TABLE III. Results for the (πK) amplitude.

	K^*_{mass} (MeV)	Γ_{K^*}	α_{K^*}	γ_{K^*}[a]	K^{**}_{mass} (MeV)	$\Gamma_{K^{**}}$	$\alpha_{K^{**}}$	$\gamma_{K^{**}}$
Calculation C	891	0.55	0.4	0.14	1265	0.16	0.75	0.02
Experiment	891[b]	0.22[b] (50 MeV)			1405[b]	0.12[b] (95 MeV)		

[a] See footnote a, Table II.
[b] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **37**, 633 (1965).

Collins and Teplitz.[12] They used an input ρ with a width of 0.43, which is about the same as was used in calculation A, but their output ρ trajectory did not quite make it to $l=1$, and no trajectory rose above $l=1.5$. Thus the effective force in the present calculation is much stronger than in the N/D calculation, even though the input forces are similar. It seems reasonable that this is because the method used here does include contributions to the force from higher terms in the Mandelstam iteration. If this conjecture be correct, then the results presented here indicate that a calculation that actually performs the iteration might be expected to be very successful.

[12] P. D. B. Collins and V. L. Teplitz, Phys. Rev. **140**, B663 (1965).

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Group Embedding for the Harmonic Oscillator

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The embedding of the algebra of the invariance group $SU(n)$ for the n-dimensional harmonic oscillator in larger algebras is considered. Among the four classical algebras of rank n, only $\tilde{su}(n+1)$ and $\tilde{sp}(2n)$ are found possible for this purpose. Specific generators and their commutation relations are examined, and the general Casimir operators are constructed. It is found that the whole spectrum of the harmonic oscillator can be embedded in one representation of $\tilde{su}(n+1)$. Depending upon the value of a partition constant c, a finite portion of the spectrum can be embedded in the compact algebra $su(n+1)$; the remainder is in the noncompact $su(n,1)$. In the case of $\tilde{sp}(2n)$, only the noncompact $sp(n,R)$ can include the $su(n)$ of the harmonic oscillator; moreover, the two useful representations of $sp(n,R)$, which are true representations of the universal covering group of $Sp(n,R)$, contain either all the even or all the odd levels of the spectrum.

I. INTRODUCTION

IN the attempt to combine internal symmetries with the Poincaré group, it has been repeatedly proposed[1–3] that noncompact groups (or algebras) be used to describe not only the dynamical symmetries, but also the internal symmetries of elementary particles. The study of the problem has led to very beautiful and mysterious results on the one hand, but, despite a large freedom, to many difficulties on the other.

In order to learn about the problem of embedding groups (or algebras) into larger ones, it is useful to examine completely soluble models provided by nonrelativistic classical and quantum-mechanical systems. The Kepler problem (or quantum-mechanically the hydrogen atom) and the harmonic oscillator are two standard examples.

It was shown[4] long ago that the Hamiltonian of the Kepler problem is invariant under the $SO(4)$ group, which explains the "accidental" degeneracy of the hydrogen atom. More recently, it has been conjectured that this $SO(4)$ group can be embedded in an $SO(4,1)$ group, whose generators have been written down for the classical case.[5] This $SO(4,1)$ group does not com-

* Work supported by the National Science Foundation.
† Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR 42-65.
[1] B. Kurşunoğlu, Phys. Rev. **135**, B761 (1964); A. Barut, Nuovo Cimento **32**, 234 (1964).
[2] R. Delbourgo, A. Salam, and J. Strathdee, Proc. Roy. Soc. (London) **A284**, 146 (1965); H. Bacry and J. Nuyts, Nuovo Cimento **37**, 1702 (1965); W. Rühl, Phys. Letters **14**, 346 (1965).
[3] Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Letters **17**, 148 (1965).
[4] W. Pauli, Z. Physik **36**, 336 (1926); V. Fock, *ibid.* **98**, 145 (1935); V. Bargmann, *ibid.* **99**, 576 (1936).
[5] H. Bacry, Nuovo Cimento **41A**, 222 (1966); E. C. G. Sudarshan, N. Mukunda, and L. O'Raifeartaigh, Phys. Letters **19**, 322 (1965).

mute with the Hamiltonian (and in this sense is not a true invariance group) but has the property that all physically meaningful representations of the $SO(4)$ group are included in one representation of this $SO(4,1)$ group.

It is also well known[6] that the Hamiltonian of the n-dimensional harmonic oscillator is invariant under the $SU(n)$ group, as we shall summarize in Sec. II. In this paper we shall show that the algebra $su(n)$ of this invariance group can be embedded in the algebras $su(n+1)$ or $su(n,1)$ [Sec. III] and in the algebra $sp(n,R)$ [Sec. IV] for both the classical and quantum oscillators.

The physical relevance of the useful representation of these algebras and their associated groups is discussed in Sec. V. This again, in our opinion, provides a very interesting and unifying description of energy levels, and, as such, may shed some light on the more general group-embedding problem in particle physics.

II. $U(n)$ INVARIANCE FOR THE HARMONIC OSCILLATOR

The Hamiltonian H of the normalized (unit mass and coupling constant) harmonic oscillator in n dimensions is

$$H=\tfrac{1}{2}\sum_{i=1}^{n}(p_i^2+q_i^2). \tag{2.1}$$

In the classical case the coordinates q_i and the momenta p_i satisfy the Poisson bracket

$$\{q_i,p_j\}=\delta_{ij}, \tag{2.2}$$

with the definition

$$\{A,B\}=\sum_{i=1}^{n}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right). \tag{2.3}$$

As usual, it is convenient to define

$$a_j=\tfrac{1}{2}(p_j-iq_j), \tag{2.4}$$

$$a_j^\dagger=\tfrac{1}{2}(p_j+iq_j), \tag{2.5}$$

and to introduce the *bracket*

$$(A,B)=i\{A,B\}=\sum_{m=1}^{n}\left(\frac{\partial A}{\partial a_m}\frac{\partial B}{\partial a_m^\dagger}-\frac{\partial A}{\partial a_m^\dagger}\frac{\partial B}{\partial a_m}\right). \tag{2.6}$$

