The problem of the mass spectrum is therewith reduced to the problem of the simpler spectrum of the α.

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Proton-Proton Bremsstrahlung*†

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We investigate proton-proton bremsstrahlung by using a field-theory, as opposed to a potential, model. We adopt the one-pion-exchange model and relate part of the matrix element to the photon-pion production matrix element, which is expressed in terms of dispersion relations. A detailed analysis shows that the Born-approximation diagrams give the dominant contribution and that the diagrams involving N^* or ρ resonances are negligible for the energy considered. We find that our results, unlike those of the potential model, seem to give satisfactory agreement with experimental data.

1. INTRODUCTION

SINCE the proposal of the meson theory of nuclear forces by Yukawa[1] a large number of theoretical as well as experimental investigations have been carried out to increase our knowledge about the nature of the nucleon-nucleon interaction.

One way to attack this basic problem is to investigate nucleon-nucleon bremsstrahlung. Although this process is more complicated than elastic nucleon-nucleon scattering, it yields more information than can be obtained from the study of the elastic case. In addition it is worthwhile to investigate this process for its own sake, since it is desirable to know the photon spectrum due to bremsstrahlung when we investigate a high-energy process such as neutral-pion production. Surprisingly enough, the experimental work[2,3] done on the nucleon-nucleon bremsstrahlung reaction has been very scanty and only recently have experiments been started to obtain measurements of good accuracy for the proton-proton case. Two preliminary results of the recent Harvard[4] and Rochester[5] experiments are now available. In these experiments the incident nucleon energy is below the threshold for pion production.

The importance of this process was first pointed out by Ashkin and Marshak.[6,7] They treat the nucleon-nucleon interaction in terms of a phenomenological Yukawa potential and show that proton-proton bremsstrahlung would be zero in the low-energy-photon limit if the recoil of the proton is neglected. Classically speaking, this corresponds to a vanishing electric-dipole transition.

Another approach involves the use of meson theory. Simon[8] calculated neutron-proton bremsstrahlung in the Born approximation for both scalar and pseudoscalar-meson theory in 1950. One interesting aspect of this approach is that a virtual charged pion can emit a real photon, a process which cannot be explained in the potential model.

At the high-energy end of the photon spectrum for nucleon-nucleon bremsstrahlung, we expect the final protons to have rather low kinetic energy. Dullemond and deSwart[9] discuss phenomenologically the effect of the $E2$ contribution in this region for proton-proton bremsstrahlung. Cutkosky[10] also discusses the final-state interactions for neutron-proton bremsstrahlung in this region.

The case of proton-nucleus bremsstrahlung[11,12] has also been investigated in phenomenological models.

The most thorough investigation based on the po-

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[12] W. C. Beckham, Lawrence Radiation Laboratory Report No. UCRL-7001 (unpublished).

tential model has been done by Sobel and Cromer.[13] These authors note that different phenomenological potentials which fit elastic nucleon-nucleon scattering will give different results for bremsstrahlung cross section. This method could therefore be used to discriminate among the known nucleon-nucleon potentials.

The scattering amplitude for nucleon-nucleon bremsstrahlung may be expressed generally in terms of that for physical nucleon-nucleon scattering in the low-energy-photon limit.[14] However, as is stressed by Sobel,[13] an actual calculation of the potential model for proton-proton bremsstrahlung involves matrix elements that are far from the proton energy shell. In fact, for an incident energy of 160 MeV the initial and final center-of-mass energies of the two protons may differ by a factor of more than 3.5.

This situation may be beyond the range of validity of the potential model. In fact, calculations based upon the potential model seem to give poor agreement with experiment. Therefore it is desirable to investigate the same process from a different point of view. We will refine Simon's early perturbation-theory treatment. Our approach may be summarized as follows.

We take the point of view that nucleon-nucleon bremsstrahlung occurs through one-pion exchange between nucleons. (See Fig. 1.) Since the pion is the lightest strongly interacting boson, we expect it to give the main contribution. The blob in the diagram represents the following virtual-pion capture process: $\pi+N\rightarrow\gamma+N$. First the off-shell effect is taken account of by writing the over-all amplitude as the physical amplitude times a pionic form factor $K(\Delta^2)$, where Δ^2 is the square of the 4-momentum of the virtual pion.

Then the physical amplitude is connected to the known photopion production amplitude with the help of dispersion relations. Since the theoretical structure of the right-hand π-N-N vertex is known, we can express all quantities in the matrix element, except $K(\Delta^2)$, in terms of observables.

The factor $K(\Delta^2)$ will be estimated in two separate ways and the results compared.

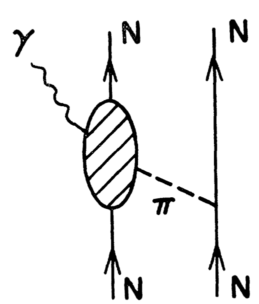

FIG. 1. A diagram for nucleon-nucleon bremsstrahlung.

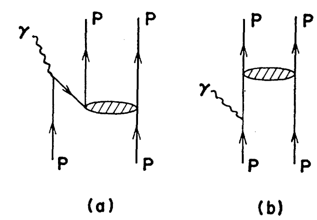

FIG. 2. Typical (time-ordered) Feynman diagrams for proton-proton bremsstrahlung.

One possibility, corresponding to ordinary perturbation theory, is to choose $K(\Delta^2)=1$. Another possibility is to accept a phenomenological form based on other experimental sources.

We found that for the energy considered the dominant contribution to the proton-proton bremsstrahlung process comes from the Born terms and not from resonance diagrams such as N^* or ρ.

The theory with a phenomenological pionic form factor seems to agree well with the preliminary results for $(d\sigma/d\Omega)_{90°}$ obtained from the Rochester experiment.

Our theory shows that corresponding to the vanishing electric-dipole transition, the photon energy spectrum of proton-proton bremsstrahlung is much reduced compared with the characteristic $1/k_0$ dependence in the low-energy region, where k_0 is the photon energy.

Before closing this section, we would like to discuss the validity of our model. We are aware that the elastic nucleon-nucleon interaction at low energies cannot be regarded as solely due to one-pion exchange. Exchange of virtual bosons (such as ρ, ω, η, etc.) between two protons should be considered as well.

In this paper, however, we assume one-pion exchange as a first, probably crude, approximation of the nucleon-nucleon interaction and neglect other contributions. The merit of doing this is that it enables us to treat the remaining part of the matrix element of the bremsstrahlung more rigorously than other theories. For instance, on the basis of perturbation theory the two (time-ordered) Feynman diagrams (a) and (b) of Fig. 2 should be treated on an equal footing. The diagram (a) is omitted in the potential-model calculation, but not in our approach.

The Sobel and Cromer theory[13] is criticized by Yennie,[15] based on the fact that their treatment does not seem to satisfy the direct result of charge conservation. Such criticism does not apply to us, since our field-theoretical treatment does not necessitate the expansion in terms of photon energy at all and the whole theory is constructed to be gauge-invariant. These may be the reasons that good agreement with experiment is obtained in our theory in spite of our crude approximation for nucleon-nucleon interactions.

[13] M. I. Sobel and A. H. Cromer, Phys. Rev. **132**, 2698 (1963); M. I. Sobel, Ph.D. thesis, Harvard University (unpublished).
[14] F. E. Low, Phys. Rev. **110**, 974 (1958).

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2. KINEMATICS AND ONE-PION-EXCHANGE FORMALISM

We denote by P_1, P_2, P_1', P_2', and k the energy momentum 4-vectors of the incoming two protons, outgoing two protons, and outgoing photon, respectively.

The T matrix is defined by

$$S_{fi}=\delta_{fi}-i\left(\frac{M^4}{2k_0E_1E_2E_1'E_2'}\right)^{1/2}(2\pi)^4 \times\delta^4(P_1'+P_2'+k-P_1-P_2)T_{fi}, \quad (2.1)$$

where E_1, E_2, E_1', E_2', and k_0 are the energies of the incoming two protons and outgoing two protons and a photon, and M is the proton mass. The differential cross section $d\sigma$ can be written in terms of T_{fi}. The relation is

$$d\sigma=\frac{M^4}{2}\frac{1}{F}\frac{1}{(2\pi)^5}\delta^4(P_1'+P_2'+k-P_1-P_2) \times\sum_f |T_{fi}|^2\frac{d^3P_1'd^3P_2'd^3k}{E_1'E_2'k_0}, \quad (2.2)$$

where the invariant flux F is given by[16]

$$F=[(P_1\cdot P_2)^2-M^4]^{1/2}. \quad (2.3)$$

$\sum$ indicates a sum (average) over final (initial) spin states and a sum over photon polarization. Experimentally the bremsstrahlung cross section is measured in terms of the photon energy and angle.

In the center-of-mass (c.m.) systems of the initial two protons, the two integrations over P_1' and P_2' can be easily done. The result is

$$\left(\frac{d\sigma}{d\Omega_\gamma dk_0}\right)_{\text{c.m.}}=\frac{M^4}{2F}\frac{k_0}{(2\pi)^5}\sum\frac{P_1'^2}{P_1'(W-k_0)+E_1'k_0\cos\theta''} \times|T_{fi}|^2\,d\cos\theta'\,d\varphi', \quad (2.4)$$

where (θ',φ') specifies the direction of P_1' relative to the incident direction in the c.m. system (see Fig. 3), Ω_γ is the solid angle of k relative to the incident direction, and $\cos\theta''$ is the angle between $\mathbf{P}_1'$ and $\mathbf{k}$. There are four possible Feynman diagrams based on the one-pion-exchange model. These are given in Fig. 4. It is convenient to introduce 4-momentum transfer variables corresponding to each diagram as follows:

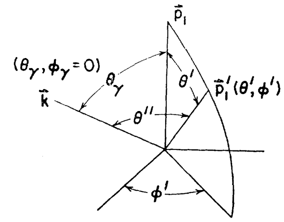

FIG. 3. Coordinates used in the calculation.

$$\Delta=P_2-P_2', \quad (2.5a)$$

$$\bar{t}=P_2-P_1', \quad (2.5b)$$

$$t=P_1-P_1', \quad (2.5c)$$

$$\bar{\Delta}=P_1-P_2'. \quad (2.5d)$$

The T-matrix element for the Fig. 4(a) process is given by

$$T(a)=\bar{u}(P_1')N(P_1',P_1;k,q=\Delta) \times u(P_1)[K'(\Delta^2)/(\Delta^2+1)]g_rK(\Delta^2)\bar{u}(P_2')\gamma_5u(P_2).$$

The factor $\bar{u}(P_1')N(P_1',P_1;k,q=\Delta)u(P_1)$ represents the amplitude for the following virtual process:

$$\pi^0(q=\Delta)+P(P_1)\rightarrow P(P_1')+\gamma(k),$$

corresponding to the blob in Fig. 4(a).

