Wave Functions for Particles of Higher Spin*

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A simple method is presented for constructing wave functions for particles of arbitrary spin. These are helicity eigenstates and satisfy the Rarita-Schwinger equation. The wave functions are given explicitly.

I. INTRODUCTION

IN several recent papers we have considered models for pion-nucleon resonances.[1] To calculate the relevant Feynman diagrams it was necessary to employ higher spin wave functions. Although we only used wave functions for spin $\frac{1}{2}$, 1, $\cdots\frac{5}{2}$, a technique was developed for constructing wave functions in a helicity representation for arbitrary spin. Since this method seems somewhat simpler than the one presented recently by Frishman and Gotsman,[2] we wish to describe it here.

We shall show that these wave functions for arbitrary spin can be constructed from the spin-$\frac{1}{2}$ and spin-1 functions by coupling with Clebsch-Gordan coefficients. By always coupling to the maximum possible spin, we automatically satisfy the symmetry requirements.

In Sec. II we discuss the spin-$\frac{1}{2}$ and spin-1 cases. The wave functions are given explicitly, and their properties are noted. Sections III and IV contain a construction of integral and half-integral spin wave functions, respectively. We prove that these satisfy the Rarita-Schwinger[3] equation.

II. WAVE FUNCTIONS FOR SPIN ONE-HALF AND SPIN ONE

We wish to construct wave functions which are eigenstates of helicity. We shall use the phase conventions of Jacob and Wick[4] with D functions as defined by Rose[5] and a metric, $g_{\mu\nu}=(+,-,-,-)$. With these definitions we have

$$U_\alpha{}^\lambda(\mathbf{p})=\sum_\beta S_{\alpha\beta}(\mathbf{p})\sum_s D_{s\lambda}{}^{1/2}(\varphi,\theta,-\varphi)U_\beta{}^s(0) \quad (1)$$

and

$$\epsilon_\mu{}^\lambda(\mathbf{p})=\sum_\nu \Lambda_\mu{}^\nu(\mathbf{p})\sum_s D_{s\lambda}{}^1(\varphi,\theta,-\varphi)\epsilon_\nu{}^s(0)\,, \quad (2)$$

where S and Λ are the 4-spinor and vector representations of a Lorentz transformation[6] along the direction, (θ,φ), of $\mathbf{p}$. $U_\alpha{}^\lambda(\mathbf{p})$ and $\epsilon_\mu{}^\lambda(\mathbf{p})$ are now wave functions for spin-$\frac{1}{2}$ and spin-1 particles, respectively, if $U_\alpha{}^s(0)$ and $\epsilon_\mu{}^s(0)$ describe the corresponding rest states with spin quantized along the z axis. In particular we have

$$U_\alpha{}^+(0)=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\quad U_\alpha{}^-(0)=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad (3)$$

$$\begin{aligned}\epsilon_\mu{}^+(0)&=\tfrac{1}{2}\sqrt{2}(0;1,i,0)\,,\\ \epsilon_\mu{}^0(0)&=(0;0,0,-1)\,,\\ \epsilon_\mu{}^-(0)&=\tfrac{1}{2}\sqrt{2}(0;-1,i,0)\,,\end{aligned} \quad (4)$$

and

$$U_\alpha{}^+(\mathbf{p})=N_p\begin{bmatrix}\cos(\frac{1}{2}\theta)\\ e^{i\varphi}\sin(\frac{1}{2}\theta)\\ \dfrac{p}{p_0+m}\begin{pmatrix}\cos(\frac{1}{2}\theta)\\ e^{i\varphi}\sin(\frac{1}{2}\theta)\end{pmatrix}\end{bmatrix},$$
$$U_\alpha{}^-(\mathbf{p})=N_p\begin{bmatrix}-e^{-i\varphi}\sin(\frac{1}{2}\theta)\\ \cos(\frac{1}{2}\theta)\\ \dfrac{-p}{p_0+m}\begin{pmatrix}-e^{-i\varphi}\sin(\frac{1}{2}\theta)\\ \cos(\frac{1}{2}\theta)\end{pmatrix}\end{bmatrix}, \quad (5)$$

