Meson Spectrum, Mass Formula, and the Quark-Antiquark Interaction

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Higher mesons fit into an $L=1$, $U(6)\otimes U(6)$ "35"-plet, $J^P=0^+$, 1^+, and 2^+ splitting by tensor and spin-orbit interactions. Quantum numbers are assigned with G parities that check and missing ones predicted. Mesons are placed on quark-antiquark $(q\bar{q})$ Regge trajectories. A general mass formula is derived which includes modifications of $SU(6)$-type splittings in higher supermultiplets, as well as additional quantum numbers related to dynamical groups. General features of an effective $(q\bar{q})$ interaction indicated by the mass spectrum are deduced. Predicted mesons are a $\pi'(640\mp40)$, an $\omega'(567\mp40)$, and a $\phi'(835\mp40)$ all with $J^P=0^+$; an $\hat{f}(1000\mp20)$ and a $\hat{\phi}(1268\mp60)$ with $J^P=1^+$; and a $\phi''(1520\mp60)$ with $J^P=2^+$. There is also an $L=1$, $SU(6)$ singlet with $J^P=1^+$ and mass about 1600 MeV. Resonances that the above system would lead to at still higher energies are discussed.

1. INTRODUCTION

THE implications of $SU(6)$[1] for quark models,[2] or several fundamental triplets with integral charge,[3] and in regard to a quark mass $M_q \gtrsim 10$ BeV have been discussed by Nambu,[4] Lipkin,[5] and others. Gell-Mann[6] has derived the orbital angular momentum **L** of quarks from a current algebra, supplementing the group $U(6)\otimes U(6)$ with $O^L(3)$. (See also Ref. 7.) For the quark-antiquark $(q\bar{q})$ representations $(6\times\bar{6})$, $U(6)\otimes U(6)$ is similar to $SU(6)$, but it also contains the intrinsic quark parity. Orbital angular momentum has been included also in various shell models of baryons based on fermion quarks,[8] three-fermion triplets[3,4] or paraquarks,[9,10] and in L-S coupling[4,8,9] or j-j coupling[4,8,10] schemes.

The assignment of the quantum numbers of higher baryons on the basis of some fundamental triplet model seems ambiguous since several orbital angular momenta must be combined in a way dependent on what type of triplet is used.[4] Also, few experimental J^P values[11] of higher baryons are known with certainty. With mesons on the other hand, quantum numbers are assigned rather unambiguously on the basis of a $(q\bar{q})$ system.[12] The spectrum is much less complicated; it is quite independent of the kind of fundamental triplet involved, and an additional quantum number, G parity, provides a check of the L and S values.[12]

In this article we consider in detail, not the relativistic and S-matrix problem of strong interactions, but the rest-frame problem of the quantum numbers and mass spectrum of static mesons. In particular: (a) the quantum numbers of mesons are assigned, compared with data and the missing ones are predicted, (b) mesons are placed on $(q\bar{q})$ Regge trajectories, (c) general features of an effective $(q\bar{q})$ potential indicated by the spectrum are deduced and used to discuss mesons that they would lead to at higher energies, and (d) a general mass formula applicable to higher mesons for $SU(6)$-type and other splittings is derived. Comparison with experimental data shows interesting symmetry-breaking effects at higher mesons understandable in terms of the internal dynamics given.

2. MESON ASSIGNMENTS

In Table I, mesons are classified according to their assigned quantum numbers. Only those mesons and resonances contained in Ref. 11 are included. The lowest mesons belonging[1] to the "1" and "35" of $SU(6)$, or rather $U(6)\otimes U(6)$, correspond to $L=0$. Since in $U(6)\otimes U(6)$, opposite quark and antiquark parities are included, the total parity is given by $P=(-1)^{L+1}$. Table I indicates that the higher mesons fit into the $L=1$ states of a $(q\bar{q})$ system.[12] According to $U(6)\otimes U(6)\otimes O^L(3)$ the entire "35" and "1" multiplets would repeat themselves, except that spin-spin tensor forces and spin-orbit coupling split the $L=1$, $S=1$ states into $J^P=0^+$, 1^+, and 2^+ or in spectroscopic notation $^3P_0$, $^3P_1$, and $^3P_2$. One obtains a $^1P_1(J^P=1^+)$ octet and three nonets ("^{38}"+"^{31}") with intrinsic $q\bar{q}$ spin 1 and corresponding to $J^P=0^+$, 1^+ and 2^+. The quantum numbers give all the known[11] G parities correctly from

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1 (a) F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964); (b) B. Sakita, Phys. Rev. **136**, B1756 (1964); and subsequent papers.
2 M. Gell-Mann, Phys. Letters **8**, 214 (1964); G. Zweig, CERN report, 1964 (unpublished).
3 See, e.g., M. Y. Han and Y. Nambu, Phys. Rev. **139**, B1006 (1965), and other references therein.
4 Y. Nambu, in *Symmetry Principles at High Energy—Second Coral Gables Conference*, edited by B. Kurşunoğlu, A. Perlmutter, and I. Sakmar (W. H. Freeman and Company, San Francisco, 1965), pp. 274–283.
5 H. J. Lipkin, Phys. Rev. **139**, B1633 (1965).
6 M. Gell-Mann, Phys. Rev. Letters **14**, 77 (1965).
7 K. T. Mahanthappa and E. C. G. Sudarshan, Phys. Rev. Letters **14**, 163 (1965) have assumed the $SU(6)\otimes O(3)$ invariance of strong interactions without models. In this symmetry, parity is given only by $(-1)^L$. It leads to nearly degenerate lowest mesons with $J^P=0^-$, 1^- and 2^-.
8 P. G. O. Freund and B. W. Lee, Phys. Rev. Letters **13**, 592 (1964).
9 O. W. Greenberg, Phys. Rev. Letters **13**, 598 (1964).
10 M. M. Miller, Phys. Rev. Letters **14**, 416 (1965).
11 A. H. Rosenfeld, A. Barbaro-Galtieri, W. H. Barkas, P. L. Bastien, J. Kirz, and M. Roos, Lawrence Radiation Laboratory, Berkeley, California, Report No. UCRL-8030, Part I, 1965 (unpublished).
12 O. Sinanoğlu, Phys. Rev. Letters **16**, 207 (1966).