If $\Delta^2=-1$, the process reduces to the physical process. However, in general the pion is not on the mass shell.

The factor $g_rK(\Delta^2)\bar{u}(P_2')\gamma_5u(P_2)$ represents the π_0PP vertex. $K(\Delta^2)$ is the form factor, introduced by Federbush *et al.*[17] to take account of the fact that the pion is off the mass shell. $K(\Delta^2)$ is normalized by setting $K(-1)=1$. Finally, $K'(\Delta^2)/(\Delta^2+1)$ is the renormalized pion propagator. [$K'(\Delta^2)$ is a ratio between the complete propagator and the free propagator.]

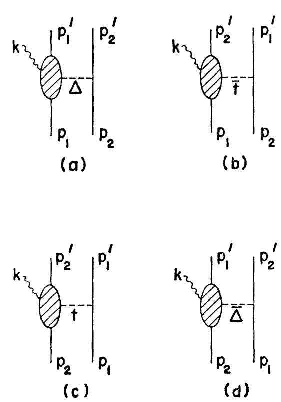

FIG. 4. Feynman diagrams for proton-proton bremsstrahlung.

[16] *Notation:* Units such that $\hbar=c=$pion mass$=1$ are used throughout. The four-vector scalar products are written as

$$(a\cdot b)=a_\mu b_\mu=\mathbf{a}\cdot\mathbf{b}+a_4b_4.$$

The spin-$\frac{1}{2}$ fermion satisfies the Dirac equation

$$(i\boldsymbol{P}+M)u(P)=0 \quad\text{where}\quad \boldsymbol{P}=\gamma_\mu P_\mu.$$

The γ matrices satisfy

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}, \quad (\mu,\nu=1,2,3,\text{ and }4).$$

We choose $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$ and $\gamma_\mu^\dagger=\gamma_\mu$, where the dagger denotes Hermitian conjugate. The asterisk will be used to represent complex conjugation. The normalization of $u(P)$ is, for positive energy, taken to be

$$\bar{u}(P)u(P)=1 \quad\text{where}\quad \bar{u}=u^\dagger\gamma_4.$$

The polarization vector of the photon, ϵ_μ, satisfies

$$\sum_{\text{pol}}\epsilon_\mu\epsilon_\nu=\delta_{\mu\nu},$$

where Σ_{pol} denotes polarization summation.

[17] P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. **112**, 642 (1958).

Similarly the amplitudes corresponding to Figs. 4(b), 4(c), and 4(d) can be written. Altogether we have

$$T_{fi}=T(a)-T(b)+T(c)-T(d), \tag{2.6}$$

where

$$T(a)=\bar{u}(P_1')N(P_1',P_1;\,k,\,q=\Delta)u(P_1)[K'(\Delta^2)/(\Delta^2+1)]g_rK(\Delta^2)\bar{u}(P_2')\gamma_5u(P_2), \tag{2.7a}$$

$$T(b)=\bar{u}(P_2')N(P_2',P_1;\,k,\,q=\bar{t})u(P_1)[K'(\bar{t}^2)/(\bar{t}^2+1)]g_rK(\bar{t}^2)\bar{u}(P_1')\gamma_5u(P_2), \tag{2.7b}$$

$$T(c)=\bar{u}(P_2')N(P_2',P_2;\,k,\,q=t)u(P_2)[K'(t^2)/(t^2+1)]g_rK(t^2)\bar{u}(P_1')\gamma_5u(P_1), \tag{2.7c}$$

$$T(d)=\bar{u}(P_1')N(P_1',P_2;\,k,\,q=\bar{\Delta})u(P_2)[K'(\bar{\Delta}^2)/(\bar{\Delta}^2+1)]g_rK(\bar{\Delta}^2)\bar{u}(P_2')\gamma_5u(P_1). \tag{2.7d}$$

The minus signs in Eq. (2.6) are due to the Pauli principle.

Next we investigate the structure of $\bar{u}(P_1')N(P_1'P_1;\,k,\,q=\Delta)u(P_1)$ corresponding to the diagram in Fig. 5. Since we express our amplitude in terms of the photopion production amplitude of Chew, Goldberger, Low, and Nambu[18] (hereafter CGLN), it is convenient to use their variables. We thus introduce the following three invariant variables:

$$\nu=-(P\cdot k)/M,\quad \nu_B=(q\cdot k)/2M,\quad q^2=\Delta^2,$$

where

$$P=(P_1+P_1')/2. \tag{2.8}$$

Note that we are considering the case $q^2=\Delta^2$ ($\neq -1$), i.e., the virtual pion is off the mass shell. The amplitude N must satisfy Lorentz invariance and gauge invariance.

Following CGLN, we define four Lorentz-invariant and gauge-invariant fundamental forms.

$$\begin{aligned}
M_1&=\tfrac{1}{2}i\gamma_5\{\gamma,\gamma\},\\
M_2&=2i\gamma_5\{P,q\},\\
M_3&=\gamma_5\{\gamma,q\},\\
M_4&=2\gamma_5(\{\gamma,P\}-\tfrac{1}{2}iM\{\gamma,\gamma\}),
\end{aligned} \tag{2.9}$$

where the abbreviation $\{a,b\}$ is used for the gauge-invariant combination $(a\epsilon)(bk)-(ak)(b\epsilon)$. Then the most general amplitude for the $\pi^0+p\to\gamma+p$ process is written as

$$N(P_1'P_1;\,k,q=\Delta)=\sum_{i=1}^{4}M_iA_i(\nu,\nu_B,\Delta^2), \tag{2.10}$$

where A_1, A_2, A_3, and A_4 are scalar functions of ν, ν_B, and Δ^2. For later convenience we will generalize A_i so that it can accommodate a charged as well as a neutral pion.

By charge independence the amplitude A_i for the process $\pi^{(\alpha)}+N\to\gamma+N$ (N, nucleon; α, isospin of a pion) can be generally decomposed as

$$A_i=\tfrac{1}{2}[\tau_\alpha,\tau_3]+A_i^{(+)}+\tfrac{1}{2}[\tau_\alpha,\tau_3]A_i^{(-)}+\tau_\alpha A_i^{(0)}, \tag{2.11}$$

where the τ_3 results from the vector part of the photon isotopic spin. For the case $\pi^0+p\to\gamma+p$ the amplitude A_i can be decomposed in terms of the above charged states as

$$A_i=A_i^{(+)}+A_i^{(0)},\quad (i=1\text{–}4). \tag{2.12}$$

3. FORMULAS FOR THE SQUARED ABSOLUTE VALUE OF THE T-MATRIX ELEMENT

To calculate the proton-proton bremsstrahlung cross section we require the squared absolute value of T:

$$|T|^2=|T(a)-T(b)+T(c)-T(d)|^2. \tag{3.1}$$

Note that we have

$$T(a)\leftrightarrow T(b)\ \text{under}\ P_1'\leftrightarrow P_2',$$
$$T(c)\leftrightarrow T(d)\ \text{under}\ P_1'\leftrightarrow P_2',$$

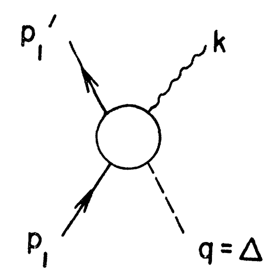

FIG. 5. A diagram for the process "virtual π"$+N\to\gamma+N$.

and that we must eventually integrate over P_1' and P_2'. Furthermore we note that $T(c)$ [or $T(d)$] can be obtained from $T(b)$ [or $T(a)$] by changing $P_1\leftrightarrow P_2$.

$|T|^2$ can be written in the form

$$\begin{aligned}
|T|^2=2(|T(a)|^2&+\{P_1\leftrightarrow P_2\})\\
&-2\,\mathrm{Re}[T(a)^*T(b)+\{P_1\leftrightarrow P_2\}]\\
&+4\,\mathrm{Re}[T(a)^*T(c)]-4\,\mathrm{Re}[T(a)^*T(d)],
\end{aligned} \tag{3.2}$$

where the $\{P_1\leftrightarrow P_2\}$ symbol represents the result of exchanging $P_1\leftrightarrow P_2$ in the previous terms.

Essentially we need only calculate the following four terms $|T(a)|^2$, $T(a)^*T(b)$, $T(a)^*T(c)$, and $T(a)^*T(d)$. The remaining terms can be obtained by the simple substitution $P_1\leftrightarrow P_2$. It will be simplest to calculate in the c.m. system of the initial protons.