One then obtains

$$H=\sum_{i=1}^{n}a_ia_i^\dagger, \tag{2.7}$$

$$(a_i,a_j^\dagger)=\delta_{ij}. \tag{2.8}$$

In the quantum case, one replaces the Poisson brackets by commutators for the operators q_j and p_j,

$$[q_j,p_k]=i\delta_{jk}. \tag{2.9}$$

Using the same definition (2.4) and (2.5) for a_j and $a_j^\dagger$, one gets

$$H=\sum_{i=1}^{n}a_i^\dagger a_i+\frac{n}{2}, \tag{2.10}$$

$$[a_i,a_j^\dagger]=\delta_{ij}. \tag{2.11}$$

From the form of (2.7) and (2.10) it is clear that the Hamiltonian, in both the classical and quantum cases, is invariant under unitary transformations in n dimensions $(U(n))$. The generators

$$E_i^j=\tfrac{1}{2}[a_i,a_j^\dagger]_+ \tag{2.12}$$

can be shown to satisfy the commutation relations

$$[E_i^j,E_l^k]=\delta_i^kE_l^j-\delta_l^jE_i^k. \tag{2.13}$$

These relations are also valid for the classical case when the commutator is replaced by the bracket. In terms of these generators H assumes the general form

$$H=E_i^i=\tfrac{1}{2}[a_i,a_i^\dagger]_+. \tag{2.14}$$

III. UNITARY EMBEDDING

The generators E_j^i which satisfy the commutation relations (2.13) form elements of the complex extension[7] $\bar{u}(n)$ of the Lie algebra. Before going to the problem of embedding this Lie algebra into a larger one, let us recall the connection between the complex Lie algebra $\bar{u}(m)$ and the many real forms associated with it: $u(k,l)$, $k+l=m$. An unitary representation of the algebra $u(k,l)$ is found if one constructs a set of E_j^i satisfying the commutation relations (2.13) and the generalized Hermiticity conditions

$$\begin{aligned}(E_j^i)^\dagger&=\eta_i\eta_jE_i^j,\\ \eta_i&=+1\quad(i=1,\cdots,k),\\ \eta_i&=-1\quad(i=k+1,\cdots,m).\end{aligned} \tag{3.1}$$

Thus there are k plus signs and $l=m-k$ minus signs. If k or l is zero, then we have

$$(E_j^i)^\dagger=E_i^j, \tag{3.2}$$

corresponding to the only compact form $u(0,m)=u(m,0)=u(m)$ of the complex extended algebra $\bar{u}(m)$.

Since for p and q Hermitian, the generators (2.12) satisfy (3.2), the harmonic oscillator is, in fact, invariant under the compact group $U(n)$.

In order to embed this compact group into a larger

[6] J. M. Jauch and E. L. Hill, Phys. Rev. **57**, 641 (1940); G. A. Baker, *ibid.* **103**, 1119 (1956); N. Mukunda, L. O'Raifeartaigh, and E. C. G. Sudarshan (to be published); A. O. Barut, Phys. Rev. **139**, B1433 (1965).

[7] As is customary in the mathematical literature, we associate capital letters with the groups and lower case letters for the corresponding algebra. We use the notation $\bar{u}(n)$ to indicate the complex extension of $u(n)$, i.e., with the number of infinitesimal generators of $u(n)$ kept fixed, we allow for nonsingular linear combinations of these generators with complex coefficients. Thus these generators span a vector space on the complex field.

Lie group, we first require the embedding of the Lie algebra into a larger Lie algebra. Depending on the Hermiticity properties of the new generators which are to be constructed, we shall be able to discuss the compactness or noncompactness of the associated group.

Let us now state the rules of embedding which we shall adopt:

(a) The larger Lie algebra is chosen to be finite and simple.

(b) The generators are built out of the p_i and q_i, or a_i and $a_i^\dagger$.

(c) As an interesting step, only algebras of rank n are considered [$su(n)$ is of rank $n-1$]. (If a simple algebra of rank m, $m\geq n+1$, contains an algebra A of rank $n-1$, it contains at least one algebra of rank n which contains A.)

In particular, we shall investigate the embedding into the classical algebras $\bar{s}\bar{u}(n+1)$, $\bar{s}\bar{o}(2n+1)$, $\bar{s}\bar{p}(2n)$ and $\bar{s}\bar{o}(2n)$. The five exceptional Lie algebras will not be considered.

In this section we construct the generators corresponding to $\bar{s}\bar{u}(n+1)$. Let us denote the new dimension of the weight diagram by the index 0. The adjoint representation (adj) of $\bar{s}\bar{u}(n+1)$ decomposes with respect to $su(n)$ according to

$$\mathrm{adj}(\bar{s}\bar{u}(n+1))=\mathrm{adj}(su(n))\oplus n\oplus\bar{n}\oplus 1\,,$$

where n and $\bar{n}$ are the n-dimensional representations of $su(n)$. The generators $E_\beta{}^\alpha(\alpha,\beta=0,\cdots,n)$ of $\bar{s}\bar{u}(n+1)$ (with the trace condition $E_\alpha{}^\alpha=0$ which we shall impose only later) decompose into $E_j{}^i$, $E_0{}^i$, $E_i{}^0$, and $E_0{}^0$ $(i=1,\cdots,n)$. They satisfy

$$[E_\alpha{}^\beta,E_\gamma{}^\delta]=\delta_\alpha{}^\delta E_\gamma{}^\beta-\delta_\gamma{}^\beta E_\alpha{}^\delta\,, \tag{3.3}$$

or, more explicitely, (2.13) and

$$[E_j{}^i,E_k{}^0]=-\delta_k{}^i E_j{}^0\,, \tag{3.4a}$$

$$[E_j{}^i,E_0{}^k]=\delta_j{}^k E_0{}^i\,, \tag{3.4b}$$

$$[E_j{}^i,E_0{}^0]=[E_k{}^0,E_l{}^0]=[E_0{}^k,E_0{}^l]=0\,, \tag{3.4c}$$

$$[E_0{}^0,E_j{}^0]=E_j{}^0\,, \tag{3.4d}$$

$$[E_0{}^0,E_0{}^j]=-E_0{}^j\,, \tag{3.4e}$$

$$[E_k{}^0,E_0{}^l]=\delta_k{}^l E_0{}^0-E_k{}^l. \tag{3.4f}$$

Let us now consider separately the classical and quantum version of this problem.

