14d}$$

$P_l'(\cos\theta)$ the derivatives of the Legendre polynomials, $P_l(\cos\theta)$. The functions $M_{l\pm}(W)$, $E_{l\pm}(W)$ are called magnetic (electric) multipole transition amplitudes, multipole referring to the state of the initial-state photon,[12] the subscript $l\pm$ meaning that the final meson-nucleon state has a relative orbital angular momentum l, $J=l+\frac{1}{2}$ or $J=l-\frac{1}{2}$. One of the differences between this case and meson-nucleon scattering is that here for scattering in only one total angular momentum state we can have two amplitudes. The reason is that consistent with the conservation of total angular momentum and of parity, there are two possible multipole states[12] for the photon. If the multipole index is denoted by $l_{\gamma\pm}=J\pm\frac{1}{2}$, then if the parity is such that $P_J=-(-1)^{l_{\gamma-}}=(-1)^{l_{\gamma+}}$, we have, by definition, a magnetic $2^{l_{\gamma-}}$ pole *and* an electric $2^{l_{\gamma+}}$ pole. For opposite parity we have a magnetic $2^{l_{\gamma+}}$ pole *and* an electric $2^{l_{\gamma-}}$ pole. The two amplitudes add coherently.

Since there are two multipole possibilities for the electromagnetic coupling, its Lagrangian, for free particles, must contain two terms. With the additional restriction of gauge-invariance,[10] the Lagrangian density may be written as

$$\mathfrak{L}_{\text{em}}(x)=\frac{eg_1}{m_\pi{}^n}\bar{\psi}(x)\gamma_\mu[\gamma_5]\psi_{\alpha_1\cdots\alpha_n}(x)\partial_{\alpha_1}\cdots\partial_{\alpha_{n-1}}F_{\alpha_n\mu}(x)+\frac{eg_2}{m_\pi{}^{n+1}}\bar{\psi}(x)\partial_\mu[\gamma_5]\psi_{\alpha_1\cdots\alpha_n}(x)\partial_{\alpha_1}\cdots\partial_{\alpha_{n-1}}F_{\alpha_n\mu}(x)+\text{H.c.}, \tag{15}$$

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[10] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

[11] The convention adopted here is the one used in Ref. 10.

[12] For a more complete discussion of multipoles, see, e.g., M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957), Chap. VII.

where $F_{\alpha_n\mu}(x)$ is the electromagnetic tensor. All other Lorentz-invariant and parity-conserving combinations are equivalent to the above by the conditions (i)–(v) of Sec. II. Of course, the boson coupling is as in the previous section. In both couplings $1/m_\pi$ to the appropriate power is inserted to make the coupling constants dimensionless (m_π is the mass of the pion). As before, the brackets around γ_5 indicate that it is included in the electromagnetic coupling whenever $n=l$; in this case γ_5 is not included in the boson coupling. Just the opposite is true when $n=l-1$.

The differential cross section depends on the J^P of the resonance and on the value of the two constants g_1 and g_2 in the Lagrangian. Our aim is to express the multipole amplitudes corresponding to the above Lagrangian in terms of these constants and, thus, obtain the differential cross section in terms of two coupling constants.

The multipole amplitudes are calculated in the same way that the partial-wave amplitudes were calculated in the previous section but with the following modification. Since for a transition of definite J two amplitudes must be calculated, we must evaluate the scattering amplitude at two different scattering angles. (Actually since J_z is conserved, we could just as easily consider the scattering amplitude at one angle but for the two projections $m=\frac{1}{2}$ and $m=\frac{3}{2}$.) For convenience we choose the angles $\theta=0$ (or equivalently $\theta=\pi$) and $\theta=\pi/2$. By Eqs. (13) one may consider only $m_\gamma=+1$ without loss of generality.