TABLE I. Meson assignments. (Predicted mesons shown in [ ].)

L	$SU(6)$ or $U(6)\otimes U(6)$	S	$SU^S(2)\otimes SU^U(3)$	J^P	Spectroscopic notation	Regge trajectory (see text)	G-parity $=(-1)^{L+S+T}$	Meson[a] (MeV)
0	"1"	0	"1"	0^-	$^1S_0$	A'	+	$X^0(959)$
	"35"	0	"8"	0^-	$^1S_0$	A	−	$\pi(138)$
								$K(496)$
							+	$\eta(549)$
		1	"^{3}8"+"^{3}1"	1^-	$^3S_1$	D	+	$\rho(769)$
								$K^*(891)$
							−	$\omega(783)$
							−	$\phi(1020)$
1	"1"	0	"1"	1^+	$^1P_1$	A'	−	[$X'(1570\mp150)$]
	"35"	0	"8"	1^+	$^1P_1$	A	+	$B(1220)$
								$K_c(1215\mp15)$
							−	$E(1420)$
		1	"^{3}8"+"^{3}1"	0^+	$^3P_0$	C	−	[$\pi'(640\mp40)$]
								$\kappa(725)$
							+	[$\omega'(567\mp40)$]?
							+	[$\phi'(835\mp60)$]?
				1^+	$^3P_1$	B	−	$A1(1072\mp8)$[b]
								[$K^{*\prime\prime}(1158\mp40)$]
							+	[$f(1000\mp20)$; (ω'?)]
							+	[$\hat{\phi}'(1268\mp60)$]?
				2^+	$^3P_2$	D	−	$A2(1324\mp9)$
								$K^{*\prime}(1410\mp10)$
							+	$f(1253\mp20)$[ω'']?
							+	[$\phi''(1520\mp60)$]?

[a] Data from Ref. 11.
[b] See, however, Ref. 22.

$G=(-1)^{L+S+T}$. The predicted G parities for the unknown ones are also shown in the table.

There are hardly any ambiguous cases in the assignments. For $B(1220)$, experimentally[11] $J\geqslant 1$, $P=?$, $G=+$. The $(q\bar{q})$ with $L=1$, $S=0^-$ gives $J^{PG}=1^{++}$ whereas $L=2$, $S=0^-$ would contradict G. An $L=2$, $S=1^-$ would give too many unobserved nearby mesons with $J^P=3^-$, 2^- and 1^-. The $A2(1324)$ is consistent with $L=1$, $S=1$ of $(q\bar{q})$, but also with $S=0^+$, $L=2$, e.g., of $(qq\bar{q}\bar{q})$. The mass changes in the table are consistent with changes in the values of L, S, and J of a $(q\bar{q})$ system as discussed in the sections on dynamics below. Table I also shows the Regge trajectories to which various $SU(3)$ multiplets belong. These are discussed in the next section. In subsequent sections we shall consider each type of mass splitting and the $(q\bar{q})$ interactions that would cause them separately. From these the missing mesons are predicted. They are shown in the table in square brackets, and will be discussed individually in appropriate sections as they come up.

3. REGGE TRAJECTORIES

The above $(q\bar{q})$ system gives the following Regge trajectories over the real L axis, i.e., for bound states with $L=0, 1, 2, \cdots$ and for fixed T, Y and other $U(6)\otimes U(6)$ or $SU(6)$ quantum numbers:

(A) $S=0^-$, $L=J$; $J^P=0^-, 1^+, 2^-, \cdots$

(B) $S=1^-$, $L=J$; $J^P=1^+, 2^-, \cdots$

(C) $S=1^-$, $L=J+1$; $J^P=0^+, 1^-, \cdots$

(D) $S=1^-$, $L=J-1$; $J^P=1^-, 2^+, 3^-, \cdots$.

These of course are different from the trajectories based say on $N\bar{N}$ of the S-matrix problem which would be in the complex part of the J plane and related also to strong decays.

In the trajectory "D" above, the missing $J^P=0^+$ would correspond to the $L=-1$, "nonsense term."[13,14] In Table I, for fixed T, Y, the $^1S_0(0^-)$ and $^1P_1(1^+)$ lie on the trajectory "A"; the $^3P_1(1^+)$ is alone on "B," and $^3P_0(0^+)$ alone on "C"; the $^3S_1(1^-)$ and $^3P_2(2^+)$ lie on "D." The 0^- and 1^+ on "A" would further split with respect to "signature," $(-1)^L$, as would 1^- and 2^+ on "D." However, the mass spectrum as discussed below indicates that any interaction of the type $|(-1)^L V_{\text{exch}}|$

[13] M. Gell-Mann, CERN report No. 533 (unpublished).
[14] See, also for example, R. G. Newton, *The Complex J-Plane* (W. A. Benjamin Company, Inc., New York, 1964).

which would lead to "signature" splitting is small (<300 MeV), if it exists at all.