A. Classical Case

The generators must satisfy the algebraic relations (3.3) and (3.4), where the commutators should be replaced by brackets. It is clear that the a_i and $a_i^\dagger$ span the spaces of the representations n and $\bar{n}$, respectively. A symmetrical combination of the a_i's,

$$A_{ijk\cdots}=a_i a_j a_k\cdots\,, \tag{3.5}$$

can be reduced to a tensor transforming as a representation n [i.e., satisfying (3.4a)] only by saturating the indices $j,k,\cdots$ with $a_j^\dagger$, $a_k^\dagger$, $\cdots$. Since a_i and $a_i^\dagger$ commute and since $H=a_i a_i^\dagger$, the most general form of the function $E_i{}^0$ is therefore

$$E_i{}^0=f(H)a_i. \tag{3.6}$$

Similarly, we have

$$E_0{}^i=g(H)a_i^\dagger\,, \tag{3.7}$$

$$E_0{}^0=h(H). \tag{3.8}$$

Conditions (3.4d)–(3.4f) then lead to the equations

$$f(H)g(H)=h(H)\,, \tag{3.9}$$

$$\partial h(H)/\partial H=-1. \tag{3.10}$$

Letting c be the integration constant, we obtain

$$h=-H+c. \tag{3.11}$$

In this classical case we deal with functions and Poisson brackets. A discussion of the commutator relations, the Hermiticity properties and of Eq. (3.9) is postponed until after the quantum case.

B. Quantum Case

Despite the fact that a_i and $a_i^\dagger$ do not commute in the quantum case, the most general form of the operators $E_i{}^0$, $E_0{}^i$, and $E_0{}^0$ is still given by (3.6), (3.7), and (3.8). These operators satisfy (3.4a, b, c).

The following identities can be established for any function $l(H)$:

$$a_i l(H)=l(H+1)a_i\,, \tag{3.12}$$

$$a_i^\dagger l(H)=l(H-1)a_i^\dagger. \tag{3.13}$$

Using these and the general form of the E's, Eqs. (3.4d), (3.4e), and (3.4f) can be transformed into

$$h(H+1)-h(H)=h(H)-h(H-1)=-1\,, \tag{3.14}$$

$$2h(H)=f(H)g(H+1)+g(H)f(H-1)\,, \tag{3.15}$$

$$-1=f(H)g(H+1)-g(H)f(H-1). \tag{3.16}$$

Because of the discreteness properties of the eigenvalues (which we shall discuss later), the solution of the difference equation (3.14) can always be written, without loss of any physical content, as

$$h(H)=-H+c. \tag{3.17}$$

Since the generators of $su(n)$ for the harmonic oscillator satisfy (3.2), η_i $(i=1,\cdots,n)$ may be taken to be $+1$; the Hermiticity condition (3.1) is therefore reduced to

$$(E_i{}^0)^\dagger=\eta_0 E_0{}^i. \tag{3.18}$$

This implies the relation

$$f^\dagger(H)=\eta_0 g(H+1). \tag{3.19}$$

Inserting this back into (3.15), one obtains

$$h(H)=\eta_0 f(H)f^\dagger(H)+\tfrac{1}{2}=-H+c\,, \tag{3.20}$$

$$h(H)=\eta_0 g(H)g^\dagger(H)-\tfrac{1}{2}=-H+c\,. \tag{3.21}$$

Since $ff^\dagger$ or $gg^\dagger$ is Hermitian positive, (3.20) and (3.21) yield the result

$$\eta_0=1: \quad H\leq c-\tfrac{1}{2}, \quad \text{compact}\ u(n+1)\,, \tag{3.22}$$

$$\eta_0=-1: \quad H\geq c+\tfrac{1}{2}, \quad \text{noncompact}\ u(n,1)\,, \tag{3.23}$$

where the inequalities mean that we take into account only the irreducible unitary representations for which all the eigenvalues of H are either $\leq c-\frac{1}{2}$ or $\geq c+\frac{1}{2}$.

We note that we are dealing with the nonsimple algebra $\bar{u}(n+1)$, since the trace condition $E_\alpha{}^\alpha=0$ has not been imposed. From (2.14) and (3.17), we see that $E_\alpha{}^\alpha=c$. Thus the simplest way to obtain $\bar{s}\bar{u}(n+1)$ from $\bar{u}(n+1)$ is to require that $c=0$. However, this turns out to conflict with the required value of c when we later consider the Casimir operator of the larger groups $SU(n,1)$ and $SU(n+1)$. The proper way to obtain the groups $SU(n+1)$ or $SU(n,1)$ is therefore to consider a new set of traceless generators $D_\beta{}^\alpha$, defined by (A4) in Appendix A. Because $D_\beta{}^\alpha$ satisfies the same algebra as $E_\beta{}^\alpha$, the results contained in (3.22) and (3.23) remain valid for the simple algebra $\bar{s}\bar{u}(n+1)$, and c takes on a denumerable set of values. The group $SU(n,1)$ is not its own universal covering group[8,9] whose algebra however is $su(n,1)$. If one does not restrict the representation of $su(n,1)$ to be true representations of $SU(n,1)$ and thus allow them to be true representations of the universal covering group of $SU(n,1)$, then the parameter c may assume a continuous set of values which includes $c=0$. A physical interpretation of these results will be given in Sec. V.

IV. SYMPLECTIC EMBEDDING

In this section we consider the embedding of the algebra $u(n)$ of the harmonic oscillator into the sympletic algebra $\bar{s}\bar{p}(2n)$. The adjoint representation of $\bar{s}\bar{p}(2n)$ decomposes with respect to $u(n)$ according to

$$\mathrm{adj}(\bar{s}\bar{p}(2n))=\mathrm{adj}(u(n))\oplus n(n+1)/2\oplus\overline{n(n+1)/2}\,, \tag{4.1}$$

where $n(n+1)/2$ is the $[n(n+1)/2]$-dimensional representation of $u(n)$ obtained by symmetrizing the product of two fundamental n-dimensional representations. $[\overline{n(n+1)/2}$ is the representation conjugate to $n(n+1)/2]$. Let us again denote the new dimension in the weight diagram by the index 0. The generators of the $\bar{s}\bar{p}(2n)$ algebra thus decompose into $E_j{}^i$, $E_{ij}{}^0$, and $E_0{}^{ij}$ where $E_{ij}{}^0$ and $E_0{}^{ij}$ are symmetrical in i and j. The trace $E_i{}^i$ can be identified with $E_0{}^0$ the new diagonal operator since the symplectic transformations in $2n$ dimensions always have[10] det+1.