First we consider $|T(a)|^2$. It is convenient to introduce

$$I\equiv\tfrac{1}{2}\sum_{\text{pol}}\sum_{\text{spin sum}}|\bar{u}(P_1')N(P_1'P_1;\,k,\,q=\Delta)u(P_1)|^2.$$

[18] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

Then

$$\sum|T(a)|^2=\tfrac{1}{2}\sum_{\text{pol}}\sum_{\text{spin sum}}|\bar{u}(P_1')N(P_1'P_1;k,q=\Delta)u(P_1)|^2\frac{[K(\Delta^2)K'(\Delta^2)]^2}{(\Delta^2+1)^2}\times\tfrac{1}{2}\sum_{\text{spin sum}}|\bar{u}(P_i')g_r\gamma_5u(P_2)|^2$$

$$=I\frac{[K(\Delta^2)K'(\Delta^2)]^2}{(\Delta^2+1)^2}\frac{g_r^2\Delta^2}{4M^2},\tag{3.3}$$

where we have summed over the spin states at the pion-nucleon vertex. Next we carry out the remaining spin and polarization summations and obtain after a straightforward but lengthy calculation the following expression for I:

$$I=\frac{1}{8M^2}\sum_{\text{pol}}\mathrm{Tr}[N(M-i\boldsymbol{P}_1)\gamma_4N^\dagger\gamma_4(M-i\boldsymbol{P}_1')]$$

$$=\frac{1}{2M^2}\sum_{i\leqslant j}B_{ij}\,\mathrm{Re}(A_iA_j^*),\quad (i,j=1\text{–}4)\tag{3.4}$$

where

$B_{11}=4(P\cdot k)^2-(q\cdot k)^2,\qquad B_{22}=\tfrac{1}{2}t^2[4(P\cdot k)^2t^2-(t^2+4M^2)(q\cdot k)^2],$

$B_{12}=4(P\cdot k)^2t^2-(t^2+4M^2)(q\cdot k)^2,\qquad B_{23}=B_{24}=0,$

$B_{13}=8M(P\cdot k)(q\cdot k),\qquad B_{33}=\tfrac{1}{2}[4(P\cdot k)^2t^2+(q\cdot k)^2(t^2+4M^2)],$

$B_{14}=4M(q\cdot k)^2,\qquad B_{34}=4t^2(P\cdot k)(q\cdot k),$

$B_{44}=\tfrac{1}{2}[4(P\cdot k)^2t^2+(q\cdot k)^2(t^2-4M^2)]\quad$ (where $q=\Delta$).

The interference term $\sum T(a)T(c)^*$ can be similarly calculated.

$$4\sum T(a)T(c)^*=4g_r^2\frac{K(\Delta^2)K'(\Delta^2)K(t^2)K'(t^2)}{(\Delta^2+1)(t^2+1)}\tfrac{1}{4}\sum_{\text{pol}}\sum_{\text{spin}}(\bar{u}(P_1')N(a)u(P_1))$$
$$\times(\bar{u}(P_1')\gamma_5u(P_1))^*(\bar{u}(P_2')\gamma_5u(P_1))(\bar{u}(P_2')N(c)u(P_2))^*,$$

where the factor $\frac{1}{4}$ comes from averaging over the spin of the initial two protons. We rewrite this as

$$4\sum T(a)T(c)^*=-g_r^2\frac{K(\Delta^2)K'(\Delta^2)K(t^2)K'(t^2)}{(\Delta^2+1)(t^2+1)}\frac{1}{(2M)^4}I,\tag{3.5}$$

where

$$I=\sum_{\text{pol}}\mathrm{Tr}[N(a)(M-i\boldsymbol{P}_1)\gamma_5(M-\boldsymbol{P}_1')]\,\mathrm{Tr}[\gamma_4N(c)^\dagger\gamma_4(M-i\boldsymbol{P}_2')\gamma_5(M-i\boldsymbol{P}_2)].$$

The result of spin and polarization summation is

$$4\sum T(a)T(c)^*=-g_r^2\frac{K(\Delta^2)K'(\Delta^2)K(t^2)K'(t^2)}{(\Delta^2+1)(t^2+1)}\frac{1}{M^4}[A_1(a)+t^2A_2(a)][A_1^*(c)+\Delta^2A_2^*(c)]$$
$$\times\{(P\cdot P')(q\cdot k)^2+[q^2-(q\cdot k)](P\cdot k)(P'\cdot k)\},$$

where

$$P'=(P_2+P_2')/2,\quad P=(P_1+P_1')/2,\quad\text{and}\quad q=\Delta.\tag{3.6}$$

Similarly the interference term $\sum T(a)T(b)^*$ can be written

$$-2\sum T(a)T(b)^*=2g_r^2\frac{K(\Delta^2)K'(\Delta^2)K(\bar{t}^2)K'(\bar{t}^2)}{4(\Delta^2+1)(\bar{t}^2+1)}\frac{I}{(2M)^4},\tag{3.7}$$

where

$$I=\sum_{\text{pol}}\mathrm{Tr}[N(a)(M-i\boldsymbol{P}_1)(\gamma_4N(b)^\dagger\gamma_4)(M-\boldsymbol{P}_2')(M+i\boldsymbol{P}_2)(M-i\boldsymbol{P}_1')].$$

After a straightforward but lengthy calculation we get the following result:

$$I=\sum_{i,j}C_{ij}A_i(a)A_j(b)^*,\quad (i,j=0,2,3,4),\tag{3.8}$$

where we have introduced $A_0(x)\equiv A_1(x)-2MA_4(x)$, ($x=a$ or b), and

$$
\begin{aligned}
C_{00}=&\,8(P_1\cdot k)f\,,\\
C_{20}=&\,8\{(\Delta\cdot k)[\Delta^2P^2(P_2'\cdot k)-(P\cdot P_2)t^2(P_1\cdot k)]+(P\cdot k)t^2[(P_1\cdot k)(P_2\cdot k)+\tfrac{1}{2}\Delta^2(\bar{t}\cdot k)]\}\,,\\
C_{22}=&\,8[\Delta^2\bar{t}^2-f]\{(\Delta\cdot k)[P^2(P_2'\cdot k)-(P\cdot P_2)(P_1\cdot k)]-(P\cdot k)[(P_1\cdot k)^2+(\bar{t}\cdot k)(P_1\cdot\Delta)]\}\,,\\
C_{30}=&\,4M\{2(P_1\cdot k)[2(P_1\cdot k)(\Delta\cdot k)+\Delta^2(\bar{t}\cdot k)]+(\Delta\cdot k)f\}\,,\\
C_{40}=&\,8M\{2(P_1\cdot k)[(P\cdot k)(P_2'\cdot k)-(\Delta\cdot k)(P\cdot\bar{t})]+(P\cdot k)f\}\,,\\
C_{32}=&\,16M(P\cdot k)\{(\tilde{P}\cdot k)[\bar{\Delta}^2(\Delta\cdot k)+(P_1'\cdot k)(\bar{t}\cdot k)]+(\bar{t}\cdot k)^2(\tilde{P}\cdot\Delta)\}\,,\\
C_{33}=&\,-4\{-(\Delta\cdot k)(\bar{t}\cdot k)(P_1\cdot k)[4M^2-2(P_1\cdot P_2)]+2(P_1\cdot k)^3(P_2\cdot k)\\
&\qquad+[\Delta^2(\bar{t}\cdot k)^2+\bar{t}^2(\Delta\cdot k)^2][(P_1\cdot P_2)-M^2]+t^2\bar{t}^2(\Delta\cdot k)(P\cdot k)+\Delta^2\bar{\Delta}^2(\bar{t}\cdot k)(\tilde{P}\cdot k)\}\,,\qquad(3.8\text{a})\\
C_{42}=&\,-16M\{(P\cdot k)[\bar{t}^2-(\bar{t}\cdot k)][(\tilde{P}\cdot\Delta)(\bar{t}\cdot k)-(\tilde{P}\cdot k)(P_1'\cdot k)]\\
&\qquad+[(P\cdot\bar{t})(\tilde{P}\cdot k)-(P\cdot\tilde{P})(\bar{t}\cdot k)][(\Delta\cdot k)(\bar{t}^2-(\bar{t}\cdot k))+2(\bar{t}\cdot k)(P\cdot k)]\}\,,\\
C_{43}=&\,-8\{M^2[-3(\bar{t}\cdot k)(P\cdot k)(P_1\cdot k)-\bar{t}^2(\Delta\cdot k)(P\cdot k)+(\bar{t}\cdot k)(\Delta\cdot k)(P\cdot\bar{t})-(P\cdot k)(\bar{t}\cdot k)^2]\\
&\qquad+\tfrac{1}{2}f[(\bar{t}\cdot k)(P^2-(P\cdot k))-2(P_1\cdot k)(P\cdot\bar{t})]+h(\bar{t}\cdot k)(P\cdot k)+\tfrac{1}{2}\Delta^2(\bar{t}\cdot k)X(P;\bar{t})+\bar{t}^2(P\cdot k)X(P_1;\bar{t})\}\,,\\
C_{44}=&\,-16\{M^2[-(P\cdot k)(\tilde{P}\cdot k)(P_1+P_2,k)+(\tilde{P}\cdot k)(\Delta\cdot k)(P\cdot\bar{t})+(P\cdot k)(\tilde{P}\cdot\Delta)(\bar{t}\cdot k)]\\
&\qquad+\tfrac{1}{2}f[-2(P_1\cdot k)\tilde{P}\cdot P)+P^2(\tilde{P}\cdot k)+\tilde{P}^2(P\cdot k)-2(P\cdot k)(\tilde{P}\cdot k)]\\
&\qquad+h(P\cdot k)(\tilde{P}\cdot k)+\tfrac{1}{2}\Delta^2(\tilde{P}\cdot k)X(P;\bar{t})+\tfrac{1}{2}\bar{t}^2(P\cdot k)X(\tilde{P};\Delta)\}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
f&=(\Delta\cdot k)\bar{t}^2+2(P_1\cdot k)(P_2\cdot k)+\Delta^2(\bar{t}\cdot k)\,,\\
h&=(P_1\cdot\Delta)\bar{t}^2+2(P_1\cdot P_2)(P_1\cdot k)+\Delta^2(P_1\cdot\bar{t})\,,\\
X(x;y)&=(k\cdot x)\tfrac{1}{2}y^2-(x\cdot y)(P_2\cdot k)+(P_2\cdot x)(k\cdot y)\,,\\
\tilde{P}&=(P_1+P_2')/2\,.
\end{aligned}
$$

The remaining C_{ji} can be obtained from the corresponding C_{ij} by the following interchange:

$$P_1'\leftrightarrow P_2'\,,$$

which also implies the interchanges:

$$P\leftrightarrow\tilde{P}\,,\qquad\Delta\leftrightarrow\bar{t}\,.$$

C_{04}, for instance, is given by

$$C_{04}=8M\{2(P_1\cdot k)[(\tilde{P}\cdot k)(P_1'\cdot k)-(\bar{t}\cdot k)(\tilde{P}\cdot\Delta)]+(\tilde{P}\cdot k)f\}\,.$$