A given trajectory, e.g., "A" or "B," would further split depending on T and Y and $SU(6)$ quantum numbers. However, no two such trajectories of a given class are expected to be parallel because of spin-unitary spin mixing of $SU(6)$ (see Sec. 6 below).

4. THE QUARK-ANTIQUARK INTERACTION

The meson mass spectrum observed in Table I is easily interpreted in terms of a nonrelativistic $(q\bar{q})$ internal dynamics, though the main features should remain valid in a relativistic discussion. We consider $(q\bar{q})$ as the first term of a Tamm-Dancoff type sum

$$c_1|q\bar{q}\rangle+c_2|qq\bar{q}\bar{q}\rangle+\cdots.$$

Each term may in addition involve arbitrary numbers of some field quanta[15] contributing to the $q\bar{q}$ interaction. For a quasi-nonrelativistic quark-antiquark model one need not have $c_1\gg c_2, \cdots$. The c_1 and c_2 could even be of comparable magnitude provided each $(q\bar{q})$ level remained in a one-to-one correspondence with the levels after the inclusion of the effects of $|qq\bar{q}\bar{q}\rangle$. On the other hand, the second and higher terms are of course essential for decays. In a state like $|qq\bar{q}\bar{q}\rangle$ a clustering is expected with strongly bound $(q\bar{q})$ and $(q\bar{q})'$ interacting less strongly with each other. Provided the mass of, say $(q\bar{q})'$, $m_{(q\bar{q})'}<m_{(q\bar{q})}$, terms like $|qq\bar{q}\bar{q}\rangle$ may be taken to contribute to an effective $q\bar{q}$ potential $V^*(r)$ with r the $q\bar{q}$ distance, and to an effective quark mass[16] M_q^*. With $M_q^*\sim 10$ BeV and $r_0\sim 10^{-13}$ cm, this effect would not invalidate the use of a $V^*(r)$ in spite of quark recoil and cutoff effects since $r>\hbar/M_q^*c$. A drawback would come up, however, if $M_q\rightarrow M_q^*$ depended strongly on $(q\bar{q})$ internal state.

As in the meson theory of nuclear forces,[17] with such arguments one may justify the use of a quasi-nonrelativistic model. As in the Lévy-Klein method,[17] on the other hand, one may also start with a Bethe-Salpeter equation for $(q\bar{q})$ (see also Ref. 15) and obtain from it a Pauli-Breit type approximation. Thus, for the $(q\bar{q})$ system, we write

$$V^*(r)\cong V_c(r)+V_S(r)\mathbf{s}_1\cdot\mathbf{s}_2+V_{LS}\mathbf{L}\cdot\mathbf{S} +V_tS_{12}+V_{SU}+V_U+V_{\text{exch}}P. \quad (1)$$

The first four terms correspond respectively to: central, spin, spin-orbit, and spin-spin tensor interactions. V_{SU} is for simultaneously spin and unitary spin-dependent interactions,[18] V_U for unitary-spin dependent ones. The $V_{\text{exch}}P$ stands for any other type of exchange interaction not included in the previous terms. The various mass splittings indicate the relative strengths of the V's to be in the order

$$V_c\gg V_S,V_{LS},V_t>V_{SU}>V_U. \quad (2)$$

Equation (1) leads to a mass formula applicable to $(q\bar{q})$ meson states, of the general form

$$m=m_0+\Delta m_D(n',L)+a\mathbf{L}\cdot\mathbf{S}+bG_t^{(1)} +\Delta m_U(S,T,Y,C_2^{(4)},\mathfrak{N},\mathcal{S}) \quad (3)$$

(for the m versus m^2 formula see Sec. 8 below and Ref. 15). The $n'=0, 1, 2, \cdots$ and $L=0, 1, 2, \cdots$ are the radial and angular "dynamical" quantum numbers. $G_t^{(1)}$ is given in Eq. (6) and is for the tensor interaction. Δm_U depends[19,20] on the $SU(6)$ quantum numbers within $q\bar{q}=6\times\bar{6}$ representations, with $\mathfrak{N}$ and $\mathcal{S}$ total nonstrange and strange quark spins. It gives the broken $SU(6)$ [or $U(6)\otimes U(6)$]-type splittings within a "35"-plet. Between Eqs. (3) and (1), m_0 and $\Delta m_D(n',L)$ depend on the detailed nature of $V_c(r)$, a and b on $V_{LS}(r)$ and $V_t(r)$, and Δm_U on the forms of $V_S(r)$, V_{SU}, and V_U. Because of the relative strengths, the main $(q\bar{q})$ wave functions are determined by $V_c(r)$ with the other terms causing mass splittings. Were the V's independent of r, the corresponding coefficients in Eq. (3) like a, b and those[19,20] in Δm_U would be constants. As it is, the mass coefficients, where they depend on r, vary with n' and L as discussed in detail in Sec. 6.