We consider first the case of $n=l-1$. Forward scattering: Here one needs the relationship

$$q_{\alpha_1}'\cdots q_{\alpha_n}'k_{\beta_1}\cdots k_{\beta_{n-1}}(\epsilon_{m_\gamma})_{\beta_n}X_{\alpha_1\cdots\alpha_n;\beta_1\cdots\beta_n}(q)=\frac{2^n(n+1)(n!)^2}{\sqrt{2}(2n+1)!}\sum_m u^{(m)}(q)\bar{u}^{(m-m_\gamma)}(q)\,, \tag{16}$$

since

$$(\epsilon_{m_n}{}^*(q))_{\beta_n}(\epsilon_{m_\gamma})_{\beta_n}=\delta_{m_nm_\gamma}\,.$$

The rest is straightforward algebra and the result for the transition matrix is

$$T_{m',-1/2}{}^1(m_R,0)=\frac{eg}{m_\pi{}^{2n}}\frac{1}{\Gamma}\frac{(n+1)(n!)^2 2^n}{(2n+1)!}|\mathbf{k}|^n|\mathbf{q}'|^{n+1}\left(\frac{E_{N_i}+m_{N_i}}{E_{N_f}+m_{N_f}}\right)^{1/2}$$
$$\times\left[g_1\left(1-\frac{\Delta m}{m_R+m_{N_i}}\right)+g_2\left(\frac{m_R}{m_\pi}\right)\right]\frac{\chi_{m'}{}^\dagger i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_1\chi_{-1/2}}{2(m_{N_i}m_{N_f})^{1/2}}\,, \tag{17}$$

where χ_m are the two-component Pauli spinors, $\Delta m=m_R-m_{N_i}$, m_{N_i} the mass and E_{N_i} the energy of the initial-state fermion, m_{N_f} the mass and E_{N_f} the energy of the final-state fermion, and m_R the mass of the resonance particle. Here we have used the relation

$$\frac{|\mathbf{k}|}{E_{N_i}+m_{N_i}}=\frac{\Delta m}{m_R+m_{N_i}}\,.$$

By Eqs. (17), (13b), (14a), and (14b) we obtain

$$-l[(l+1)M_{l-}(m_R)-(l-1)E_{l-}(m_R)]=\frac{eg}{4\pi m_R}\frac{1}{\Gamma}\frac{(n+1)(n!)^2 2^n}{(2n+1)!}\left(\frac{E_{N_i}+m_{N_i}}{E_{N_f}+m_{N_f}}\right)^{1/2}$$
$$\times\left[g_1\left(1-\frac{\Delta m}{m_R+m_{N_i}}\right)+g_2\left(\frac{m_R}{m_\pi}\right)\right]\left(\frac{|\mathbf{k}|}{m_\pi}\right)^n\left(\frac{|\mathbf{q}'|}{m_\pi}\right)^{n+1}m_\pi\,. \tag{18}$$

Scattering at $\pi/2$:

Since in this case $\mathbf{q}'=(|\mathbf{q}'|,0,0)$, it is not difficult to convince oneself that the following is true

$$q_{\alpha_1}'\cdots q_{\alpha_n}'u_{\alpha_1\cdots\alpha_n}{}^{(m)}(q)=(-1)^s\binom{n}{s}\left[\frac{(n+\frac{1}{2}+m)!(n+\frac{1}{2}-m)!}{(2n+1)!}\right]^{1/2}(\mathbf{q}\cdot\boldsymbol{\varepsilon}_{-1})^n u(q)^{(m+n-2s)}\,, \tag{19}$$

where $\binom{n}{s}\equiv n!/s!(n-s)!$ and s is a number from 0 to n such that for definite m, $(m+n-2s)$ is either $+\frac{1}{2}$ or $-\frac{1}{2}$. One obtains Eq. (19) by using the property of the Clebsch-Gordan coefficients mentioned previously,[8] i.e.,

$$\langle j1m_sm_v|j+1\,M\rangle=\left[\frac{2(2j)!}{(2(j+1))!}\frac{(j+1+M)!(j+1-M)!}{(j+m_s)!(j-m_s)!(1+m_v)!(1-m_v)!}\right]^{1/2}.$$

Using Eq. (19) and choosing $m=\frac{3}{2}$, the transition matrix is

$$T_{m',1/2}{}^{1}(m_R,\pi/2)=\frac{eg}{m_\pi{}^{2n}}\frac{1}{\Gamma}\frac{n+2}{n}\frac{(n+1)(n!)^2}{(2n+1)!}(-1)^s\binom{n}{s}\frac{|\mathbf{q}'|^{n+1}|\mathbf{k}|^n}{(2m_{N_i}m_{N_f})^{1/2}}$$

$$\times\left(\frac{E_{N_i}+m_{N_i}}{E_{N_f}+m_{N_f}}\right)^{1/2}\left[g_1\left(1+\frac{\Delta m}{m_R+m_{N_i}}\right)+g_2\left(\frac{m_R}{m_\pi}\right)\right]\delta_{-m';\frac{3}{2}+n-2s}. \quad (20)$$

Hence if $n=$even, then $\frac{3}{2}+n-2s=-\frac{1}{2}$ and $s=(n+2)/2$. If $n=$odd, then $\frac{3}{2}+n-2s=\frac{1}{2}$ and $s=(n+1)/2$.