Finally the interference term $-4T(a)T(d)^*$ can be written

$$-4T(a)T(d)^*=g_r^2\frac{K(\Delta^2)K'(\Delta^2)K(\bar{\Delta}^2)K'(\bar{\Delta}^2)}{(\Delta^2+1)(\bar{\Delta}^2+1)}\frac{I}{(2M)^4}\,,\qquad(3.9)$$

where

$$I=\sum_{\text{pol}}\mathrm{Tr}[N(a)(M-i\boldsymbol{P}_1)(\gamma_4N(d)^\dagger\gamma_4)(M-i\boldsymbol{P}_2')\times(M+i\boldsymbol{P}_2)(M-i\boldsymbol{P}_1')]\,.$$

We write

$$I=\sum_{i,j}D_{ij}A_i(a)A_j(d)^*\,,\quad(i,j=0,2,3,4)\,.\qquad(3.10)$$

The explicit form of D_{ij} can be obtained, up to a phase from the corresponding C_{ij} by the following interchanges:

$$P_1\leftrightarrow -P_1'\quad\text{and}\quad P_2\leftrightarrow -P_2'\,,$$

which also imply

$$P\leftrightarrow -P\,,\quad q\leftrightarrow q\,,\quad \bar{t}\leftrightarrow\bar{\Delta}\,,\quad\text{and}\quad\tilde{P}\leftrightarrow -P\,,$$

where

$$\bar{P}=(P_2+P_1')/2\,.$$

Taking into account the phase ϵ_{ij}, we write

$$D_{ij}=\epsilon_{ij}C_{ij}\begin{pmatrix}P_1\leftrightarrow -P_1'\\ P_2\leftrightarrow -P_2'\end{pmatrix}\quad\text{(no summation)}\,,\qquad(3.10\text{a})$$

where

$$\epsilon_{23}=\epsilon_{32}=\epsilon_{43}=\epsilon_{34}=\epsilon_{13}=\epsilon_{31}=-1\,;$$

other ϵ's$=+1$.

4. VIRTUAL PION PHOTOPRODUCTION AND ITS INVERSE PROCESS

We must relate the $\pi+N\to\gamma+N$ amplitude to the amplitude for pion photoproduction.

First we construct the dispersion relations for virtual pion photoproduction:

$$\gamma(k')+N(P_1)\to N(P_1')+\pi^{(\alpha)}(q')\,,$$

where α denotes the isospin of the pion.

The scattering process which involves one off-shell particle was investigated first by Fubini, Nambu, and Wataghin[19] for the case of electropion production. The case of pion-nucleon scattering was also investigated by Ferrari and Selleri[20] and by Iizuka and Klein.[21]

We modify CGLN[18] and briefly sketch the formalism.

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[19] S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. **111**, 329 (1957).

[20] E. Ferrari and F. Selleri, Nuovo Cimento **21**, 1028 (1961). See also E. Ferrari and F. Selleri, *ibid.* **27**, 1450 (1963).

[21] J. Iizuka and A. Klein, Progr. Theoret. Phys. (Kyoto) **25**, 1017 (1961).

It is convenient to introduce the following four Lorentz-invariant gauge-invariant forms:

$$\begin{aligned} M_A&=\tfrac{1}{2}i\gamma_5\{\gamma,\gamma\}\,,\\ M_B&=2i\gamma_5\{P,q'\}\,,\\ M_C&=\gamma_5\{\gamma,q'\}\,,\\ M_D&=2\gamma_5(\{\gamma,P\}-\tfrac{1}{2}iM\{\gamma,\gamma\})\,, \end{aligned} \tag{4.1}$$

where $P=(P_1+P_1')/2$ and the abbreviation $\{a,b\}$ is used for the gauge-invariant combination $(a\epsilon)(bk')-(ak')(b\epsilon)$. In terms of these forms, the general amplitude H is written as

$$H=M_AA+M_BB+M_CC+M_DD. \tag{4.2}$$

Each of the invariant coefficients can be further decomposed according to its isotopic spin dependence.

$$A=\tfrac{1}{2}[\tau_\alpha,\tau_3]_+A^{(+)}+\tfrac{1}{2}[\tau_\alpha,\tau_3]A^{(-)}+\tau_\alpha A^{(0)}\text{ etc.} \tag{4.3}$$

The resulting 12 coefficients which we designate H_i can be considered as functions of the scalars,

$$\nu'=-(P\cdot k')/M\,,\quad \nu_B'=(q'\cdot k')/2M\,,\quad\text{and}\quad q'^2=\Delta^2, \tag{4.4}$$

and satisfy dispersion relations of the form

$$\mathrm{Re}H_i(\nu',\nu_B',\Delta^2)=R(\Delta^2)\left[\frac{1}{\nu_B'-\nu'}\pm\frac{1}{\nu_B'+\nu'}\right]+\frac{1}{\pi}\int_{\nu_0'}^{\infty}d\nu''\left[\frac{1}{\nu''-\nu'}\pm\frac{1}{\nu''+\nu'}\right]\mathrm{Im}H_i(\nu',\nu_B',\Delta^2)\,, \tag{4.5}$$

where $\nu_0'=\nu_B'+1+1/2M$. The $\pm$ sign is chosen so that H satisfies crossing symmetry. Therefore $A^{(+,0)}$, $B^{(+,0)}$, $C^{(-)}$, and $D^{(+,0)}$ must be even functions of ν, while $A^{(-)}$, $B^{(-)}$, $C^{(+,0)}$, and $D^{(-)}$ are odd functions.

The residues R at the poles turn out to be

$$\begin{aligned} R[A^{\pm,0}]&=-\tfrac{1}{2}e_rf_rK(\Delta^2)\,,\\ R[B^{\pm,0}]&=\left(\frac{1}{2M\nu_B'}\right)\tfrac{1}{2}e_rf_rK(\Delta^2)\,,\\ R[C^{\pm}]&=R[D^{\pm}]=\tfrac{1}{2}f_r(\mu_{pr}'-\mu_{nr})K(\Delta^2)\,,\\ R[C^0]&=R[D^0]=\tfrac{1}{2}f_r(\mu_{pr}'+\mu_{nr})K(\Delta^2)\,, \end{aligned} \tag{4.6}$$

where μ_{pr}' (μ_{nr}) is the anomalous magnetic moment of the proton (neutron). The residues are obtained by evaluating one nuclear intermediate state contribution to the unitarity relation. Since the mass-shell condition for the pion $q'^2=-1$ is not satisfied, we have not used this condition at all. This is the reason $K(\Delta^2)$, defined before, appears explicitly. Now we have

$$\begin{aligned} \nu_B'&=(q'\cdot k')/2M\\ &=-(k'-q')^2/4M+(k'^2+q'^2)/4M\\ &=(\Delta^2-t^2)/4M\,,\quad t^2=(P_1'-P_1)^2\,, \end{aligned}$$

while

$$\nu_B'=(-1-t^2)/4M\quad\text{for real-pion production}.$$

In terms of final-state total energy W, H_i can be written in the form

$$H_i(W,\nu_B',\Delta^2)=2MR(\Delta^2)\left[-\frac{1}{W^2-M^2}\pm\frac{1}{W^2-M^2+4M\nu_B'}\right]+\frac{1}{\pi}\int_{(M+1)^2}^{\infty}dW'^2\,\mathrm{Im}H_i(W',\nu_B',\Delta^2)\left[\frac{1}{W'^2-W^2}\pm\frac{1}{W'^2+W^2-2M^2+4M\nu_B'}\right]. \tag{4.7}$$

Now we shall express the amplitude "virtual π"$+N\rightarrow\gamma+N$ in terms of the amplitude $\gamma+N\rightarrow$ "virtual π"$+N$. From the T matrix for,

$$\gamma(k')+N(P_1)\rightarrow\pi^{(\alpha)}(q')+N(P_1')\,,$$

expressed in the form

$$H_{\alpha3}(P_1',P_1,q',k')\,,$$

the T matrix N for

$$\pi^{(\alpha)}(q)+N(P_1)\rightarrow\gamma(k)+N(P_1')$$

can be obtained, by using the substitution rule,[22] as follows:

$$N=H_{3\alpha}(P_1',\,P_1,\,-q,\,-k)\,,$$

where the subscripts denote the isospin dependence of the matrix element and it is understood that we are to sum over photon polarization.

By substituting the explicit forms of H and N [see Eqs. (4.2) and (2.10)] and comparing both sides, we can relate A_1, A_2, A_3, and A_4 of Eq. (2.10) to A, B, C, and D of Eq. (4.2) as follows:

$$\begin{aligned} A_1^{(+,0)}(\nu,\nu_B,\Delta^2)&=-A^{(+,0)}(\nu'\rightarrow-\nu,\,\nu_B'\rightarrow\nu_B,\,\Delta^2)\,,\\ A_2^{(+,0)}(\nu,\nu_B,\Delta^2)&=+B^{(+,0)}(\nu'\rightarrow-\nu,\,\nu_B'\rightarrow\nu_B,\,\Delta^2)\,,\\ A_3^{(+,0)}(\nu,\nu_B,\Delta^2)&=+C^{(+,0)}(\nu'\rightarrow-\nu,\,\nu_B'\rightarrow\nu_B,\,\Delta^2)\,,\\ A_4^{(+,0)}(\nu,\nu_B,\Delta^2)&=-D^{(+,0)}(\nu'\rightarrow-\nu,\,\nu_B'\rightarrow\nu_B,\,\Delta^2)\,. \end{aligned}$$

Similarly $A_i^{(-)}$ $(i=1,\,2,\,3,\,4)$ can be obtained by taking the negative of corresponding relations. Using crossing symmetry for A, B, C, and D, we can further simplify

[22] See, for example, J. M. Jauch, and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., London, 1955).

these relations to

$$A_1{}^{(\pm,0)}(\nu,\nu_B,\Delta^2) = -A^{(\pm,0)}(\nu'\to\nu,\ \nu_B'\to\nu_B,\ \Delta^2)\,,\ \text{etc.} \tag{4.8}$$

As is seen from the previous treatment, the Δ^2 dependence of the matrix element of the virtual process is entirely either in K or ν_B'. Therefore we assert that under certain conditions, to be discussed below, the amplitude for photo virtual pion can be factored as

$$T(\gamma+N\to\text{virtual }\pi+N) = K(\Delta^2)T(\gamma+N\to\text{real }\pi+N)\,, \tag{4.9}$$

where $K(\Delta^2)$ represents the pionic form factor and the expression for T is that given by CGLN[18] with ν_B' interpreted as $(\Delta^2-t^2)/4M$. This relation is exact to lowest order Born approximation, since we take account of the virtual pion at the π-N-N vertex by merely replacing g_r by $g_rK(\Delta^2)$.