We shall now discuss the various mass splittings of Table I successively. First we examine the ${}^3P_0$, ${}^3P_1$, ${}^3P_2$ splittings and obtain the unsplit ${}^3P_c$ "central" masses for each T, Y. Next we evaluate the $SU(6)$-type splittings within the "35"-plets of each L. These yield information on the corresponding V's. Finally the n', L "dynamical quantum numbers" dependence of degenerate supermultiplet masses are obtained showing the form of $V_c(r)$ and the dynamical group (i.e., the group that yields the energy levels as well as the degeneracies[21]) of the $(q\bar{q})$ system.

5. SPIN-ORBIT AND TENSOR FORCES

We assume the splitting of the $J^P=0^+$, 1^+, and 2^+ (${}^3P_0$,${}^3P_1$,${}^3P_2$) states of each nonet component in Table I to be due to spin-orbit and tensor potentials $V_{LS}(r)\mathbf{L}\cdot\mathbf{S}+V_t(r)S_{12}$, where

$$S_{12}=\mathbf{s}_1\cdot\mathbf{s}_2-\frac{3(\mathbf{s}_1\cdot\mathbf{r})(\mathbf{s}_2\cdot\mathbf{r})}{r^2}. \quad (4)$$

The V_{LS} and V_t could depend on similar regions of r and would be expected to contribute equally. Taking their effect on meson masses to first order one has ap-

[15] A number of pre-$SU(6)$ models with fundamental triplets along with field quanta were considered by F. Gürsey, T. D. Lee, and M. Nauenberg, Phys. Rev. **135**, B467 (1964).

[16] See also H. J. Lipkin and A. Tavkhelidze, Phys. Letters **17**, 331 (1965).

[17] See, e.g., R. J. N. Phillips, Rept. Progr. Phys. **22**, 562 (1959).

[18] T. K. Kuo and L. A. Radicati, Phys. Rev. **139**, B746 (1965).

[19] M. A. B. Bég and V. Singh, Phys. Rev. Letters **13**, 418 (1964).

[20] T. K. Kuo and T. Yao, Phys. Rev. Letters **13**, 415 (1964).

[21] A. O. Barut, Phys. Rev. **135**, B839 (1964).

TABLE II. Mass splitting by spin-orbit and tensor interactions.

L	S	J^P	Δm_{LS}	$\Delta m_t \cong bG_t^{(1)}$
0	0	0^-	0	0
0	1	1^-	0	0
1	0	1^+	0	0
1	1	0^+	$-2a$	$-b$
1	1	1^+	$-a$	$+\frac{1}{2}b$
1	1	2^+	$+a$	$-\frac{1}{10}b$
2	0	2^-	0	0
2	1	1^-	$-3a$	$-\frac{1}{2}b$
2	1	2^-	$-a$	$+\frac{1}{2}b$
2	1	3^-	$+2a$	$-\frac{1}{7}b$

proximately as in Eq. (3),

$$\Delta m_{LS}=a\mathbf{L}\cdot\mathbf{S}=\tfrac{1}{2}a[J(J+1)-L(L+1)-S(S+1)], \quad (5)$$

$$\Delta m_t \cong bG_t^{(1)}=b[(2\mathbf{s}_1\cdot\mathbf{s}_2+\tfrac{3}{2})\mathbf{L}^2-\tfrac{3}{2}(\mathbf{S}\cdot\mathbf{L}) -3(\mathbf{S}\cdot\mathbf{L})^2]/(2L+3)(2L-1). \quad (6)$$

The a and b are spin-orbit and tensor interaction $(q\bar{q})$ parameters. They should be nearly constant for the zero-order $(q\bar{q})$, $V_c(r)$ states with $L=0$, 1, 2, and $n'=0$, 1, since the considerations of Sec. 8 indicate the deviation of r from the average $(q\bar{q})$ distance r_0 to be small for these states. In Table I only the a and b appropriate to $n'=0$, $L=1$ are involved. The first-order splittings given by Eqs. (5) and (6) in terms of a and b are shown in Table II. To this order, J, J_3, L, and S remain good quantum numbers. The second-order effect of V_tS_{12} (e.g., from the mixing of ${}^3S_1$ with ${}^3D_1$) is neglected; it is estimated below.

To evaluate the a and b which are independent of T and Y, we fit Eqs. (5) and (6) (Table II) to the experimental mass differences

$${}^3P_2[K^{*\prime}(1410)]-{}^3P_0[\kappa(725)],$$

and (7)

$${}^3P_2[A2(1324)]-{}^3P_1[A1(1072)],$$

thereby obtaining

$$a=177\mp20 \text{ MeV}, \quad b=171\mp20 \text{ MeV}. \quad (8)$$

These values are very reasonable in view of Table I, though the existence of the $A1$ resonance is now uncertain.[22]

In Table II, second-order effect of V_tS_{12} was neglected. We now see this neglect was justified. The second-order tensor interaction mass shift, $\Delta m_t^{(2)}$, of ${}^3S_1$ would be roughly $-b^2/\Delta E$, where ΔE is the ${}^3S_1$ to ${}^3D_1(q\bar{q})$ excitation energy. Section 8 below yields this ΔE as ca. 1800 MeV, so that $\Delta m_t^{(2)}\sim16$ MeV. Similar second-order shifts on $L=1$ masses would be even less.

[22] B. C. Shen, G. Goldhaber, S. Goldhaber, and J. A. Kadyk, Phys. Rev. Letters **15**, 731 (1965).