The commutation relations are (2.13) and

$$[E_j{}^i,E_{kl}{}^0]=-\delta_k{}^iE_{jl}{}^0-\delta_l{}^iE_{jk}{}^0\,, \tag{4.2a}$$

$$[E_j{}^i,E_0{}^{kl}]=\delta_j{}^kE_0{}^{il}+\delta_j{}^lE_0{}^{ik}\,, \tag{4.2b}$$

$$[E_0{}^{kl},E_0{}^{mn}]=[E_{kl}{}^0,E_{mn}{}^0]=0\,, \tag{4.2c}$$

$$[E_0{}^{kl},E_{mn}{}^0]=\delta_m{}^kE_n{}^l+\delta_n{}^lE_m{}^k+\delta_n{}^kE_m{}^l+\delta_m{}^lE_n{}^k\,. \tag{4.2d}$$

There is a freedom in the choice of the over-all sign of the right-hand side of (4.2d). The particular sign chosen is convenient, as will become apparent later. It follows from (4.2) and the identification $E_0{}^0=E_i{}^i$ that

$$\begin{aligned}[][E_0{}^0,E_{kl}{}^0]&=-2E_{kl}{}^0\,,\\ [E_0{}^0,E_0{}^{kl}]&=2E_0{}^{kl}\,.\end{aligned} \tag{4.3}$$

The generalized Hermiticity conditions can be written as

$$(E_{kl}{}^0)^\dagger=\eta_0\eta_k\eta_lE_0{}^{kl}\,. \tag{4.4}$$

As we have shown that only the compact $u(n)$ underlies the harmonic oscillator (3.2) (i.e., $\eta_i=+1$), we are simply left with the two possibilities

$$(E_{kl}{}^0)^\dagger=\eta_0E_0{}^{kl}\,, \tag{4.5}$$

where $\eta_0=+1$ corresponds to the compact case $sp(2n)$ and $\eta_0=-1$ to the noncompact form $sp(n,n)$. In the literature this noncompact form is sometimes also referred to as $sp(n,R)$. A different choice of sign for (4.2d) would result in the opposite correspondence.

Let us again discuss the classical and quantum cases separately.

A. Classical Case

The most general form of the operators $E_{ij}{}^0$ and $E_0{}^{ij}$ is quite evidently

$$\begin{aligned}E_{ij}{}^0&=r(H)a_ia_j\,,\\ E_0{}^{ij}&=s(H)a_i{}^\dagger a_j{}^\dagger\,.\end{aligned} \tag{4.6}$$

They satisfy (4.2a, b, c), where the commutators are replaced by brackets. In the same way (4.2d) reduces to

$$r(H)s(H)=-1\,. \tag{4.7}$$

B. Quantum Case

The same general form (4.6) also satisfies (4.2a, b, c). Equation (4.2d) reduces to

$$r(H)s(H+2)=s(H)r(H-2)=-1\,, \tag{4.8}$$

quite similar to (4.7).

We remark that $E_0{}^0=E_i{}^i=H$, in fact, satisfies (4.3) in both the classical and quantum cases. In view of the hermiticity condition (4.5) we see that (4.7) and (4.8)

[8] S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic Press Inc., New York, 1962).

[9] L. C. Biedenharn, J. Nuyts, and N. Straumann, Ann. Inst. Henri Poincaré **III A**, 13 (1965).

[10] H. Weyl, *The Classical Groups* (Princeton University Press, Princeton, New Jersey, 1946), p. 166.

lead, in both cases, to the same condition

$$\eta_0 r(H) r^\dagger(H) = -1\,, \tag{4.9}$$

which can be satisfied only with $\eta_0 = -1$. This embedding thus allows only for the noncompact form $sp(n,n)$. The representations we have constructed and which we shall discuss in Sec. V are, however, not true representations of the group $Sp(n,n)$ but of its universal covering group.

V. DISCUSSION AND CONCLUSION

Before discussing briefly the interpretation of the results obtained in the preceding two sections, we note that there are two other series of classical algebras $\bar{so}(2n)$ and $\bar{so}(2n+1)$ which contain the algebra of $su(n)$. However, this embedding is not possible for the harmonic oscillator for $n>3$. This can be seen quite simply by noting that both algebras $\bar{so}(2n)$ and $\bar{so}(2n+1)$ require the existence of operators E_{ij} which are antisymmetrical in i and j, but which cannot be built out of the commuting a_i and a_j. For $n=3$, $su(3)$ can be embedded in $\bar{so}(6)$, which is isomorphic to $\bar{su}(4)$, because E_{ij} can be defined as $\epsilon_{ijk}E_0{}^k$, but not in $\bar{so}(7)$. For $n=2$, $su(2)$ can be embedded in $\bar{so}(5)$ isomorphic to $\bar{sp}(4)$, while the embedding of $su(2)$ into $\bar{so}(4)$ cannot be achieved.

To discuss the representations of the larger Lie algebra, one evidently must find the explicit values of the Casimir operators [of $\bar{su}(n+1)$ and $\bar{sp}(2n)$ in the present problem]. It is necessary also to decompose the representations corresponding to these values with respect to the maximal compact subalgebra $u(n)$. For $\bar{su}(n+1)$ the generators of the maximal compact subalgebra can conveniently be chosen to be composed of the algebra of $su(n)$,

$$D_j{}^i = E_j{}^i - \frac{1}{n}\delta_j{}^i H \tag{5.1}$$

and of a generalized "hypercharge"

$$Y = \frac{1}{n+1}(E_i{}^i - nE_0{}^0) = H - nc/(n+1)\,. \tag{5.2}$$

For $\bar{sp}(2n)$ the maximal compact subalgebra is composed of (5.1) and $E_0{}^0 = H$.