In general, when more than 1 virtual pion is involved in the process this factoring may be allowed only if higher order contributions do not introduce drastic Δ^2 dependence. For example, it is easily seen that this factorization is true for the dominant partial amplitude M_{1+} in the static approximation. Using Eq. (4.8), we get the formulas

$$A_1{}^{(\pm,0)}(\nu,\nu_B,\Delta^2) = -K(\Delta^2)A_{\text{CGLN}}{}^{(\pm,0)}(\nu'\to\nu,\ \nu_B'\to\nu_B)\,,\ \text{etc.}\quad \text{with } \nu_B=(\Delta^2-t^2)/4M\,. \tag{4.10}$$

We are only interested in the processes which involve a neutral pion. These amplitudes can be decomposed as

$$H(\gamma+p\to p+\pi^0)=H^{(+)}(\gamma+N\to\pi+N)+H^{(0)}(\gamma+N\to\pi+N)\,,$$
$$M(\pi^0+p\to p+\gamma)=M^{(+)}(\pi+N\to\gamma+N)+M^{(0)}(\pi+N\to\gamma+N)\,.$$

The amplitudes A_i ($i=1$–4) for "virtual π^0"$+p\to p+\gamma$ may be expressed in terms of the amplitudes $A^{(+,0)}$, $B^{(+,0)}$, $C^{(+,0)}$, and $D^{(+,0)}$ for $\gamma+N\to$ "real $\pi^{(\alpha)}$"$+N$ as

$$\begin{aligned}
A_1(\nu,\nu_B,\Delta^2)&=-K(\Delta^2)[A^{(+)}{}_{\text{CGLN}}+A^{(0)}{}_{\text{CGLN}}]_{\nu'\to\nu,\nu_B'\to\nu_B}\,,\\
A_2(\nu,\nu_B,\Delta^2)&=+K(\Delta^2)[B^{(+)}{}_{\text{CGLN}}+B^{(0)}{}_{\text{CGLN}}]_{\nu'\to\nu,\nu_B'\to\nu_B}\,,\\
A_3(\nu,\nu_B,\Delta^2)&=-K(\Delta^2)[C^{(+)}{}_{\text{CGLN}}+C^{(0)}{}_{\text{CGLN}}]_{\nu'\to\nu,\nu_B'\to\nu_B}\,,\\
A_4(\nu,\nu_B,\Delta^2)&=-K(\Delta^2)[D^{(+)}{}_{\text{CGLN}}+D^{(0)}{}_{\text{CGLN}}]_{\nu'\to\nu,\nu_B'\to\nu_B}\,.
\end{aligned} \tag{4.11}$$

Now we consider the dispersion integral for photo (real) pion production. This problem has been fully discussed by CGLN[18] and Ball.[23] With these authors we assume that it is only necessary to keep the large $I=\frac{3}{2}$, $J=\frac{3}{2}$ magnetic-dipole ($M_{1+}{}^{3/2}$) amplitude. Then we obtain the following expressions[23]:

Dispersion integral of

$$A^{(\pm)}{}_{\text{CGLN}}=\pm\binom{2}{1}\frac{1}{3\pi}\int_{(M+1)^2}^{\infty}dW'^2\,C(W')[q_0'(W'+M)-3t^2-1]\,\text{Im}M_{1+}{}^{3/2}(W')\times\left[\frac{1}{W'^2-W^2}\pm\frac{1}{W'^2+W^2-2M^2+4M\nu_B'}\right],$$

Dispersion integral of

$$B^{(\pm)}{}_{\text{CGLN}}=\mp\binom{2}{1}\frac{1}{\pi}\int_{(M+1)^2}^{\infty}dW'^2\,C(W')\,\text{Im}M_{1+}{}^{3/2}(W')\left[\frac{1}{W'^2-W^2}\pm\frac{1}{W'^2+W^2-2M^2+4M\nu_B'}\right],$$

Dispersion integral of

$$C^{(\pm)}{}_{\text{CGLN}}=\pm\binom{2}{1}\frac{1}{3\pi}\int_{(M+1)^2}^{\infty}dW'^2\,C(W')\,\text{Im}M_{1+}{}^{3/2}(W')\left[-\frac{3}{2}\frac{t^2+1}{W'+M}+q_0'-(W'+M)\right]\times\left[\frac{1}{W'^2-W^2}\mp\frac{1}{W'^2+W^2-2M^2+4M\nu_B'}\right],$$

Dispersion integral of

$$D^{(\pm)}{}_{\text{CGLN}}=\pm\binom{2}{1}\frac{1}{3\pi}\int_{(M+1)^2}^{\infty}dW'^2\,C(W')\,\text{Im}M_{1+}{}^{3/2}(W')\left[-\frac{3}{2}\frac{t^2+1}{W'+M}+q_0'+2(W'+M)\right]\times\left[\frac{1}{W'^2-W^2}\pm\frac{1}{W'^2+W^2-2M^2+4M\nu_B'}\right], \tag{4.12}$$

[23] J. S. Ball, Phys. Rev. 124, 2014 (1961).

where

$$C(W')=(4\pi/|q'||k'|)[(W'+M)^2-1]^{-1/2}.$$

The dispersion integrals for $A^{(0)}$, $B^{(0)}$, $C^{(0)}$, and $D^{(0)}$ are zero in the $M_{1+}{}^{3/2}$ dominance approximation.

The factors $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ are due to the following isotopic-spin decompositions:

$$M_{1+}{}^{(+)}=\tfrac{1}{3}M_{1+}{}^{1/2}+\tfrac{2}{3}M_{1+}{}^{3/2}\approx\tfrac{2}{3}M_{1+}{}^{3/2}, \qquad M_{1+}{}^{(-)}=\tfrac{1}{3}M_{1+}{}^{1/2}-\tfrac{1}{3}M_{1+}{}^{3/2}\approx-\tfrac{1}{3}M_{1+}{}^{3/2}. \tag{4.13}$$

Next, following CGLN,[18] we write the relation

$$M_{1+}{}^{3/2}/k'q'=[(\mu_{pr}-\mu_{nr})/2f_r](f_{33}/q'^2), \tag{4.14}$$

where f_{33} is the $I=J=\frac{3}{2}$ amplitude for pion-nucleon scattering. Since the structure of f_{33} is well known experimentally, we can thus determine the $M_{1+}{}^{3/2}$ amplitude.

To simplify further calculations, we shall adopt the narrow-resonance approximation for f_{33}:

$$\mathrm{Im} f_{33}(W')/q'^2=\tfrac{4}{3}\pi f^2 2W_r\delta(W'^2-W_r^2), \quad f^2=0.08, \tag{4.15}$$

where W_r is the c.m. energy at the 33 resonance, chosen to be roughly $M+2$. Then

$$\mathrm{Im} M_{1+}{}^{3/2}(W')=q'(W_r)k'(W_r)\frac{\mu_{pr}-\mu_{nr}}{2}\frac{g_r}{3}\frac{W_r}{M}\delta(W'^2-W_r^2). \tag{4.16}$$

The approximation of dominance of the $M_{1+}{}^{3/2}$ amplitude corresponds physically to taking only the N^* intermediate-state contribution. Introducing

$$D(W_r)=(4W_r/9M)[(W_r+M)^2-1]^{-1/2}, \tag{4.17}$$

$$\nu_r-\nu_B=(W_r^2-M^2)/2M, \tag{4.18}$$

finally we obtain the scattering amplitude for "virtual π^0"$+P$ $(P_1)\to P$ $(P_1')+\gamma$ in the form

$$\begin{aligned}
A_1(\nu,\nu_B,\Delta^2)&=-K(\Delta^2)\left[-\frac{e_r g_r}{M}\frac{\nu_B}{\nu_B^2-\nu^2}+\frac{\mu_{pr}-\mu_{nr}}{M}g_r D(W_r)\frac{\nu_r}{\nu_r^2-\nu^2}[q_0(W_r)(W_r+M)-3t^2-1]\right],\\
A_2(\nu,\nu_B,\Delta^2)&=+K(\Delta^2)\left[\frac{e_r g_r}{2M^2(\nu_B^2-\nu^2)}-3\frac{\mu_{pr}-\mu_{nr}}{M}g_r D(W_r)\frac{\nu_r}{\nu_r^2-\nu^2}\right],\\
A_3(\nu,\nu_B,\Delta^2)&=-K(\Delta^2)\left[\frac{g_r\mu_{pr}'}{M}\frac{\nu}{\nu_B^2-\nu^2}+\frac{\mu_{pr}-\mu_{nr}}{M}g_r D(W_r)\frac{\nu}{\nu_r^2-\nu^2}\left(-\frac{3}{2}\frac{t^2+1}{W_r+M}+q_0(W_r)-(W_r+M)\right)\right],\\
A_4(\nu,\nu_B,\Delta^2)&=-K(\Delta^2)\left[\frac{g_r\mu_{pr}'}{M}\frac{\nu_B}{\nu_B^2-\nu^2}+\frac{\mu_{pr}-\mu_{nr}}{M}g_r D(W_r)\frac{\nu_r}{\nu_r^2-\nu^2}\left(-\frac{3}{2}\frac{t^2+1}{W_r+M}+g_0(W_r)+2(W_r+M)\right)\right],
\end{aligned} \tag{4.19}$$

where

$$u_{pr}'=1.78(e_r/2M), \quad \mu_{pr}=2.78(e_r/2M), \quad \mu_{nr}=-1.91(e_r/2M).$$

We abbreviate these expressions as $A_i(a)$, the a referring to the fact that they describe the process depicted in Fig. 4(a). Then the $A_i(b)$, corresponding to Fig. 4(b), can be obtained from the $A_i(a)$ by the interchange $P_1'\leftrightarrow P_2'$. This interchange is equivalent to the following substitutions:

$$\begin{gathered}
\nu\equiv-(P\cdot k)/M\leftrightarrow\bar{\nu}\equiv-(\bar{P}\cdot k)/M=-(P_1+P_2',k)/M,\\
\nu_B\equiv(\Delta\cdot k)/M\leftrightarrow\bar{\nu}_B\equiv(\bar{t}\cdot k)/M=(P_2-P_1',k)/M,\\
\Delta^2\leftrightarrow\bar{t}^2, \qquad t^2\leftrightarrow\bar{\Delta}^2.
\end{gathered}$$

Similarly the $A_i(d)$ can be obtained from the $A_i(a)$ by interchange $P_1\leftrightarrow P_2$. Finally, the $A_i(c)$ can be obtained from the $A_i(a)$ by the interchanges $P_1\leftrightarrow P_2$ and $P_1'\leftrightarrow P_2'$.