Using Eqs. (8) and Table II we may predict the missing members of the $L=1$, $S=1$ nonets. One obtains a $J^{PG}=0^{+-}$, $\pi'(640\mp40)$, and a $J^P=1^+$, $K^{*\prime\prime}(1158\mp40)$. It is uncertain whether $f(1253)$ is the ω-like or ϕ-like component of the $J^P=2^+$ nonet, however its quite low mass makes it a more likely candidate for ω''. In any case, Eqs. (8) and Table II yield additional $T=0$, $Y=0$ members (ω's?), $\hat{f}(1000\mp20)$ for $J^{PG}=1^{++}$ and another at 567 ∓ 40 MeV for 0^{++}. The error estimates are from the reported experimental uncertainties[11] of the known meson masses used. In Sec. 6, we shall also estimate the masses of ϕ-like mesons based on $SU(6)$-type splittings assuming the above to be ω-like, though these will be less certain.

Equations (8) and Table II give the $L=1$, $S=1$ nonet masses, a ${}^3P_c$ for each nonet component, before V_{LS} and V_t caused them to further split into 0^+, 1^+, and 2^+. These unsplit masses are

$${}^3P_c(\pi')=1164\mp40 \text{ MeV}, \quad (9)$$

$${}^3P_c(K',\bar{K}')=1250\mp40 \text{ MeV}, \quad (10)$$

$${}^3P_c(\eta';[\omega']?)=1093\mp40 \text{ MeV}, \quad (11)$$

$${}^3P_c(\eta';[\phi']?)=1360\mp40 \text{ MeV}. \quad (12)$$

The last two values are subject to the ω, ϕ uncertainty mentioned above. Equation (11) is obtained from $f(1253)$, Eq. (12) from the $SU(6)$-type splittings of the next section.

Equations (9)–(12) give the unsplit masses of the components of an over-all $L=1$, $S=1$ nonet. This ${}^3P_c$ nonet, along with the $J^P=1^+$, ${}^1P_1$ octet in Table I, makes up a single $L=1$, $SU(6)$ or $U(6)\otimes U(6)$ "35"-plet. In this form it is now suitable for examining the additional intrinsic quark spin and unitary spin-dependent $SU(6)$-type splittings and for comparing them with the $L=0$ ones.

6. GENERAL MASS FORMULA FOR $SU(6)$-TYPE SPLITTINGS AND HIGHER MESONS

In the original, $L=0$, $SU(6)$ "35"-plet,[1] the main symmetry breaking is proportional to $S(S+1)$. The corresponding sizable mass differences are

$$\Delta m[{}^3S_1(\rho)-{}^1S_0(\pi)]=630 \text{ MeV}, \quad \Delta m[{}^3S_1(K^*)-{}^1S_0'(K)]=395 \text{ MeV}. \quad (13)$$

Smaller mass splittings involve the other $SU(6)$ quantum numbers in Δm_U, in Eq. (3). Bég and Singh[19] and Kuo and Yao[20] have derived the phenomenological mass formula for the broken $SU(6)$ lowest "35"-plet,[23]

$$\Delta m_U^2=cS(S+1)+e[2\mathcal{S}(\mathcal{S}+1)-C_2^{(4)}+\tfrac{1}{4}Y^2] +f[T(T+1)-\tfrac{1}{4}Y^2+\mathfrak{N}(\mathfrak{N}+1)-\mathcal{S}(\mathcal{S}+1)]. \quad (14)$$

[23] We shall assume that as far as just the $(q\bar{q})$ "35"-plets are concerned the broken $SU(6)$ mass formula could be used even though we are actually dealing with $U(6)\otimes U(6)$.

Mass relations[20,19] based on such a formula, for example,

$$m_\phi^2+\tfrac{1}{2}(m_\rho^2+m_\omega^2)=2m_{K^*}^2 \qquad (15)$$

are obeyed to 5–8% for the $L=0$, "35"-plet mesons (cf. Table I).

In terms of the $(q\bar{q})$ model, the c, e, and f terms of Eq. (14) would arise from V_S, V_{SU}, and V_U. The nonrelativistic model would give m, rather than m^2 formulas (see also Sec. 8). The m and m^2 formulas should be equivalent to first order,[15] though empirically the m^2 formulas are most often used.[19,20] Now with the higher, $L=1$, mesons c, e, and f would be the same constants as in the $L=0$, "35"-plet, if V_S, V_{SU}, and V_U were constants, independent of r. However, since in Eq. (1) all the potentials can be functions of r, the mass formula, Eq. (14) becomes generalized to any dynamical state of $(q\bar{q})$ as

$$\begin{aligned}\Delta m_U^2=c(n',L)S(S+1)+e(n',L)[2\mathcal{S}(\mathcal{S}+1)-C_2^{(4)}+\tfrac{1}{4}Y^2]\\+f(n',L)[T(T+1)-\tfrac{1}{4}Y^2\\+\mathfrak{N}(\mathfrak{N}+1)-\mathcal{S}(\mathcal{S}+1)].\end{aligned} \qquad (16)$$

For $L\neq 0$, $S\neq 0$, this formula applies to "central" masses like the ${}^3P_c$ ones in Eqs. (9)–(12) unsplit by any $\mathbf{L}\cdot\mathbf{S}$ and tensor force effects.