The ground-state energy of the harmonic oscillator can be obtained from group-theoretic considerations. In Appendix A we have constructed the Casimir operators of the algebra $\bar{su}(q)$. Applying the special properties of the harmonic oscillator, specified by the relations (2.11)–(2.14), to the results obtained there, we get

$${}^1I_j{}^i = D_j{}^i\,,\quad i,j=1,\cdots,n\,, \tag{5.3}$$

$${}^2I_j{}^i = xD_j{}^i + y\delta_j{}^i\,, \tag{5.4}$$

where

$$x = (1-2/n)H\,, \tag{5.5}$$

$$y = ((n-1)/n^2)(H^2 - \tfrac{1}{4}n^2)\,. \tag{5.6}$$

Defining

$${}^{m+1}I_j{}^i = \phi_m(x,y)D_j{}^i + \psi_m(x,y)\delta_j{}^i \tag{5.7}$$

and substituting this into the recursion formula (A5), we obtain

$$\psi_{m+1} = y\phi_m\,, \tag{5.8}$$

$$\phi_{m+1} = x\phi_m + y\phi_{m-1}\,, \tag{5.9}$$

whose solution is

$$\phi_m(x,y) = \sum_{i=0} C_{m-i}{}^i x^{m-2i} y^i\,, \tag{5.10}$$

where $C_{m-i}{}^i = (m-i)!/[i!(m-2i)!]$ and the upper limit of the summation is $m/2$ or $(m-1)/2$ according as m is even or odd. Thus the mth-order Casimir operator B_m of $su(n)$ describing the harmonic oscillator is

$$B_m = {}^mI_i{}^i = ny\phi_{m-2}\,. \tag{5.11}$$

Since the eigenvalues of the second-order Casimir operator of a compact algebra is positive, we have $B_2 \geq 0$. In the case of the trivial representation, all B_m vanish, which is possible in the present problem only if y equals zero, which being the lowest possible value for B_2 is, according to (5.6), the lowest possible value for H^2. This implies that $H^2 = n^2/4$; the positive definiteness of H then selects the solution $H = n/2$. This is therefore the energy of the ground state (as it should); this state is invariant under the transformation of the group $su(n)$, (5.1).

If one applies on a state of definite energy the generalized raising and lowering operators $E_i{}^0$, $E_0{}^j$, $E_{ij}{}^0$, $E_0{}^{ij}$, one obtains states separated by one unit in the case of $\bar{su}(n+1)$ and by two units in the case of $\bar{sp}(2n)$. This is quite evident by inspection of the commutation relations (3.4d, e) and (4.3).

To learn further about the representations of the algebras $\bar{su}(n+1)$ or $\bar{sp}(2n)$ which describe the harmonic oscillator, we must first determine the Casimir operators for these larger algebras and then evaluate them for the present problem. In Appendix A the Casimir operators in question have been constructed.

The evaluation of them for $\bar{su}(n+1)$ follows the same procedure as we have developed earlier in this section for $su(n)$. Equations (5.7)–(5.10) remain valid; we need only replace i, j by α, β, where $\alpha,\beta = 0, 1, \cdots, n$. The values of x and y are, however, altered. Writing

$${}^2I_\beta{}^\alpha = xD_\beta{}^\alpha + y\delta_\beta{}^\alpha\,, \tag{5.12}$$

where

$$D_\beta{}^\alpha = E_\beta{}^\alpha - (1/n+1)E_\gamma{}^\gamma\delta_\beta{}^\alpha\,, \tag{5.13}$$

we obtain, after using various appropriate relations in Secs. II and III,

$$x = c(n-1)/(n+1)\,,\qquad y = n(c/(n+1))^2 - \tfrac{1}{4}n\,. \tag{5.14}$$

The Casimir operator B_m for $\bar{s}\bar{u}(n+1)$ is then

$$B_m = {}^mI_\alpha{}^\alpha = (n+1)y\phi_{m-2}, \tag{5.15}$$

where ϕ_m is defined in (5.10). Since $\phi_0=1$, we see that

$$B_m = B_2\phi_{m-2}. \tag{5.16}$$

For the symplectic algebra $sp(n,n)$, the independent Casimir operators are of even order

$$C_{2p} = {}^{(2p)}K_i{}^i, \tag{5.17}$$

where the operators ${}^mK_j{}^i$ are defined recursively in (A9)–(A12). By direct substitution of (A9) into (A10), one obtains, with the help of the results in Sec. IV, that

$${}^2K_j{}^i = z\delta_j{}^i, \tag{5.18}$$

where

$$z = -(n+\tfrac{1}{2})/2. \tag{5.19}$$

Thus we have, by generalization, that

$${}^{2p}K_j{}^i = z^p\delta_j{}^i, \tag{5.20}$$

and

$$C_{2p} = n[-(n+\tfrac{1}{2})/2]^p. \tag{5.21}$$

In Appendix B we have analyzed in detail the problem for the two-dimensional harmonic oscillators, based on the results of Ref. 9 for the group $SU(2,1)$. For $c\leq\frac{1}{2}$, the states can be described completely by one representation of $su(2,1)$. For $c\geq\frac{3}{2}$, the states corresponding to

$$1\leq H\leq c-\tfrac{1}{2} \tag{5.22}$$

are embedded into one representation of the compact form $su(3)$, while the states corresponding to

$$H\geq c+\tfrac{1}{2} \tag{5.23}$$

belong to one representation of the noncompact $su(2,1)$.

Conclusion

Without going into the details of the calculations whose premises have been given in (5.16) and (5.21), let us simply state the results for the embedding of the n-dimensional harmonic oscillator.

In the case of $\bar{s}\bar{u}(n+1)$, the spectrum can be embedded into the two forms $su(n,1)$ and $su(n+1)$, the partition being dependent upon the integration constant c. Conveniently, c can be written as

$$c=\tfrac{1}{2}(n-1)+\nu, \tag{5.24}$$

where ν is an integer for allowed representations of the group $SU(n,1)$ or $SU(n+1)$. For $\nu\geq 1$, there are ν states in the interval

$$\tfrac{1}{2}n\leq H\leq \tfrac{1}{2}n+\nu-1 \tag{5.25}$$

which can be embedded in one unitary representation of $su(n+1)$. States with

$$H\geq\tfrac{1}{2}n+\nu \tag{5.26}$$

are embedded in one representation of the group $SU(n,1)$. For $\nu<1$, all the states, starting with $H=\frac{1}{2}n$, are embedded in one representation of the $su(n,1)$ algebra corresponding to the $SU(n,1)$ group. In this case, since no compact part is allowed, a new freedom arises. The noncompact $SU(n,1)$ group is not its own universal covering group. In analogy with the $SU(2,1)$ problem discussed in Appendix B, the parameter ν may assume, continuously, any value <1.