5. NUMERICAL CALCULATION AND DISCUSSION OF RESULTS

First we rewrite the cross section Eq. (2.4), in the form

$$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{\text{c.m.}}=\left(\frac{Mk_0}{F}\right)\left(\frac{e_r^2}{4\pi}\right)\left(\frac{g_r^2}{4\pi}\right)^2\frac{M}{2\pi^2}\frac{1}{2}\int_0^{2\pi}d\varphi'\int_{-1}^{1}d(\cos\theta')\left(\frac{M^2}{e_r^2g_r^4}|T|^2\right)J, \tag{5.1}$$

where the Jacobian

$$J=\frac{P_1'^2}{P_1'(W-k_0)+E_1'k_0\cos\theta''}. \tag{5.1a}$$

The factor $\frac{1}{2}$ was added to take account of the fact that the two outgoing protons are indistinguishable. Previously we have used $\hbar=c=\text{pion mass}=1$.

The cross section in terms of cm^2 per MeV, is given by

$$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{\text{c.m.}}=\frac{2\times10^{-26}}{140}\left(\frac{Mk_0}{F}\right)\left(\frac{e_r^2}{4\pi}\right)\left(\frac{g_r^2}{4\pi}\right)^2\frac{M}{(2\pi)^2}\int_0^{2\pi}d\varphi'\int_{-1}^{+1}d\cos\theta'\left\{\frac{M^2}{e_r^2g_r^4}|T|^2\right\}J\quad\left(\frac{\text{cm}^2}{\text{MeV sr}}\right).$$

Substituting

$$M=6.71,\quad g_r^2/4\pi=15,\quad e_r^2/4\pi=1/137,$$

we get

$$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{\text{c.m.}}=0.6958\times10^{-28}\left(\frac{k_0M}{F}\right)\int_0^{2\pi}d\varphi'\int_{-1}^{+1}d(\cos\theta')\left\{\frac{M^2}{e_r^2g_r^4}|T^2|\right\}J. \tag{5.2}$$

According to Eq. (3.2), $|T|^2$ is the sum of direct terms and interference terms. We write, correspondingly,

$$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{\text{c.m.}}=\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{\text{direct}}+\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{ab}+\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{ac}+\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{ad}, \tag{5.3}$$

where

$$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{\text{direct}}=2\int_0^{2\pi}d\varphi'\int_{-1}^{1}d\cos\theta'\,[|T(a)|^2+\{P_1\leftrightarrow P_2\}]Y,$$

$$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{ab}=-2\int_0^{2\pi}d\varphi'\int_{-1}^{1}d\cos\theta'\,\text{Re}[T(a)T(b)^*+\{P_1\leftrightarrow P_2\}]Y,$$

$$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{ac}=4\int_0^{2\pi}d\varphi'\int_{-1}^{1}d\cos\theta'\,\text{Re}[T(a)T(c)^*]Y,$$

$$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{ad}=-4\int_0^{2\pi}d\varphi'\int_{-1}^{1}d\cos\theta'\,\text{Re}[T(a)T(d)^*]Y,$$

with

$$Y=0.6958\times10^{-28}(k_0M^3/Fe_r^2g_r^4)J.$$

$|T(a)|^2$, $-2\,\text{Re}[T(a)T(b)^*]$, $4\,\text{Re}[T(a)T(c)^*]$, and $-4\,\text{Re}[T(a)T(d)^*]$ are given by Eqs. (3.3), (3.7), (3.5), and (3.9), respectively. The terms with P_1 and P_2 interchanged will be discussed later.

We shall compute the proton-proton bremsstrahlung cross section for incident protons with a laboratory kinetic energy of 200 MeV, corresponding to the experiment being carried out at Rochester. We choose the incident direction along the Z axis. (See Fig.3.) The initial 4-momenta P_1 and P_2 are specified quantities. P_2' has been eliminated by energy-momentum conservation already and the photon momentum k is taken to be fixed. Therefore we must only integrate over the direction of P_1'. The direction of P_1' is specified by (θ',φ'). From energy and momentum conservation, E_1' can be written as

$$E_1'=\tfrac{1}{2}(W^2-2k_0W+k_0^2\sin^2\theta'')^{-1}\{(W-k_0)(W^2-2k_0W)-k_0\cos\theta''[(W^2-2k_0W)(W^2-2k_0W-4M^2)-4k_0^2M^2\sin^2\theta'']^{1/2}\},$$

with

$$\cos\theta''=\mathbf{P_1'k}/|\mathbf{P_1'}||\mathbf{k}|=\cos\theta_\gamma\cos\theta'+\sin\theta_\gamma\sin\theta'\cos\varphi'. \tag{5.4}$$

This formula is correct when the energy of the emitted photon is smaller than the critical energy

$$k_c=W(W-2M)/2(W-M).$$

This covers almost all the region of interest; for instance, the maximum value of the photon energy $k_{0\,\max}=(W^2-4M^2)/2W$ is 95.1 MeV and $k_c=92.9$ MeV for a laboratory kinetic energy of 200 MeV.

Therefore we shall consider only the case[24]

$$0\leq k_0\leq k_c=92.9 \text{ MeV}$$

hereafter. To avoid possible inaccuracy, the Jacobian of Eq. (5.1a) will be rewritten as

$$J=2|\mathbf{P}_1'|^2[(W^2-2k_0W)(W^2-2k_0W-4M^2)-4k_0^2M^2\sin^2\theta'']^{-1/2}. \quad (5.5)$$

The formula is obtained by substituting the explicit expression of E_1'. By definition the variables which appear in $|T(a)|^2$ can be written as

$$(P\cdot k)=-M\nu=\tfrac{1}{2}[(P_1\cdot k)+(P_1'\cdot k)],$$
$$(q\cdot k)=2M\nu_B=(P_1'\cdot k)-(P_1\cdot k),$$
$$t^2=(P_1-P_1')^2=-2M^2-2(P_1\cdot P_1'),$$
$$\Delta^2=(P_2-P_2')^2=2[(P_1'\cdot k)-(P_1\cdot k)-(P_1\cdot P_1')-M^2],$$

where $2(q\cdot k)=\Delta^2-t^2$. The scalar products on the right-hand side can be expressed as

$$(P_1'\cdot k)=[(E_1'^2-M^2)^{1/2}\cos\theta''-E_1']k_0,$$
$$(P_1'\cdot P_1)=|\mathbf{P}_1|(E_1'^2-M^2)^{1/2}\cos\theta'-E_1E_1',$$
$$(P_1\cdot k)=(|\mathbf{P}_1|\cos\theta_\gamma-E_1)k_0,$$
$$(P_2\cdot k)=(-|\mathbf{P}_1|\cos\theta_\gamma-E_1)k_0.$$

Thus all quantities in $|T(a)|^2$ can be expressed in terms of $(\theta'\varphi')$. Next we observe that the interchange $P_1\leftrightarrow P_2$ is equivalent to[25]

$$\theta_\gamma\leftrightarrow\pi-\theta_\gamma.$$

(See Fig. 3.) The calculation of interference terms can be done quite analogously. Numerical calculation was carried out at $\cos\theta_\gamma=0.508$, -0.310, and -0.834 (c.m.) corresponding to laboratory angles $\theta_{\gamma L}=45°$, 90°, and 135°, respectively, for an incident laboratory kinetic energy of 200 MeV. These are the energy and angles of the experimental setup at Rochester.[5]

The IBM 7074 electronic computer at Rochester was used for numerical calculations.

[24] If k_0 satisfies the condition $k_{0\,\max}>k_0>k_c$ then E_1' becomes double valued:

$$E_1'=\tfrac{1}{2}(W^2-2k_0W+k_0^2\sin^2\theta'')^{-1}\{(W-k_0)(W^2-2k_0W)\pm k_0\cos\theta''\times[(W^2-2k_0W)(W^2-2k_0W-4M^2)-4k_0^2M^2\sin^2\theta'']^{1/2}\}.$$

Furthermore, only the particular angular range

$$\pi\geq\theta''\geq\theta_c(\geq\pi/2),$$

where

$$4k_0^2M^2\sin^2\theta_c''=(W^2-2k_0W)(W^2-2k_0W-4M^2)$$

is physically allowed. Because of these complexities we do not consider this region.

[25] This can be easily seen as follows: After the integrations over two outgoing protons are carried out, the cross section will only depend on the three invariant variables $(P_1\cdot k)$, $(P_2\cdot k)$, and $(P_1\cdot P_2)$. Therefore $P_1\leftrightarrow P_2$ means essentially $(P_1\cdot k)\leftrightarrow(P_2\cdot k)$, which can be obtained by $\cos\theta_\gamma\leftrightarrow-\cos\theta_\gamma$.

TABLE I. Bremsstrahlung cross sections$\times10^{32}$ in units of cm^2/(str MeV), predicted by the theory with off-shell effects neglected. Incident (lab) energy = 200 MeV.