For the $L=1$ "35"-plet we may compare the $S=0$ to $S=1$ mass differences using Eqs. (9), (10), and the known ${}^1P_1$ masses in Table I. We have[24]

$$\Delta m[{}^3P_c(\pi')-{}^1P_1(B)]\cong 0 \qquad (17)$$

and

$$\Delta m[{}^3P_c(K')-{}^1P_1(K_c)]\cong 0,$$

compared to the large values in Eqs. (13), of the $L=0$ case. Thus, we conclude that $V_S(r)$ in Eq. (1) is strongly dependent on r perhaps large only for smaller values of r such that

$$\langle n'L|V_S(r)|n'L\rangle\approx 0 \quad \text{for } n',\ L\neq 0. \qquad (18)$$

It is also possible however, though less likely, that the effect may be caused by additional exchange forces dependent on L, S, and U simultaneously.

What about now the other $SU(6)$ splittings for fixed S? How are they affected in going from $n'=0$, $L=0$, to $n'=0$, $L=1$? For the $L=1$, unsplit ${}^3P_c$ and ${}^1P_1$ supermultiplet, mass relations like Eq. (15), for fixed S, continue to hold though less accurately than in the $L=0$ case. The masses in Eqs. (9), (10), and (11) satisfy the relations to 10–20%. From these masses we may also calculate the $c(0,1)$, $e(0,1)$, and $f(0,1)$ using Eq. (16). The values obtained are shown in Table III. For comparison we have also calculated the $L=0$ values. The $L=1$ values are naturally much cruder. Nevertheless the data appear consistent with

$$\begin{aligned}e(0,1)\approx e(0,0),\\ f(0,1)\approx f(0,0).\end{aligned} \qquad (19)$$

[24] A similar comparison for the $T=0$, $Y=0$ components would be too uncertain because of the ω', ϕ' ambiquity.

TABLE III. Comparison of $SU(6)$-type mass splittings (in MeV^2) within $L=0$ and $L=1$ supermultiplets [cf. Eqs. (3) and (16)].

	$n'=0, L=0$	$n'=0, L=1$
c	27.7×10^4	≈ 0
e	3.77×10^4	$(5\mp 1.5)\times 10^4$
f	0.85×10^4	≈ 0

On a quark model of $L=0$ baryons, Kuo and Radicati[18] had shown that simultaneous spin–unitary-spin exchange and unitary-spin exchange forces would lead to broken $SU(6)$. Equations (19) indicate that such interactions, V_{SU} and V_U, if they led to e and f may be longer range than V_S and yield $\langle n'L|V_{SU}+V_U|n'L\rangle$ quite constant for $L=0$ and 1. On the other hand, a V_U causing $SU(3)$ splittings could also be due to N' and Λ' quark mass differences.[2] The resulting mass coefficients would then also be independent of n' and L, at least in the nonrelativistic approximation.

In spite of the ω'', ϕ'' ambiquity mentioned above in assigning $f(1253)$ (Table I), we now assume (a) that within $L=1$, $S=1$, ${}^3P_c$ nonet [Eqs. (9)–(12)], the broken $SU(6)$ mass relations[20,19] like Eq. (15) remain valid, (b) that $f(1253)$ is ω'', and (c) that nonet splittings are the same within each of $J^P=0^+$, 1^+, and 2^+ independent of $\mathbf{L}\cdot\mathbf{S}$ and V_t interactions. With these assumptions, we use Eqs. (9), (10), and (11) in Eq. (15) which give the ${}^3P_c(\phi')$ value in Eq. (12). Then using Table II, we predict a $J^P=0^+$ $\omega'(567)$, a $J^P=0^+$ $\phi'(835)$, a $J^P=1^+$ $\hat{\phi}'(1268)$, and a $J^P=2^+$ $\phi''(1520)$, all shown in Table I.

7. SUPERMULTIPLET ORBITAL EXCITATION ENERGY

From Table I, the unsplit over-all $L=0$, "35"-plet mass is calculated. Several ways of obtaining it give (see also Ref. 19),

$$\bar{m}(L=0,\text{ "35"})=605\mp 50\text{ MeV}. \qquad (20)$$

Similarly, for the $L=1$, "35"-plet one gets roughly, from Eqs. (9)–(12) and the ${}^1P_1$ masses in Table I,

$$\bar{m}(L=1,\text{ "35"})=1213\mp 100\text{ MeV}. \qquad (21)$$

With these values, we have for the over-all $L=0$ to $L=1$ excitation energy of the $(q\bar{q})$ supermultiplet:

$$\Delta m_L=608\mp 150\text{ MeV}. \qquad (22)$$

If we now assume that the interactions responsible for the splitting of the $L=0$ $(q\bar{q})$ $SU(6)$ singlet "1," $X^0(959)$ from the $L=0$, "35"-plet are unaffected by $(q\bar{q})$ excitation into the $L=1$ state, we predict from Eq. (22), an $L=1$, $SU(6)$ [or $U(6)\otimes U(6)$] "1," i.e., the $X'(1570\mp 150)$ shown in Table I. This completes the missing mesons of the classification.

8. CENTRAL POTENTIAL AND THE DYNAMICAL QUANTUM NUMBERS

In Eq. (3), the m_0 of a $(q\bar{q})$ bound state is given by

$$m_0=2M_q^*-|\epsilon_D|\,, \qquad (23)$$

where $|\epsilon_D|$ is the $(q\bar{q})$ $V_c(r)$ potential well depth including any zero-point energies. If $M_q^*\gtrsim 10$ BeV, we have $|\epsilon_D|\gtrsim 20$ BeV. The form of $\Delta m_D(n',L)$, on the other hand, and the dynamical group[21] describing the $(q\bar{q})$ mesons depend on the general shape of $V_c(r)$. We consider three types of $V_c(r)$: (i) a hydrogen-like, or more generally an attractive Yukawa-type well, (ii) a "coreless" finite well such as an isotropic three-dimensional harmonic-oscillator (h.o.) potential, and (iii) an attractive well with a minimum at $r=r_0$ and repulsions for $r<r_0$. We shall refer to (iii) as a "cored" or "hard-core" type for convenience, though the repulsive region need not extend to infinity for $r\to 0$, nor does it need to be very steep. Is the observed meson spectrum consistent with one definite type of $V_c(r)$?