For the algebra of $sp(n,n)$ there exists no analogous c parameter. The representation to which the spectrum of the harmonic oscillator corresponds is not a true representation of the group $Sp(n,n)$ but rather a representation of the universal covering group of $Sp(n,n)$. This can be exemplified by the following remark: in $Sp(n,n)$ an $SU(n)$ singlet, or more generally, an $SU(n)$ Young diagram of zero boxes modulo n, has a Y value of integral multiple of n whereas for the harmonic oscillator the $SU(n)$ singlet ground states has $Y=H=n/2$. The spectrum has either the form

$$H=\tfrac{1}{2}n+2\beta \tag{5.27}$$

or

$$H=\tfrac{1}{2}n+2\beta+1, \tag{5.28}$$

where β is a positive integer. All these states of (5.27) or (5.28) are contained in only one representation of $sp(n,n)$, whose Casimir operators are by (5.21). Thus, although the algebra $su(n)$ can be embedded in $sp(n,n)$, only every other state of the harmonic oscillator is included in the relevant representation of $sp(n,n)$.

For any value of H ($H=\frac{1}{2}n+\alpha$, α positive integer) the degenerative states form the representation of $su(n)$ corresponding to a one-row Young diagram of α boxes. These representations are completely symmetric and form the generalized triangular weight diagram in $n-1$ dimensions.

In conclusion, we see that the embedding of the harmonic oscillator into a noninvariance group offers a very interesting way of analyzing a "badly broken symmetry," provided that the breaking term (here the energy) has reasonably simple transformation properties. In this light we may perhaps hope that the many propositions to frame the elementary particle spectrum in badly broken symmetries based on compact or even noncompact groups may be more reasonably founded and more interesting than one might imagine from *a priori* considerations.

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APPENDIX A: CASIMIR OPERATORS FOR $\bar{u}(q)$, $\bar{s}\bar{u}(q)$, AND $\bar{s}\bar{p}(2q)$ ALGEBRAS

In terms of the generators $E_\beta{}^\alpha(\alpha,\beta=1,\cdots,q)$ given in Sec. II which satisfy the commutation relations

$$[E_\beta{}^\alpha, E_\delta{}^\gamma] = \delta_\beta{}^\gamma E_\delta{}^\alpha - \delta_\delta{}^\alpha E_\beta{}^\gamma, \tag{A1}$$

let us introduce the following recursion formula

$$
\begin{aligned}
{}^{0}J_{\beta}{}^{\alpha}&=\delta_{\beta}{}^{\alpha}\,,\\
{}^{1}J_{\beta}{}^{\alpha}&=E_{\beta}{}^{\alpha}\,,
\end{aligned}
\tag{A2}
$$

$$
{}^{m+1}J_{\beta}{}^{\alpha}=\tfrac{1}{4}[{}^{m}J_{\gamma}{}^{\alpha}E_{\beta}{}^{\gamma}+{}^{m}J_{\beta}{}^{\gamma}E_{\gamma}{}^{\alpha}+E_{\gamma}{}^{\alpha}\,{}^{m}J_{\beta}{}^{\gamma}+E_{\beta}{}^{\gamma}\,{}^{m}J_{\gamma}{}^{\alpha}]\,.
$$

The q Casimir operators of $\bar{u}(q)$ can be shown to be

$$C_{m}={}^{m}J_{\alpha}{}^{\alpha}. \tag{A3}$$

For the algebra of $\bar{s}\bar{u}(q)$ whose generators are

$$D_{\beta}{}^{\alpha}=E_{\beta}{}^{\alpha}-\frac{1}{q}\delta_{\beta}{}^{\alpha}E_{\gamma}{}^{\gamma}\,, \tag{A4}$$

one defines, in an analogous fashion,

$$
\begin{aligned}
{}^{0}I_{\beta}{}^{\alpha}&=\delta_{\beta}{}^{\alpha}\,,\\
{}^{1}I_{\beta}{}^{\alpha}&=D_{\beta}{}^{\alpha}\,,
\end{aligned}
\tag{A5}
$$

$$
{}^{m+1}I_{\beta}{}^{\alpha}=\tfrac{1}{4}[{}^{m}I_{\gamma}{}^{\alpha}D_{\beta}{}^{\gamma}+{}^{m}I_{\beta}{}^{\gamma}D_{\gamma}{}^{\alpha}+D_{\gamma}{}^{\alpha}\,{}^{m}I_{\beta}{}^{\gamma}+D_{\beta}{}^{\gamma}\,{}^{m}I_{\gamma}{}^{\alpha}]\,,
$$

and obtains for the $(q-1)$ Casimir operators

$$B_{m}={}^{m}I_{\alpha}{}^{\alpha}. \tag{A6}$$

We remark that the $D_{\beta}{}^{\alpha}$ and $E_{\beta}{}^{\alpha}$ satisfy exactly the same commutation relations (A1). Moreover, $D_{\beta}{}^{\alpha}$ has the property

$$B_{1}=D_{\alpha}{}^{\alpha}=0. \tag{A7}$$

The generators $E_{j}{}^{i}$, $E_{ij}{}^{0}$, $E_{0}{}^{ij}$ $(i,\ j=1,\ \cdots,\ p)$ of the symplectic algebra $\bar{s}\bar{p}(2p)$ defined in Sec. IV satisfy the commutation relations

$$
\begin{aligned}
&[E_{j}{}^{i},E_{l}{}^{k}]=\delta_{j}{}^{k}E_{l}{}^{i}-\delta_{l}{}^{i}E_{j}{}^{k}\,,\\
&[E_{j}{}^{i},E_{kl}{}^{0}]=-\delta_{k}{}^{i}E_{jl}{}^{0}-\delta_{l}{}^{i}E_{jk}{}^{0}\,,\\
&[E_{j}{}^{i},E_{0}{}^{kl}]=\delta_{j}{}^{k}E^{li}+\delta_{j}{}^{l}E^{ki}\,,\\
&[E_{kl}{}^{0},E_{ij}{}^{0}]=0\,,\\
&[E_{0}{}^{kl},E_{ij}{}^{0}]=\delta_{i}{}^{k}E_{j}{}^{l}+\delta_{j}{}^{l}E_{i}{}^{k}+\delta_{j}{}^{k}E_{i}{}^{l}+\delta_{i}{}^{l}E_{j}{}^{k}.
\end{aligned}
\tag{A8}
$$