$$\begin{aligned}\epsilon_\mu{}^+(\mathbf{p})&=\tfrac{1}{2}\sqrt{2}e^{i\varphi}(0;\cos\theta\cos\varphi-i\sin\varphi,\\&\qquad\cos\theta\sin\varphi+i\cos\varphi,-\sin\theta)\,,\\ \epsilon_\mu{}^0(\mathbf{p})&=((p/m);-(p_0/m)\sin\theta\cos\varphi,\\&\qquad-(p_0/m)\sin\theta\sin\varphi,-(p_0/m)\cos\theta)\,,\\ \epsilon_\mu{}^-(\mathbf{p})&=\tfrac{1}{2}\sqrt{2}e^{-i\varphi}(0;-\cos\theta\cos\varphi-i\sin\varphi,\\&\qquad-\cos\theta\sin\varphi+i\cos\varphi,\sin\theta)\,,\end{aligned} \quad (6)$$

where $N_p=(p_0+m/2m)^{1/2}$ and $p_\mu=(p_0;-p\sin\theta\cos\varphi,-p\sin\theta\sin\varphi,-p\cos\theta)$.

These states satisfy the usual equations of motion,

$$(\not p-m)U^\lambda=0\,, \quad (7)$$

and

$$\begin{aligned}(p^2-m^2)\epsilon_\mu{}^\lambda&=0\,,\\ p^\mu\epsilon_\mu{}^\lambda&=0\,,\end{aligned} \quad (8)$$

and have a normalization such that

$$\bar U_\alpha{}^\lambda(p)U_\alpha{}^{\lambda'}(p)=\delta_{\lambda\lambda'}\,, \quad (9)$$

and

$$g^{\mu\nu}\epsilon_\mu{}^{\lambda\dagger}(p)\epsilon_\nu{}^{\lambda'}(p)=-\delta_{\lambda\lambda'}\,. \quad (10)$$

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[5] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).
[6] We work in a representation where

$$\gamma^0=\begin{pmatrix}I&0\\0&-I\end{pmatrix},\quad \gamma^i=\begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix},\quad \gamma_5=\begin{pmatrix}0&I\\I&0\end{pmatrix}.$$

III. WAVE FUNCTIONS FOR INTEGRAL SPIN

For particles of spin L we want a helicity wave function, $\Phi^{\Lambda}{}_{\mu_1\cdots\mu_L}(\mathbf{p})$, which satisfies

$$(p^2-m^2)\Phi^{\Lambda}{}_{\mu_1\cdots\mu_L}=0\,, \tag{11}$$

$$p^{\mu}\Phi^{\Lambda}{}_{\mu\mu_2\cdots\mu_L}=0\,, \tag{12}$$

$$g^{\mu\nu}\Phi^{\Lambda}{}_{\mu\nu\mu_3\cdots\mu_L}=0\,, \tag{13}$$

and is symmetric in the 4-vector indices, $\mu_1\cdots\mu_L$. Such a wave function is given by

$$\Phi^{\Lambda}{}_{\mu_1\cdots\mu_L}(\mathbf{p})=\sum_{m,\lambda}\langle L-1,\,m,\,1,\,\lambda\,|\,L-1,\,1,\,L,\,\Lambda\rangle \times\Phi^{m}{}_{\mu_1\cdots\mu_{L-1}}(\mathbf{p})\epsilon_{\mu_L}{}^{\lambda}(\mathbf{p})\,, \tag{14}$$

where

$$\Phi_{\mu_1}{}^{\Lambda}(\mathbf{p})=\epsilon_{\mu_1}{}^{\Lambda}(\mathbf{p})\,.$$

The normalization is such that

$$\Phi^{\Lambda}{}_{\mu_1\cdots\mu_L}{}^{\dagger}(\mathbf{p})\Phi^{\Lambda'\mu_1\cdots\mu_L}(\mathbf{p})=(-1)^L\delta_{\Lambda\Lambda'}\,. \tag{15}$$

This wave function is obviously an eigenstate of helicity due to our Clebsch-Gordan coupling. Also, since we always couple to maximum L, it is symmetric in $\mu_1\cdots\mu_L$. Equations (11) and (12) follow directly from Eq. (8). It is not obvious, however, that Eq. (13) is satisfied, but from (14) it is clear that if it is true for $L=2$, then it holds in general. As it is easily checked and shown to be correct for $L=2$, we do not include this verification here. Thus we have shown that Eq. (14) provides a simple construction of Φ for arbitrary L.