The observed mesons corresponded to $(n'L)=(0s)$ and $(0p)$ states of $(q\bar{q})$. With an infinite hydrogen-like well or a Yukawa one with a range about $2r_0$, on the other hand, the first excited $(q\bar{q})$ supermultiplet would have corresponded to $n'=1$, $L=0$ giving ${}^1S_0'$, ${}^3S_1'$; $J^P=0^{-\prime}$, $1^{-\prime}$ mesons. Since such a repetition of the lowest multiplets is not observed up to about 1500 MeV, a $V_c(r)$ of the type (i) is ruled out. For this argument it is necessary, however, that effective quark mass shifts $M_q\to M_q^*$ (Sec. 4) due for example to $|qq\bar{q}\bar{q}\rangle$ or other field effects be quite independent of n' and L.

For a deep well of the "hard-core" type with $r_0\sim 10^{-13}$ cm, $|\epsilon_0|>20$ BeV, the n' and L dependence of $\Delta m_D(n',L)$ in Eq. (3) would be separable with $\Delta m_D(n',L)\cong\hbar^2L(L+1)/M_q^*r_0^2$ for fixed n' (e.g., $n'=0$). This is the case when the nodes of the radial wave function $R_{n'L}(r)$ nearly coincide for the first few L values and deviations of r from r_0 are small. More generally,[25] for a "coreless" potential with r_0 a constant between two nodes, we have for fixed n':

$$\Delta m_D(n',L)>\hbar^2L(L+1)/M_q^*r_0^2\,. \qquad (24)$$

For a typical "coreless" $V_c(r)$, for example three-dimensional h.o. well, i.e., $V_c(r)\cong-|\epsilon_0|+\frac{1}{2}kr^2$, one has

$$\Delta m_D(n',L)=2m_1n'+m_1L\,. \qquad (25)$$

If the $(q\bar{q})$ potential were such a "coreless" one, we would have $m_1=608\mp150$ MeV from Eq. (22) above. Using this in Eq. (25) and the splittings evaluated using Table II and Eqs. (8), one would then predict the following multiplets: For $n'=0$, $L=2$,

$$\begin{aligned} &J^P=1^-,\quad {}^3D_1,\quad 1200\mp150 \text{ MeV (nonet)}\\ &J^P=2^-,\quad {}^3D_2,\quad 1700\mp150 \text{ MeV (nonet)}\\ &J^P=3^-,\quad {}^3D_3,\quad >1800 \text{ MeV (nonet)}\end{aligned} \qquad (26)$$

and

$$J^P=2^-,\quad {}^1D_2,\quad 1800\mp150 \text{ MeV (octet)}.$$

Also for $n'=1$, $L=0$, one would have a new "35"-plet [$J^P=0^{-\prime}({}^1S_0')$ and $1^{-\prime}({}^3S_0')$] with a center at 1800 ∓ 100 MeV. Although most of this rich spectrum would lie above 1600 MeV, it is very likely that the $J^P=1^-({}^3D_1)$ nonet mesons would have been observed—with their G parities helping in their identification[12]—had they been there. Thus, it seems a type (ii) $V_c(r)$ too may be ruled out.

For the "hard-core" type (iii) potential

$$\Delta m_D(n',L)\cong m_1'n'+m_2L(L+1)\,, \qquad (27)$$

where

$$m_2=\hbar^2/M_q^*r_0^2\,. \qquad (28)$$

From Eq. (22) one now gets $m_2\cong 304$ MeV. This leads to the spectrum of Table I with the $n'=0$, $L=0$ and 1 as the lowest states. The next higher mesons are predicted as another "35"-plet $({}^1D,{}^3D)$ with $n'=0$, $L=2$ centered at 2400 ∓ 150 MeV. Spin-orbit and tensor forces split this center [Eqs. (8) and Table II] into

$$\begin{aligned} &J^P=1^-,\quad {}^3D_1,\quad 1800\mp150 \text{ MeV (nonet)}\\ &J^P=2^-,\quad {}^3D_2,\quad 2300\mp150 \text{ MeV (nonet)}\\ &J^P=3^-,\quad {}^3D_3,\quad 2700\mp150 \text{ MeV (nonet)}.\end{aligned} \qquad (29)$$

There is also a

$$J^P=2^-,\quad {}^1D_2,\quad 2400\mp150 \text{ MeV (octet)}$$

(see also Table III).

None of the above lie in the range of observed mesons. For the (iii), deep-well, "hard-core" $V_c(r)$, one also expects the n' excitations in Eq. (27) to lie as high or higher than Eq. (29). Thus the present data, Table I, are in favor of a deep and "cored" type well. Such a well with $|\epsilon_0|\gtrsim 20$ BeV would support roughly 3 to 10 n' levels for $L=0$ or for $L=1$ and a $L_{\max}$ of about 8 for $n'=0$. With respect to $V_c(r)$, the $(q\bar{q})$ system seems neither like hydrogen atom, nor like deuteron, but more like a quantum diatomic molecule.