Let us construct the following quantities which satisfy recursion formulas:

$$
\begin{aligned}
{}^{0}K_{j}{}^{i}&=\delta_{j}{}^{i}\,, & {}^{0}K_{ij}&=0\,,\\
{}^{1}K_{j}{}^{i}&=E_{j}{}^{i}\,, & {}^{1}K_{ij}&=E_{ij}{}^{0}\,,
\end{aligned}
\tag{A9}
$$

$$
\begin{aligned}
{}^{(m+1)}K_{j}{}^{i}=\tfrac{1}{4}[&{}^{m}K_{l}{}^{i}E_{j}{}^{l}+{}^{m}K^{il}E_{lj}{}^{0}+E_{l}{}^{i}\,{}^{m}K_{j}{}^{l}\\
&+E_{0}{}^{il}\,{}^{m}K_{jl}+{}^{m}K_{j}{}^{l}E_{l}{}^{i}+{}^{m}K_{jl}E_{0}{}^{li}\\
&+E_{j}{}^{l}\,{}^{m}K_{l}{}^{i}+E_{jl}{}^{0}\,{}^{m}K^{il}]\,,
\end{aligned}
\tag{A10}
$$

$$
\begin{aligned}
{}^{(m+1)}K_{ij}=\tfrac{1}{4}[&{}^{m}K_{i}{}^{l}E_{lj}{}^{0}-{}^{m}K_{il}E_{j}{}^{l}+E_{i}{}^{l}\,{}^{m}K_{lj}\\
&+(-1)^{m}E_{il}{}^{0}\,{}^{m}K_{j}{}^{l}+(-1)^{m}\,{}^{m}K_{j}{}^{l}E_{li}\\
&+{}^{m}K_{lj}E_{i}{}^{l}-E_{i}{}^{l}\,{}^{m}K_{il}+E_{jl}{}^{0}\,{}^{m}K_{i}{}^{l}]\,,
\end{aligned}
\tag{A11}
$$

$$
\begin{aligned}
{}^{(2p)}K_{ij}&=-{}^{(2p)}K_{ji}\,,\\
{}^{(2p+1)}K_{ij}&={}^{(2p+1)}K_{ji}\,.
\end{aligned}
\tag{A12}
$$

They satisfy the commutation relations

$$
\begin{aligned}
&[{}^{m}K_{j}{}^{i},E_{t}{}^{s}]=\delta_{j}{}^{s}\,{}^{m}K_{t}{}^{i}-\delta_{t}{}^{i}\,{}^{m}K_{j}{}^{s}\,,\\
&[{}^{m}K_{ij},E_{t}{}^{s}]=\delta_{i}{}^{s}\,{}^{m}K_{tj}+\delta_{j}{}^{s}\,{}^{m}K_{it}\,,\\
&[{}^{m}K_{j}{}^{i},E_{st}]=-\delta_{s}{}^{i}\,{}^{m}K_{jt}-\delta_{t}{}^{i}\,{}^{m}K_{js}\,,\\
&[{}^{m}K^{ij},E_{st}]=(-1)^{m+1}\delta_{s}{}^{i}\,{}^{m}K_{t}{}^{j}+(-1)^{m+1}\delta_{t}{}^{i}\,{}^{m}K_{s}{}^{j}\\
&\qquad\qquad+\delta_{s}{}^{j}\,{}^{m}K_{t}{}^{i}+\delta_{t}{}^{j}\,{}^{m}K_{s}{}^{i}\,,\\
&[{}^{m}K_{ij},E_{st}]=0.
\end{aligned}
\tag{A13}
$$

It is easy to see that the Casimir operator of an even order is

$$C_{2p}={}^{(2p)}K_{i}{}^{i}. \tag{A14}$$

The Casimir operators of odd orders are linearly dependent on the ones of even and lower orders.

APPENDIX B: UNITARY EMBEDDING FOR THE TWO-DIMENSIONAL HARMONIC OSCILLATOR

In this Appendix we treat in detail the embedding of the algebra $su(2)$ for the 2-dimensional harmonic oscillator in the algebra $\bar{s}\bar{u}(3)$, and discuss the different useful representations of the larger algebra. For this purpose, the work of Ref. 9 on the unitary representations of $su(2,1)$ will be referred to extensively. Let us record here, in the notations of this paper, some of the fundamental formulas.

The *discrete* set of representations of $\bar{s}\bar{u}(3)$ is characterized by the set of three integer numbers d_1, d_2,

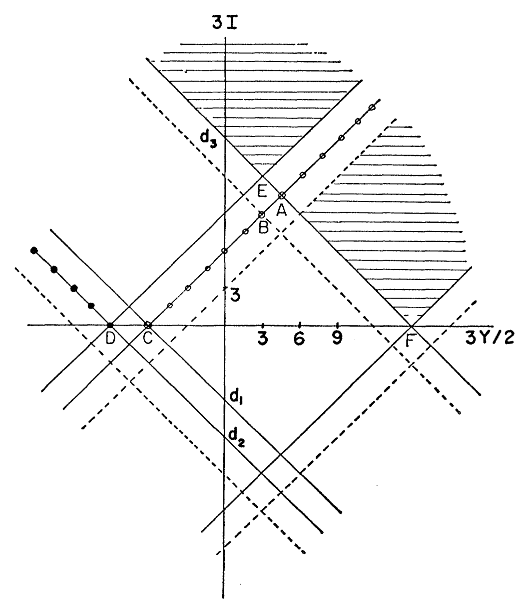

FIG. 1. Unitary representation of $\bar{s}\bar{u}(3)$ for the case $\nu\geq 1$ integer, here $\nu=7$: $d_1=-\nu+1=-6$, $d_2=-\nu-2=-9$, $d_3=2\nu+1=15$. Open circles correspond to degenerate states of the spectrum of the harmonic oscillator. Between B and C the states are in the compact $su(3)$; above A the states are in the noncompact $su(2,1)$. The construction of this plot is explained in Ref. 9.