IV. WAVE FUNCTIONS FOR HALF-INTEGRAL SPIN

The method here is just as straightforward as that of Sec. III. Now we must satisfy the Rarita-Schwinger equations[3]:

$$(\not{p}-m)_{\alpha\beta}\Psi_{\beta}{}^{\Lambda}{}_{\mu_1\cdots\mu_L}=0\,, \tag{16}$$

$$\gamma_{\alpha}{}^{\mu}{}_{\beta}\Psi_{\beta}{}^{\Lambda}{}_{\mu\mu_2\cdots\mu_L}=0\,, \tag{17}$$

for spin $L+\frac{1}{2}$ where again we require symmetry in $\mu_1\cdots\mu_L$. The wave function also satisfies

$$p^{\mu}\Psi_{\alpha}{}^{\Lambda}{}_{\mu\mu_2\cdots\mu_L}=0 \tag{18}$$

and

$$g^{\mu\nu}\Psi_{\alpha}{}^{\Lambda}{}_{\mu\nu\mu_3\cdots\mu_L}=0\,, \tag{19}$$

which are useful but are in fact consequences of (16) and (17). Analogously to Eq. (14), we write

$$\Psi_{\alpha}{}^{\Lambda}{}_{\mu_1\cdots\mu_L}(\mathbf{p})=\sum_{m,\lambda}\langle L-\tfrac{1}{2},\,m,\,1,\,\lambda\,|\,L-\tfrac{1}{2},\,1,\,L+\tfrac{1}{2},\,\Lambda\rangle \times\Psi_{\alpha}{}^{m}{}_{\mu_1\cdots\mu_{L-1}}(\mathbf{p})\epsilon_{\mu_L}{}^{\lambda}(\mathbf{p})\,, \tag{20}$$

where

$$\Psi_{\alpha}{}^{\Lambda}(\mathbf{p})=U_{\alpha}{}^{\Lambda}(\mathbf{p})\,.$$

In this case the normalization is

$$\bar{\Psi}_{\alpha}{}^{\Lambda}{}_{\mu_1\cdots\mu_L}(\mathbf{p})\Psi_{\alpha}{}^{\Lambda'\mu_1\cdots\mu_L}(\mathbf{p})=(-1)^L\delta_{\Lambda\Lambda'}\,. \tag{21}$$

Equation (16) is satisfied because of Eq. (7), and Eq. (17) will hold for all L if it is true for $L=1$. Here again we omit the explicit verification for $L=1$, but it is easily shown algebraically. Since we couple to maximum L, Ψ is symmetric in $\mu_1\cdots\mu_L$, and is by construction a helicity state. Equations (14) and (20) thus provide an explicit form for helicity functions of any spin. For completeness, we note here that because we couple to maximum angular momentum, this scheme is independent of the order of coupling. This allows us to write Ψ as

$$\Psi_{\alpha}{}^{\Lambda}{}_{\mu_1\cdots\mu_L}(\mathbf{p})=\sum_{m,\lambda}\langle L,m,\tfrac{1}{2},\lambda\,|\,L,\,\tfrac{1}{2},\,L+\tfrac{1}{2},\,\Lambda\rangle \times\Phi^{m}{}_{\mu_1\cdots\mu_L}(\mathbf{p})U_{\alpha}{}^{\lambda}(\mathbf{p})\,. \tag{22}$$

V. CONCLUSION

We refer the reader to our previous work[1] for applications of these wave functions. Possible couplings among particles are limited in form by Eqs. (12), (13), (17), (18), and (19). Actual calculation of vertices is straightforward; and we add in conclusion, that as illustrated by earlier papers, it is advantageous to express these vertices in terms of D functions. In particular when one leg of the vertex is at rest, the wave functions, (14) and (20), lend themselves to this very well and hence allow for helicity projections of entire Feynman graphs with a minimum of effort.