$\cos\theta_\gamma$ (c.m.) / k_0 (MeV) (c.m.)	0.508 Born terms	0.508 Total	−0.310 Born terms	−0.310 Total	−0.834 Born terms	−0.834 Total
20	0.978	0.993	0.692	0.699	1.173	1.175
30	0.817	0.836	0.601	0.624	1.019	1.030
40	0.747	0.786	0.568	0.611	0.971	0.995
50	0.683	0.745	0.534	0.593	0.920	0.963
50	0.591	0.673	0.458	0.542	0.822	0.883
70	0.457	0.545	0.349	0.441	0.660	0.725
80	0.280	0.351	0.207	0.282	0.423	0.471
90	0.086	0.109	0.057	0.083	0.138	0.145

We shall list the results corresponding to the different approaches. First we consider the case where the off-shell effect has been neglected, taking

$$K^2(\Delta^2)K'(\Delta^2)=1. \quad (5.6)$$

The numerical results are listed for the three angles in Table I, where the contributions from the Born terms have been explicitly separated out.

We observe that although the N^* contribution, the difference between total and Born terms, increases with photon energy and slightly modifies the spectrum corresponding to the Born terms at the high-energy end, it never becomes significant for the integrated proton-proton bremsstrahlung cross section. (See Fig. 6.)

Owing to the Pauli principle there is a huge cancellation between the diagrams. To see this we refer, for example, to Table II where the detailed results at $\cos\theta_\gamma=-0.310$ are listed for each individual contribution from the (Born+N^*) terms. We found that this cancellation occurs at other angles. This should make the proton-proton bremsstrahlung much smaller than the neutron-proton bremsstrahlung. This is consistent with a vanishing proton-proton bremsstrahlung cross section when the recoil of the proton is neglected, as mentioned in the Introduction. A similar cancellation

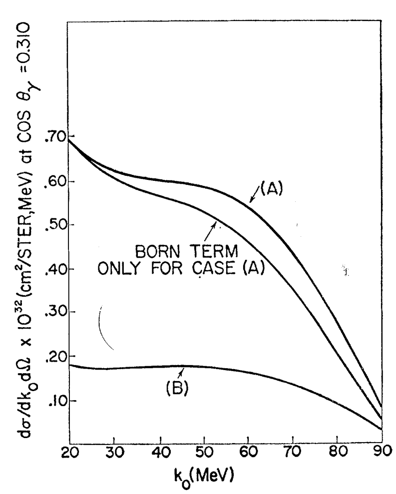

FIG. 6. Numerical results at $\theta_L=90°$ for incident energy 200 MeV. (A): Energy spectrum predicted by the theory with off-shell effects neglected. (B): Energy spectrum predicted by the theory with the pionic form factor.

TABLE II. Bremsstrahlung cross sections$\times 10^{32}$ in units of $cm^2/(sr\ MeV)$, predicted by the theory with off-shell effects neglected. Incident (lab) energy=200 MeV.

	$\cos\theta_\gamma$(c.m.)=−0.310				
(MeV) (c.m.)	$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{\text{direct}}$	$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{ac}$	$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{ab}$	$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{ad}$	Total
20	12.3420	−11.8873	1.4280	−1.1841	0.6968
30	7.4650	−7.0490	1.0233	−0.8158	0.6235
40	5.0906	−4.6598	0.7669	−0.5876	0.6106
50	3.7050	−3.2449	0.5459	−0.4132	0.5928
60	2.7974	−2.3095	0.3192	−0.2647	0.5424
70	2.1353	−1.6353	0.0738	−0.1325	0.4413
80	1.5698	−1.0977	−0.1721	−0.0183	0.2817
90	0.9042	−0.5696	−0.3043	0.0527	0.0831

was also observed in the case of the potential model by Sobel and Cromer.[13]

Next we take account of the off-shell effect for the exchanged pion. Although the structure of $K(\Delta^2)$ was theoretically investigated by Federbush *et al.*,[17] we take the phenomenological point of view about the form factor following Ferrari and Selleri.[20] It is well known that the peripheral model with one-pion exchange can explain most of the inelastic processes at high energies quite well. To fit the various phenomena, it was necessary to take account of the off-shell effect of the exchanged pion. We observe that the form factor $K^2(\Delta^2)K'(\Delta^2)$ which appears in the peripheral one-pion-exchange model should be the same as the form factor appearing in the bremsstrahlung process. For instance, we consider the process $N+N\to\pi+N+N$, which is shown in Fig. 7 assuming the one-pion-exchange model. This should be compared with the bremsstrahlung process of Fig. 1. We may also note that the off-shell form factor is a function of squared four-momentum transfer only and not a function of energy. Thus, following Amaldi and Selleri,[26] we shall take the following phenomenological expression for $K^2(\Delta^2)K'(\Delta^2)$:

$$K^2(\Delta^2)K'(\Delta^2)=\frac{0.72}{1+(1/4.73)(\Delta^2+1)}+\frac{0.28}{1+[(1/32)(\Delta^2+1)]^2}. \quad (5.7)$$

This gives a strong damping ffect for large Δ^2 where we do not expect the one-pion-exchange model to hold so well.

In order to see the cancellation of diagrams for the theory with the pionic form factor, we list in Table III contributions from the direct and the separate interference terms at $\cos\theta_\gamma=-0.310$. The two dominant

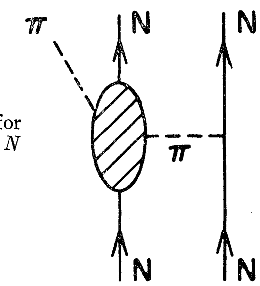

FIG. 7. A diagram for the process $N+N\to N+N+\pi$.

terms, direct and 4 Re$[R(a)^*T(c)]$ interference, have a characteristic $1/k_0$ dependence in the low-energy-photon region. However, owing to the huge cancellation, the energy spectrum becomes rather flat. (See Fig. 6). Since the cancellation due to the Pauli principle will not occur in general for neutron-proton bremsstrahlung, we expect that its cross section will be larger than the proton-proton cross section, by a factor of 10. Also the spectrum

TABLE III. Bremsstrahlung cross section$\times 10^{32}$ in units of $cm^2/(sr\ MeV)$, predicted by the theory with the pionic form factor. Incident (lab) energy=200 MeV.

	$\cos\theta_\gamma$(c.m.)=−0.310				
(MeV) (c.m.)	$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{\text{direct}}$	$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{ac}$	$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{ab}$	$\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)_{ad}$	Total
20	2.469	−2.350	0.3673	−0.305	0.181
30	1.592	−1.478	0.2751	−0.220	0.169
40	1.165	−1.042	0.215	−0.167	0.172
50	0.915	−0.778	0.160	−0.124	0.174
60	0.750	−0.598	0.098	−0.084	0.165
70	0.622	−0.461	0.024	−0.046	0.139
80	0.497	−0.338	−0.057	−0.009	0.093
90	0.310	−0.193	−0.106	0.017	0.028

[26] U. Amaldi and F. Selleri, Nuovo Cimento 31, 360 (1964). A similar form was used in Ref. 6.

TABLE IV. Bremsstrahlung cross sections$\times 10^{32}$ in units of $cm^2/(sr\ MeV)$, predicted by the theory with the pionic form factor. Incident (lab) energy$=200$ MeV.

(MeV) (c.m.) \ $\cos\theta_\gamma$ (c.m.)	0.508	−0.301	−0.834
20	0.257	0.181	0.300
30	0.227	0.169	0.281
40	0.223	0.172	0.279
50	0.221	0.174	0.284
60	0.209	0.165	0.276
70	0.178	0.139	0.225
80	0.120	0.093	0.168
90	0.042	0.028	0.058

for neutron-proton bremsstrahlung may have a characteristic $1/k_0$ dependence.[3,27]

The final numerical results at the three angles are listed in Table IV for the calculation including form-factor effect. Next we compare the results with the Rochester experiment. The integrated proton-proton bremsstrahlung cross section at θ_γ (lab)$=90°$ was obtained as

$$(d\sigma/d\Omega_\gamma)=(0.035_{-0.015}^{+0.04})\times 10^{-30}\ cm^2/sr \quad \text{for } k_{0\,\text{lab}}\geq 35\ \text{MeV}.$$

Because of poor statistics the photon spectrum has not yet been determined. The theory gives the following results[28] at $\cos\theta_\gamma=-0.310$ [corresponding to $\theta_\gamma(\text{lab})=90°$]:

$$\int\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)dk_0=0.219\times 10^{-30}\ cm^2/sr \quad \text{for the } K^2(\Delta^2)K'(\Delta^2)=1, \tag{5.8}$$

$$\int\left(\frac{d\sigma}{dk_0 d\Omega_\gamma}\right)dk_0=0.070\times 10^{-30}\ cm^2/sr \quad \text{for the phenomenological } K^2(\Delta^2)K'(\Delta^2). \tag{5.9}$$

The lower limit of integration is taken to be consistent with experimental limitations.

Within the accuracy of the experiment, the theory with the pionic form factor seems to give reasonably good agreement. Unfortunately at the time of writing of this paper, the cross sections at other angles are not yet available. Comparison with the only other available experiment, at Harvard,[4] is rather difficult because they

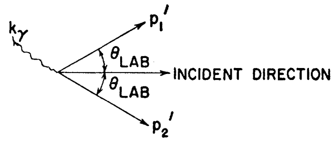

FIG. 8. Geometry of the Harvard experiment.

do not directly measure photon energy. They measure only coplanar events with two protons emerging at a symmetric angle θ_{lab} about the incident direction, as shown in Fig. 8. At $\theta_{\text{lab}}=35°$, where most of the data are taken, they get

$$d\sigma/d\Omega_1 d\Omega_2=5\times 10^{-30}\ cm^2/(sr)^2,$$

where $d\Omega_1$ and $d\Omega_2$ are the solid angles for the outgoing protons. This was compared with the theoretical prediction, $47\times 10^{-30}\ cm^2/(sr)^2$ of Sobel and Cromer,[13] based on the Yale potential. A large discrepancy between Sobel and Cromer's theory and experiment seems to exist.