A relativistic increase in quark mass, or a $V_{\text{exch}}P=(-1)^LV_{\text{exch}}$ in Eq. (1) if it existed, would lower the $L=2$ states below those of Eqs. (29) and (27), but the same would apply to a "coreless" case. Hence the above conclusion would remain unchanged.

If beyond 1500 MeV only an $n'=1$, $L=0$, $J^P=0^{-\prime}$, $1^{-\prime}$ new "35"-plet were found in future data, this would continue to support the "cored" model as would the lack of any new meson until about 1700 MeV. On the other hand, an $L=2$, "35"-plet, especially a $J^P=1^-$ $({}^3D_1)$ nonet, below 1500 MeV, would come out in favor of a "coreless" $(q\bar{q})$.

[25] See, e.g., A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic Press Inc., New York, 1963), pp. 31–36.

Equations (27), (28), and (22) above give

$$M_q^* r_0^2 \gtrsim 1300 \text{ MeV F}^2,$$

with the inequality holding for the "coreless" case [Eq. (24)]. This value is compatible with $M_q^* \gtrsim 10$ BeV, $r_0 \sim \frac{1}{2} \times 10^{-13}$ cm, but also with $M_q^* \gtrsim 10^3$ MeV (see Ref. 16) and $r_0 \sim 10^{-13}$ cm. The present discussion also indicates any $V_{\text{exch}} P_M = (-1)^L V_{\text{exch}}$ would have to be such that $|V_{\text{exch}}| < 300$ MeV regardless of the range of $V_{\text{exch}}(r)$ relative to that of $V_c(r)$.

The n' quantum number does not seem to play a role so far for mesons, but it may have nonzero values already for the low-lying baryons. This would be especially so if strong two-body repulsions[1a,18] were involved in baryons perhaps as $V_{qq} \cong -V_{q\bar{q}}$.

We now discuss the dynamical group[21] corresponding to the $V_c(r)$ indicated. Barut and Böhm[26] have shown that nonrelativistic rigid rotor levels are obtained from $\mathfrak{L}_{3+1}$ (homogeneous Lorentz group in 3 space $+1$ dimensions) as dynamical group by contraction. Each dimension of a (nonrelativistic) h.o., on the other hand, is obtained[27] from $\mathfrak{L}_{2+1}$. Then the dynamical group of the "hard core" $(q\bar{q})$, [Eq. (27)] would be $\mathfrak{L}_{2+1} \otimes \mathfrak{L}_{3+1}$ whereas the one for the "coreless" $3D$. h.o. would have been $\mathfrak{L}_{2+1} \otimes \mathfrak{L}_{2+1} \otimes \mathfrak{L}_{2+1}$. For a relativistic rotor[26] a mass formula of the form $m^2 \propto j(j+1)$ is found from the $\mathfrak{R}_{4+1}$ de Sitter group, which should correspond to extreme j-j coupling[28] since j can be a half-integer. Though in an extreme relativistic case, n' and L terms would no longer be expected to be separable, an approximate group there may still be $\mathfrak{R}_{3+1} \otimes \mathfrak{R}_{4+1}$. The nonrelativistic $(q\bar{q})$ gives an m formula, e.g., Eq. (27), instead of m^2 which should be equivalent to it however to first order[15] as mentioned in Secs. 4 and 6.

[26] A. O. Barut and A. Böhm, Phys. Rev. **139**, B1107 (1965).
[27] A. O. Barut, Phys. Rev. **139**, B1433 (1965).
[28] j-j coupling did not seem to apply to magnetic moments and to meson-baryon vertex as well as $SU(6)$ [see Ref.[8]], however recent work of A. O. Barut [International Centre for Theoretical Physics Report No. IC/65/66, 1965 (unpublished)] showed interesting mass dependences of the total j type.

9. CONCLUSION

Section 1 assigned the higher mesons into an $L=1$, $S=0$, 1, "35"-plet. The assignments agree with known G parities and the mass spectrum (Table I). The $J^P=0^+$, 1^+, and 2^+ multiplets split both by spin-orbit and spin-spin tensor forces. A general quark-antiquark potential seems justified in spite of some relativistic effects (Sec. 3). The interaction [Eq. (1)] leads to a general meson mass formula, Eqs. (3), (16), and (27). Broken $SU(6)$-type mass splittings within the $L=1$ and $L=0$ supermultiplets are compared (Table III) and indicate the nature of the corresponding interactions.

Missing members of various multiplets and missing quantum numbers are predicted (Table I; Secs. 5, 6, and 7). Mesons are placed on $(q\bar{q})$ Regge trajectories (Sec. 3).

The mass spectrum indicates the $(q\bar{q})$ central potential to have a deep well with a minimum around $r_0 \sim 0.5 \times 10^{-13}$ cm and with repulsions at shorter r's. Other, "coreless"-type potentials may be ruled out based on a discussion of higher $(q\bar{q})$ states and dynamical quantum numbers (Sec. 8). Dynamical groups corresponding to above interactions are also discussed.

Note added in proof. At the time these calculations were made only the March 1965 edition of UCRL-8030 Part I of the Berkeley Data Compilation was available and listed only the $J^P=2^+$, A2(1324), $K^{*\prime}(1410)$ and $f(1253)$ mesons. The calculations gave a 2^+, $\phi''(1520 \mp 60)$ [Table I]. Since then an $f_l(1500)$ with the same quantum numbers seems well established [see A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **37**, 633 (1965)].

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