Using Eqs. (20), (13a), and (14c) we obtain for $n=$even, $l=$odd

$$[M_{l-}(m_R)+E_{l-}(m_R)]P_{l-1}''(0)=(-1)^{(n+2)/2}\binom{n}{(n+2)/2}\left(\frac{n+2}{n}\right)\left[g_1\left(1+\frac{\Delta m}{m_R+m_{N_i}}\right)+g_2\left(\frac{m_R}{m_\pi}\right)\right]G, \quad (21a)$$

where

$$G=\frac{eg}{4\pi}\frac{1}{m_R\Gamma}\frac{(n+1)(n!)^2}{(2n+1)!}\left(\frac{E_{N_i}+m_{N_i}}{E_{N_f}+m_{N_f}}\right)^{1/2}\left(\frac{|\mathbf{q}'|}{m_\pi}\right)^{n+1}\left(\frac{|\mathbf{k}|}{m_\pi}\right)^n m\,.$$

Using Eqs. (20), (13c), and (14d) we obtain for $n=$odd, $l=$even

$$-[E_{l-}(m_R)+M_{l-}(m_R)]P_l''(0)=(-1)^{(n+1)/2}\binom{n}{(n+1)/2}\left(\frac{n+2}{n}\right)\left[g_1\left(1+\frac{\Delta m}{m_R+m_{N_i}}\right)+g_2\left(\frac{m_R}{m_\pi}\right)\right]G. \quad (21b)$$

By Eqs. (18), (21a), and (21b) we obtain

$$E_{l-}(m_R)=\frac{2^n}{n(n+1)}\left[g_1\left(1+\frac{1}{n+1}\frac{\Delta m}{m_R+m_{N_i}}\right)+g_2\left(\frac{m_R}{m_\pi}\right)\right]G, \quad (22a)$$

$$M_{l-}(m_R)=\frac{2^n}{(n+1)^2}g_1\frac{\Delta m}{m_R+m_{N_i}}G. \quad (22b)$$

Here we have used[13]

$$P_r''(0)=-r(r+1)P_r(0)=\frac{1}{2^r}(r+2)(r+1)(-1)^{(r+2)/2}\binom{r}{(r+2)/2},\quad r=\text{even}$$

$$=0,\qquad r=\text{odd}.$$

The case $n=l$ is treated in exactly the same manner and we bring only the results.

$$E_{l+}(m_R)=\frac{2^n}{(n+1)^2}\frac{\Delta m}{m_R+m_{N_i}}\left[g_1+g_2\left(\frac{m_R}{m_\pi}\right)\right]G', \quad (23a)$$

$$M_{l+}(m_R)=-\frac{2^n}{n(n+1)}\left[g_1\left(1+\frac{1}{n+1}\frac{\Delta m}{m_R+m_{N_i}}\right)+\frac{1}{n+1}\frac{\Delta m}{m_R+m_{N_i}}g_2\left(\frac{m_R}{m_\pi}\right)\right]G', \quad (23b)$$

where

$$G'=\frac{eg}{4\pi}\frac{(n+1)(n!)^2}{(2n+1)!}\frac{1}{m_R\Gamma}[(E_{N_i}+m_{N_i})(E_{N_f}+m_{N_f})]^{1/2}\left(\frac{|\mathbf{q}'|}{m_\pi}\right)^n\left(\frac{|\mathbf{k}|}{m_\pi}\right)^n.$$

The differential cross section may be expressed in terms of $M_{l\pm}(m_R)$ and $E_{l\pm}(m_R)$.