At other angles $\theta_{\text{lab}}=40°$ and $\theta_{\text{lab}}=30°$ the Harvard experiments are consistently smaller than the prediction of Sobel and Cromer. Here for the sake of comparison we evaluate the Jacobian which gives the following transformation:

$$dE_1'd\Omega_1=J(E_1',\cos\theta_1,\phi_1/k_\gamma,\cos\theta_\gamma,\phi_\gamma)dk_\gamma d\Omega_\gamma.$$

We are considering the case with the Harvard group where all outgoing particles are in one plane with $\varphi_1=0$, $\varphi_2=\pi$, and $\varphi_\gamma=0$ as is shown in Fig. 9. The Jacobian was calculated nonrelativistically as[29]

$$J=-\left(\frac{\partial\phi_1}{\partial\phi_\gamma}\right)\frac{P_2-M\cos(\theta_2+\theta_\gamma)}{P_1\cos(\theta_1+\theta_2)-P_2}\left(\frac{k_\gamma}{M}\right)\frac{\sin\theta_1}{\sin\theta_\gamma},$$

where

$$-\frac{\partial\phi_1}{\partial\phi_\gamma}=\frac{k_\gamma\sin\theta_\gamma-P_1\sin\theta_1}{P_1\sin\theta_1+P_2\sin\theta_2-k_\gamma\sin\theta_\gamma}.$$

Using the above Jacobian, we obtain the following relation:

$$d\sigma/d\Omega_2 dk_\gamma d\Omega_\gamma=(d\sigma/d\Omega_1 d\Omega_2 dE_1')J.$$

If we make a rough comparison at $k_{\gamma L}=35.2$ MeV, $\theta_{\gamma L}=85.3°$ ($\theta_{1L}=\theta_{2L}=35°$) in the laboratory system, then Sobel and Cromer's theory, based on the Yale potential, seems to predict very roughly 10 times larger

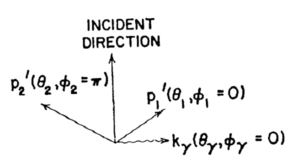

FIG. 9. Coordinates used by Sobel and Cromer.

[27] F. E. Low, Phys. Rev. **110**, 974 (1958).

[28] For evaluating the integrated cross section theoretically, the upper limit of integration was taken to be 90 MeV (c.m.) instead of $k_{0\,\max}(\approx 95$ MeV, c.m.). (See Ref. 1 on this point.) Therefore, the experimental data with photon energy higher than 90 MeV (c.m.) should not be included when a comparison is made with the theory. However, since such high-energy photons are not observed experimentally, this does not cause any trouble.

[29] This result is obtained using the method given in a private communication from A. H. Cromer to E. H. Thorndike.

TABLE V. Bremsstrahlung cross sections$\times 10^{32}$ in units of $cm^2/(sr\ MeV)$, predicted by the theory with the pionic form factor. Incident (lab) energy$=160$ MeV.

k_0 (MeV) (c.m.) \ $\cos\theta_\gamma$ (c.m.)	0.532	−0.280	−0.824
20	0.217	0.143	0.232
30	0.199	0.137	0.203
40	0.190	0.139	0.238
50	0.170	0.125	0.219
60	0.125	0.091	0.167
70	0.054	0.038	0.072
73	0.029	0.019	0.039

than our result.[30] A similar result was observed at $k_{\gamma L}=45.7$ MeV, $\theta_{\gamma L}=90°$ ($\theta_{1L}=\theta_{2L}=30°$).

In Table V we list the results for $d\sigma/dk_0d\Omega_\gamma$ in the c.m. system at $\cos\theta_\gamma=0.532$, -0.280, and -0.824 (which correspond to a $\theta_{\gamma L}=45°$, $90°$, and $135°$ in the laboratory system), based on the theory with the pionic form factor for an incident (lab) energy of 160 MeV.

The possible effects of ρ-meson exchange will be discussed in the Appendix. We only note here that the ρ contribution turns out to be very small.

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APPENDIX: CONTRIBUTION OF THE ρ RESONANCE DIAGRAM

Since a light particle can be easily accelerated, we expect it to yield a large bremsstrahlung cross section. This situation is observed in nature—the pion-nucleon bremsstrahlung gives a much larger cross section[31] than the nucleon-nucleon case. For proton-proton bremsstrahlung on the basis of the one-pion-exchange model the exchange pion has zero charge and hence cannot emit a photon. However, the next lightest virtual particle, the ρ meson, might emit a photon. In the following we shall consider this effect, which corresponds to the diagram in Fig. 10.

First we must investigate the ρ^0 contribution to (real) pion photoproduction. This is discussed by Ball[23] and by Gourdin, Lurie, and Martin[32] using the Mandelstam representation.[33] In terms of CGLN notation[18] the ρ contribution to the scattering amplitude for

$$N(P_1)+\gamma(k)\to N(P_1')+\pi^\alpha(q)$$

is given by

$$A^{(0)}=\frac{1}{\pi}\int_4^\infty dt'\frac{t'h(t')g_2{}^v(t')}{t'-t},$$

$$B^{(0)}=-\frac{1}{\pi}\int_4^\infty dt'\frac{h(t')g_2{}^v(t')}{t'-t},$$

$$C^{(0)}=0,$$

$$D^{(0)}=-\frac{1}{\pi}\int_4^\infty dt'\frac{h(t')g_1{}^v(t')}{t'-t},$$

where $t=-(P_1'-P_1)^2$ is the square of the c.m. energy in the t channel. This should not be confused with our variable $t^2=-(P_1'-P_1)^2$.

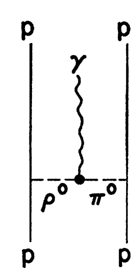

FIG. 10. A diagram for proton-proton bremsstrahlung.

[30] For an incident energy of 160 MeV, $\theta_{1L}=\theta_{2L}=35°$, $E_{1L}'=73.0$ MeV, $E_{2L}'=51.8$ MeV (which corresponds to $\theta_{\gamma L}=85.3°$, $k_{0L}=35.2$ MeV), in the laboratory system Sobel gives the following result:

$$d\sigma/d\Omega_1dE_2'd\Omega_2=1.363\times10^{-30}\ cm^2/(sr)^2\ MeV \quad \text{for Yale potential.}$$

The Jacobian is calculated to be 0.046 at this point. Therefore

$$d\sigma/d\Omega_2'dk_0d\Omega_\gamma=6.27\times10^{-32}\ cm^2/(sr)^2\ MeV.$$

Since the outgoing protons are identical, the case where E_{1L}' and E_{2L}' are interchanged should also be considered for the same given photon angle and photon energy. This is calculated similarly to be

$$d\sigma/d\Omega_2dk_0d\Omega_\gamma=1.63\times10^{-32}\ cm^2/(sr)^2\ MeV.$$

Adding these together, we obtain

$$d\sigma/d\Omega_2dk_0d\Omega_\gamma\approx7.9\times10^{-32} \quad \text{at} \quad \theta_{\gamma L}=85.3°,\ k_{0L}=35.2\ \text{MeV}.$$

To roughly compare their result with ours, we write

$$d\sigma/dk_0d\Omega_\gamma\approx\langle d\Omega_2\rangle 7.9\times10^{-32}\ cm^2/sr\ MeV,$$

where $\langle d\Omega_2\rangle$ may be of the order of unity and vary up to 2π if the distribution is isotropic. Our result is

$$d\sigma/dk_0d\Omega_\gamma=0.138\times10^{-32}\ cm^2/sr\ MeV.$$

[31] V. E. Barnes *et al.*, CERN Report 63-27, 1963 (unpublished).
[32] M. Gourdin, D. Lurie, and A. Martin, Nuovo Cimento **18**, 933 (1960).
[33] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

$h(t)$ represents the $\gamma\pi\rho$ amplitude and g_i^v is related to the vector part of the nucleon form factors as follows[23,32]:

$$G_i^v(t)=\frac{1}{\pi}\int_4^\infty dt'\frac{g_i^v(t')}{t'-t}, \quad (i=1,2).$$

Although the $\gamma\pi\rho$ amplitude $h(t)$ is not well established yet, we shall determine its form by assuming ρ dominance in the $\pi^0\to 2\gamma$. We follow Wong[34] and Ball[23] and choose

$$h(t)=\lambda'/(t+a) \quad \text{with} \quad a=5,$$

where λ' is an unknown parameter. Now the ρ contribution to the amplitude for the photon emission process $\pi^{(\alpha)}(q)+N(P_1)\to\gamma(k)+N(P_1')$ is, in terms of the previous notation,

$$A_1^{(0)}=-\lambda'\{G_2^v(-t^2)-[a/(a-t^2)] \times[G_2^v(-t^2)-G_2^v(-a^2)]\},$$

$$A_2^{(0)}=-\lambda'(a-t^2)^{-1}[G_2^v(-t^2)-G_2^v(-a)],$$

$$A_3^{(0)}=0,$$

$$A_4^{(0)}=\lambda'(a-t^2)^{-1}[G_1^v(-t^2)-G_1^v(-a)].$$

[34] H. S. Wong, Phys. Rev. Letters **5**, 70 (1960).

The vector parts of the nucleon form factors G_i^v may be approximately expressed as [35]

$$G_i^v(x)=G_i^v(0)(1+\alpha x), \quad (i=1,2) \quad \text{with} \quad \alpha=0.08.$$

Since the ρ contribution to the pion photoproduction is is consistent with experimental data if λ' satisfies[23]

$$-2.65\leq\lambda'\leq 2.65,$$

we shall consider the limiting cases

$$\lambda'=\pm 2.65$$

to obtain an upper limit on the possible ρ contribution to proton-proton bremsstrahlung.

We find that at the angle $\cos\theta_\gamma=-0.310$ the θ contribution to $d\sigma/dk_0 d\Omega_\gamma$ is almost negligible (less than 2%) compared with the Born terms at any photon energy for both choices of sign. Because of this smallness and the ambiguity in the $\gamma\pi\rho$ amplitude we do not consider this effect any further.

[35] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1609 (1960).