and d_3, related by

$$d_1 \equiv d_2 \equiv d_3 = 0 \pmod 1, \tag{B1}$$

$$d_1 \equiv d_2 \equiv d_3 \pmod 3, \tag{B2}$$

$$d_1 + d_2 + d_3 = 0. \tag{B3}$$

The quadratic and cubic Casimir operators are

$$B_2 = -2[(d_1d_2 + d_2d_3 + d_1d_3)/9 + 1], \tag{B4}$$

$$B_3 = d_1d_2d_3/9. \tag{B5}$$

From the considerations of Sec. V, these Casimir operators for the 2-dimensional harmonic oscillator can also be found to be

$$B_2 = 2(c^2 - 9/4)/3, \tag{B6}$$

$$B_3 = 2c(c^2 - 9/4)/9. \tag{B7}$$

Comparing the two forms, we have the general correspondence

$$d_1 = -c + \tfrac{3}{2}, \tag{B8}$$

$$d_2 = -c - \tfrac{3}{2}, \tag{B9}$$

$$d_3 = 2c. \tag{B10}$$

It is clear then that c must be half-integral; let us write it as

$$c = \nu + \tfrac{1}{2}, \tag{B11}$$

where ν is an integer.

Given a value of ν, and therefore of c, we have a particular representation of $\bar{s}\bar{u}(3)$, characterized by the values of B_2 and B_3 as determined according to (B6) and (B7). From the corresponding values of the d_i's such a representation can be exhibited pictorially in a $(3I, 3Y/2)$ plot, as shown in Figs. 1 and 2, where I and Y are defined by [cf. (5.6) and (5.11)]

$$I(I+1) = \tfrac{1}{2}D_j{}^iD_i{}^j = (H^2 - 1)/4, \quad i, j = 1, 2, \tag{B12}$$

$$Y = \tfrac{1}{3}(E_i{}^i - 2E_0{}^0). \tag{B13}$$

In the case of the harmonic oscillator, (B13) implies

$$H = Y + 2c/3 = Y + (2\nu + 1)/3. \tag{B14}$$

The open circles in these figures stand for the degenerate states of the harmonic oscillators, the full spectrum of which is thus seen embedded in one representation of $\bar{s}\bar{u}(3)$. Figures 1 and 2 correspond to the two cases $\nu \geq 1$ and $\nu \leq 0$, respectively.

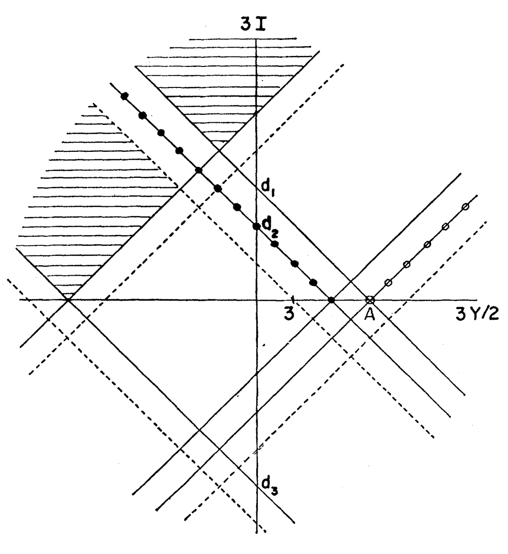

FIG. 2. Unitary representation of $\bar{s}\bar{u}(3)$ for the case $\nu < 1$, here $\nu = -8$: $d_1 = -\nu + 1 = 9$, $d_2 = -\nu - 2 = 6$, $d_3 = 2\nu + 1 = -15$. Open circles correspond to degenerate states of the spectrum of the harmonic oscillator; they are all in the noncompact $su(2,1)$. The construction of this plot is explained in Ref. 9.

$\nu \geq 1$

In this case, we have $d_3 > d_1 > d_2$. The point A ($I = \nu/2$, $H = \nu + 1$) corresponds to the lowest states (of multiplicity $3\nu + 1$) of the spectrum belonging to an unitary representation of the noncompact algebra $su(2,1)$. The points B ($I = (\nu - 1)/2$, $H = \nu$) and C ($I = 0$, $H = 1$) are the extreme states of a triangular representation of the compact $su(3)$. The fact that the lowest ν states are embedded in the compact form of $\bar{s}\bar{u}(3)$ is in agreement with our earlier result (3.22), (3.23). The representations characterized by the extreme points D (dark dots) and E (cross-hatched) clearly are not suitable for our problem because they involve always states of negative energy. (These representations extend infinitely to the left.) The point F corresponds to $I = 0$, $H = 2\nu + 1 \neq 1$, which is incompatible with (B12); hence, the representations (cross-hatched region) characterized by it as an extreme point are also not suitable.

$\nu \leq 0$

In this case, we have $d_1 > d_2 \geq d_3$ except when $\nu = 0$, in which case $d_1 = d_3 > d_2$. Figure 2 shows that only the noncompact representation is allowed with its spectrum starting from the ground state at the point A ($I = 0$, $H = 1$). This is again in agreement with (3.22), (3.23).

As has been discussed in the Appendix of Ref. 9, the group $SU(2,1)$ and quite generally all of the groups $SU(p,q)$, p, $q > 0$ are not their own universal covering group. In order to obtain the representations of the universal covering group which are projective (up to phase) representations of $SU(2,1)$, one has simply to relax, in some definite way, the requirement that d_1, d_2, d_3 be integers. In the present case, the harmonic oscillator can be described only by a figure similar to Fig. 2 provided that ν be any real number less than one. The open circle with lowest value of Y corresponds to $H = 1$ for any value of c compatible with $\nu < 1$. (See point A of Fig. 2.)