$$\left.\frac{d\sigma}{d\Omega}\right|_{n=l-1}=\frac{1}{2}\frac{|\mathbf{q}'|}{|\mathbf{k}|}\{A_l{}^1(\theta)|M_{l-}(m_R)|^2+A_l{}^2(\theta)|E_{l-}(m_R)|^2-A_l{}^3(\theta)\,\mathrm{Re}M_{l-}{}^*(m_R)E_{l-}(m_R)\}, \quad (24a)$$

[13] See, e.g., I. N. Sneddon, *Special Functions of Mathematical Physics and Chemistry* (Oliver and Boyd Ltd., London, 1961), Chap. III, especially p. 92.

where

$$A_l^1(\theta)=2(l+1)^2P_{l-1}'^2+2l^2P_l'^2-4l(l+1)\cos\theta P_{l-1}'P_l'-2(l+1)\sin^2\theta P_{l-1}'P_l''$$
$$-2\sin^2\theta\cos\theta P_{l-1}''P_l''+\sin^2\theta(P_{l-1}''^2+P_l''^2)+2l\sin^2\theta P_l'P_{l-1}'',$$
$$A_l^2(\theta)=2P_{l-1}'^2-2\sin^2\theta P_{l-1}'P_l''-2\sin^2\theta\cos\theta P_{l-1}''P_l''+\sin^2\theta(P_{l-1}''^2+P_l''^2),$$
$$A_l^3(\theta)=-4(l+1)P_{l-1}'^2+4l\cos\theta P_{l-1}'P_l'+2(l+2)\sin^2\theta P_{l-1}'P_l'' \tag{24b}$$
$$+4\sin^2\theta\cos\theta P_{l-1}''P_l''-2\sin^2\theta(P_{l-1}''^2+P_l''^2)-2l\sin^2\theta P_l'P_{l-1}''.$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{n=l}=\frac{1}{2}\frac{|\mathbf{q}'|}{|\mathbf{k}|}\{B_l^1(\theta)|M_{l+}(m_R)|^2+B_l^2(\theta)|E_{l+}(m_R)|^2+B_l^3(\theta)\,\mathrm{Re}M_{l+}^*(m_R)E_{l+}(m_R)\},$$

where

$$B_l^1(\theta)=2l^2P_{l+1}'^2+2(l+1)^2P_l'^2-4l(l+1)\cos\theta P_l'P_{l+1}'+2l\sin^2\theta P_l''P_{l+1}'$$
$$-2\sin^2\theta\cos\theta P_l''P_{l+1}''+\sin^2\theta(P_{l+1}''^2+P_l''^2)-2(l+1)\sin^2\theta P_l'P_{l+1}'',$$
$$B_l^2(\theta)=2P_{l+1}'^2-2\sin^2\theta P_l''P_{l+1}'-2\sin^2\theta\cos\theta P_l''P_{l+1}''+\sin^2\theta(P_{l+1}''^2+P_l''^2),$$
$$B_l^3(\theta)=4lP_{l+1}'^2-4(l+1)\cos\theta P_l'P_{l+1}'-2(l-1)\sin^2\theta P_l''P_{l+1}'$$
$$+4\sin^2\theta\cos\theta P_l''P_{l+1}''-2\sin^2\theta(P_{l+1}''^2+P_l''^2)+2(l+1)\sin^2\theta P_l'P_{l+1}''.$$

V. SUMMARY

We have seen that by explicitly constructing the Rarita-Schwinger wave functions for fermions of arbitrary spin [Eqs. (1) or (1′)], many technical difficulties encountered in calculating widths and resonance amplitudes can be overcome. This method is particularly simple for calculating the width of a resonance decaying into spin-$\frac{1}{2}$ baryons and spin-0 mesons [Eq. (7)] and the associated meson-nucleon scattering amplitude [Eq. (11)]; it is also easily used for calculating photomeson production via a resonance particle [Eqs. (22) and (23)].

The relation of the phenomenological coupling constants to the decay widths [Eq. (7)] might prove helpful in determining the classification of higher spin resonances into, say, SU_3 groups. An analysis using Eq. (7) for the case of $F_{5/2}$ resonances has been performed in Ref. 2.

Photoproduction processes, on the other hand, cannot be completely described even within this simplified isobar model. The two coupling constants [Eq. (15)] must be obtained from experimental data in which there is generally a non-negligible interference between the resonant and the nonresonant parts, particularly since there is experimental evidence that resonances are less strongly excited by photons than they are by pseudoscalar mesons.

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